A FAMILY OF RELATIVISTIC BARGMANN-TYPE TRANSFORMS ATTACHED TO MAASS LAPLACIANS ON THE POINCARÉ DISK

Z. MOUAYN

ABSTRACT. We construct a set of coherent states through special superpositions of eigenstates of a relativistic pseudoharmonic oscillator. In each superposition the coefficients are chosen to be L^2-eigenfunctions of a σ-weight Maass Laplacian on the Poincaré disk, which are associated with the eigenvalue $4m(\sigma-1+m)$, $m=0,1,\cdots,[(\sigma-1)/2]$. For each integer m the obtained coherent states transform constitutes a of relativistic Bargmann-type transform whose integral kernel is expressed in terms of a special Appel-Kampé de Fériet's hypergeometric function.

1. INTRODUCTION

In [1] V. Bargmann has introduced a second transform labeled by a parameter $\delta>0$ as

$$B_\delta : L^2\left(\mathbb{R}_+, \frac{x^\delta}{\Gamma(1+\delta)}dx\right) \to \mathcal{A}^{\delta+1}(\mathbb{D}) \tag{1.1}$$

defined by

$$B_\delta[\phi](z) := \frac{\left(\frac{\delta}{\pi}\right)^{\frac{1}{2}}}{\Gamma(\delta+1)(1-z)^{\delta+1}}\int_0^{+\infty}\exp\left(-\frac{x}{2}\left(\frac{1+z}{1-z}\right)\right)\phi(x)\,x^\delta dx \tag{1.2}$$

where

$$\mathcal{A}^{(\delta+1)}(\mathbb{D}) := \left\{\psi \text{ analytic on } \mathbb{D}, \int_{\mathbb{D}}|\psi(z)|^2\left(1-|z|^2\right)^{\delta-1}d\mu(z)<+\infty\right\} \tag{1.3}$$

denotes the weighted Bergman space on the unit disk $\mathbb{D}=\{z\in\mathbb{C}; |z|<1\}$ and $d\mu(z)$ being the Lebesgue measure on it. Thus, setting $\delta=\sigma-1>0$ and using the isometry form $L^2(\mathbb{R}_+, dx)$ onto $L^2\left(\mathbb{R}_+, \Gamma^{-1}(\sigma)\,x^{\sigma-1}dx\right)$, one extends B_δ to the transform

$$\mathcal{B}^{\mathrm{II}}_\sigma : L^2(\mathbb{R}_+, dx) \to \mathcal{A}^\sigma(\mathbb{D}) \tag{1.4}$$

defined by

$$\mathcal{B}^{\mathrm{II}}_\sigma[\phi](z) := \frac{\left(\frac{(\sigma-1)}{\pi\Gamma(\sigma)}\right)^{\frac{1}{2}}}{(1-z)^\sigma}\int_0^{+\infty}\exp\left(-\frac{x}{2}\left(\frac{1+z}{1-z}\right)\right)\phi(x)\,x^{\frac{1}{2}(\sigma-1)}dx. \tag{1.5}$$

The involved kernel function in (1.5) corresponds to the generating function of Laguerre polynomials [2]. The latter ones turn out to be fundamental pieces in expressing wave functions of the eigenstates of the pseudoharmonic oscillator Hamiltonian (see [3] and references therein). Now, after being observed that the role of the Laguerre polynomials can be replaced by the continuous dual Hahn polynomials [4] which are involved in the wave functions of the eigenstates of a relativistic pseudoharmonic oscillator Hamiltonian [5], here we propose a "*relativistic*" version of the second Bargmann transform (1.4), which also will include a generalization of the arrival space $\mathcal{A}_m^{\sigma}$ in (1.4). The latter one will be replaced by the eigenspace ([6, 7]):

$$\mathcal{A}_m^{\sigma}(\mathbb{D}) := \left\{\psi : \mathbb{D} \to \mathbb{C}, \Delta_\sigma \psi = \epsilon_m^{\sigma}\psi, \int_{\mathbb{D}} |\psi(z)|^2 (1-z\bar{z})^{\sigma-2} d\mu(z) < +\infty\right\} \tag{1.6}$$

of the second order differential operator

$$\Delta_\sigma := -4(1-z\bar{z})\left((1-z\bar{z})\frac{\partial^2}{\partial z\partial \bar{z}} - \sigma\bar{z}\frac{\partial}{\partial \bar{z}}\right) \tag{1.7}$$

with the eigenvalue (*hyperbolic Landau level*):

$$\epsilon_m^{\sigma} := 4m(\sigma-1-m), m = 0,1,2,\cdots,\left[\frac{\sigma-1}{2}\right], \tag{1.8}$$

where $[x]$ denotes the greatest integer less than x. The operator in (1.6) can be unitarly intertwined to represent the Schrödinger operator of a charged particle evolving in the Poincaré disk under influence of a uniform magnetic field with a strength proportional to σ. For $m = 0$, the space $\mathcal{A}_0^{\sigma}(\mathbb{D})$ in (1.5) coincides with the Bergman space $\mathcal{A}^{\sigma}(\mathbb{D})$ in (1.3) where $\delta = \sigma - 1$. In this paper, we precisely construct a family of integral transforms of the form $\mathcal{B}_{c,m}^{\text{II}} : L^2(\mathbb{R}_+) \to \mathcal{A}^{2(\gamma+m)}(\mathbb{D})$ defined by

$$\mathcal{B}_{c,m}^{\text{II}}[f](z) := \frac{(2\gamma-1)^{\frac{1}{2}}}{\sqrt{\pi m!}(1-z)^{2\gamma}}\frac{\sqrt{2\Gamma(m+2\gamma)}(i)^{\gamma}}{\Gamma(2\gamma)\Gamma\left(\gamma+\frac{1}{2}\right)}\left(\frac{\bar{z}-1}{(1-z)(1-z\bar{z})}\right)^m \tag{1.9}$$

$$\times\int_0^{+\infty}\frac{\Gamma^2(\gamma+i\xi)\left(c^{-4}\right)^{i\xi}}{\Gamma(i\xi)}F_5\left(\begin{array}{ccc}\gamma+i\xi, & \gamma-i\xi: & 2\gamma+m \\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array} \mid \frac{1-\bar{z}z}{(1-\bar{z})(z-1)}, \frac{1}{1-\bar{z}}\right)f(\xi)d\xi,$$

where $\sigma = 2(\gamma+m), c > 0, \gamma = \gamma(c) = (1+\sqrt{1+2c^4})/2$ and F_5 is a special Appel-Kampé de Fériet hypergeometric function [8]. Our method in constructing the transform (1.8) is based on a coherent states analysis by adopting a general probabilistic scheme *"à la Gazeau"* [9]. In the analytic case which corresponds to the particular value $m = 0$, we prove that the transform (1.9) reduces to the following one

$$\mathcal{B}_{c,0}^{\text{II}} : L^2(\mathbb{R}_+) \to \mathcal{A}^{2\gamma}(\mathbb{D}) \tag{1.10}$$

defined by

$$\mathcal{B}_{c,0}^{\text{II}}[f](z) = \frac{\sqrt{2}(-i)^{\gamma}\left(\frac{2\gamma-1}{\pi\Gamma(2\gamma)}\right)^{\frac{1}{2}}}{\Gamma\left(\gamma+\frac{1}{2}\right)(1-z)^{\gamma}} \tag{1.11}$$

$$\times\int_0^{+\infty}\frac{\Gamma^2(\gamma-i\xi)\Gamma^{-1}(-i\xi)}{\left(c^{-4}\right)^{i\xi}(1-z)^{i\xi}}\cdot{}_2F_1\left(\begin{array}{c}\gamma-i\xi, \frac{1}{2}-i\xi \\ \gamma+\frac{1}{2}\end{array} \mid z\right)f(\xi)d\xi,$$

where ${}_2F_1(\cdot)$ denotes Gauss hypergeometric function [10]).

The paper is organized as follows. In Section 2, we recall briefly some needed tools from the spectral theory of the σ-weight Maass Laplacians on the Poincaré disk. Section 3 deals with the construction of a set of coherent states in the framework of a probabilistic Hilbertian schem without specifying the corresponding Hamiltonian system . In section 4 we summarize some required information on a relativistic model for the pseudoharmonic oscillator. In section 5 we particularize the constructed coherent states for the relativistic pseudoharmonic oscillator and we obtain expressions for their wave functions in an explicit way. Section 6 is devoted to establish the corresponding coherent states transforms.

2. Maass Laplacians on the Poincaré disk

Let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ be unit disk endowed with its usual Khäler metric $ds^2 = -\partial\overline{\partial}Log\,(1 - z\overline{z})\,dz \otimes d\overline{z}$. The Bergman distance on $\mathbb{D}$ is given by

$$\cosh^2 d\,(z, w) = \frac{(1 - z\overline{w})(1 - \overline{z}w)}{(1 - z\overline{z})\,(1 - w\overline{w})} \tag{2.1}$$

and the volume element reads

$$d\mu(z) = \frac{1}{(1 - z\overline{z})^2} d\nu(z) \tag{2.2}$$

with the Lebesgue measure $d\nu(z)$. Let us consider the differential $1-$form on $\mathbb{D}$ defined by $\theta = -i\left(\partial - \overline{\partial}\right) Log\,(1 - z\overline{z})$ to which the Schrödinger operator

$$H_\sigma := \left(d + i\frac{\sigma}{2}ext\,(\theta)\right)^* \left(d + i\frac{\sigma}{2}ext\,(\theta)\right) \tag{2.3}$$

can be associated. Here $\sigma \geq 0$ is a fixed number, d denotes the usual exterior derivative on differential forms on $\mathbb{D}$ and $ext\,(\theta)$ is the exterior multiplication by θ while the symbol $*$ stands for the adjoint operator with respect to the Hermitian scalar product induced by the Bergman metric ds^2 on differential forms. Actually, the operator H_σ is acting on the Hilbert space $L^2\,(\mathbb{D}, d\mu(z))$ and can be unitarily intertwined as

$$(1 - z\overline{z})^{\frac{1}{2}\sigma}\,\Delta_\sigma\,(1 - z\overline{z})^{-\frac{1}{2}\sigma} = H_\sigma \tag{2.4}$$

in terms of the second order differential operator Δ_σ introduced in (1.7). The latter one is acting on the Hilbert space $L^{2,\sigma}\,(\mathbb{D}) = L^2\left(\mathbb{D}, (1 - z\overline{z})^{\sigma-2}\,d\nu(z)\right)$. Note that this operator is an elliptic densely defined operator on $L^{2,\sigma}\,(\mathbb{D})$ and admits a unique self-adjoint realization that we denote also by Δ_σ. The part of its spectrum is not empty if and only if $\sigma > 1$. This discrete part consists of eigenvalues occurring with infinite multiplicities and having the expression $\epsilon_m^\sigma = 4m\,(\sigma - m - 1)$ in (1.8) for varying $m = 0, 1, \cdots, [(\sigma - 1)/2]$. Moreover, it is well known ([6, 7, 11]) that the functions given in terms of Jacobi polynomials [2] by

$$\Phi_k^{\sigma,m}(z) := \sqrt{\frac{(\sigma - 2m - 1)\,\Gamma\,(\sigma - m)\,k!}{\pi m!\Gamma\,(\sigma - 2m + k)}}\frac{(-1)^k\,\overline{z}^{m-k}}{(1 - z\overline{z})^m}P_k^{(m-k,\sigma-2m-1)}\,(1 - 2z\overline{z}) \tag{2.5}$$

constitute an orthonormal basis of the eigenspace

$$\mathcal{A}_m^\sigma\,(\mathbb{D}) := \left\{\phi \in L^{2,\sigma}\,(\mathbb{D})\,, \Delta_\sigma\phi = \epsilon_m^\sigma\phi\right\}. \tag{2.6}$$

of Δ_σ associated with the eigenvalue ϵ_m^σ in (2.6). Finally, the L^2-eigenspace $\mathcal{A}_0^\sigma\,(\mathbb{D}) = \left\{\phi \in L^{2,\sigma}\,(\mathbb{D})\,, \Delta_\sigma\phi = 0\right\}$ corresponding to $m = 0$ and associated to $\epsilon_0^\sigma = 0$ in (2.7) reduces further to the weighted Bergman space consisting of holomorphic functions ϕ: $\mathbb{D} \to \mathbb{C}$ with the growth condition

$$\int_{\mathbb{D}} |\phi(z)|^2\,(1 - z\overline{z})^{\sigma-2}\,d\nu(z) < +\infty. \tag{2.7}$$

This is why the eigenspaces in (2.6) are also called generalized Bergman spaces on the complex unit disk.

Remark 2.1. *.The spectral analysis of Δ_σ have been studied by many authors, see* [6] *and references therein and it can also be obtained from the $\sigma-$weight Maass Laplacian $y^2\left(\partial_x^2+\partial_y^2\right)-i\sigma y\partial_x$ on the Poincaré upper half-plane* [12]*. The condition $\sigma>1$ ensuring the existence of the eigenvalues ϵ_m^σ in* (1.8) *should implies that the magnetic field $B=\sigma\Omega(z)$, where Ω stands for the Khäler 2-form on $\mathbb{D}$, has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the classical orbit of the particle will intercept the disk boundary whose points stands for $\{\infty\}$ which means escaping to infinity (see* [13, p.189]*).*

3. Coherent states in a Hilbertian probabilistic schem

The negative binomial states [14] are labeled by points $z\in\mathbb{D}$ and are of the form

$$\mid z\,,\sigma,0>:=(1-z\overline{z})^{\frac{1}{2}\sigma}\sum_{k=0}^{+\infty}\sqrt{\frac{\Gamma\left(\sigma+k\right)}{\Gamma\left(\sigma\right)k!}}z^k\mid\psi_k> \tag{3.1}$$

where $\sigma>1$ is a fixed parameter and the *kets*$\mid\psi_k>$ are for instance elements of an abstract Hilbert space $\mathcal{H}$. Their photon-counting probability distribution is given by

$$\Pr\left(X=k\right):=\left|<\psi_k\mid z\,,\sigma,0>\right|^2=(1-z\overline{z})^\sigma\left(z\overline{z}\right)^k\frac{\Gamma\left(\sigma+k\right)}{\Gamma\left(\sigma\right)k!}$$

which obeys the negative binomial probability distribution [15]. Observe that the coefficients in the superposition (3.1):

$$\Phi_k^{\sigma,0}(z):=\sqrt{\frac{\Gamma\left(\sigma+k\right)}{\pi\Gamma\left(\sigma\right)k!}}z^k,k=0,1,2,\cdots, \tag{3.2}$$

constitute an orthonormal basis of the eigenspace $\mathcal{A}_0^\sigma\left(\mathbb{D}\right)$ associated with the first eigenvalue $\epsilon_0^\sigma=0$ and consisting of analytic functions on $\mathbb{D}$ with the growth condition (2.7). For instance, let $\sigma>1$ and $m=0,1,\cdots,\left[\left(\sigma-1\right)/2\right]$ be fixed parameters and let $\{\mid\psi_k>\}_{k=0}^\infty$ be a set of Fock states in a Hilbert space $\mathcal{H}$. Then, adopting the Hilbertian probabilistic schem of coherent sates in ([9, p.74, Eq. (58)]), we state the following.

Definition 3.1. *A class of coherent states can be defined as*

$$\mid z,\sigma,m>=\left(\mathcal{N}_{\sigma,m}(z)\right)^{-\frac{1}{2}}\sum_{k=0}^{+\infty}\overline{\Phi_k^{\sigma,m}(z)}\mid\psi_k> \tag{3.3}$$

where $\mathcal{N}_{\sigma,m}(z)$ is a normalization factor and $\left\{\Phi_k^{\sigma,m}(z)\right\}_{k=0}^\infty$ is the orthonormal basis (2.5) *of the generalized Bergman space $\mathcal{A}_m^\sigma\left(\mathbb{D}\right)$.*

Now, one of the important task to do is to determine is the overlap relation between two coherent states.

Proposition 3.2. *Let $\sigma>1$ and $m=0,1,\cdots,\left[\left(\sigma-1\right)/2\right]$. Then, for every $z,w\in\mathbb{D}$, the overlap relation between two coherent states is given through the scalar product*

$$<w,\sigma,m\mid z\,,\sigma,m>_{\mathcal{H}}=\frac{(\sigma-2m-1)\Gamma\left(\sigma-m\right)\left(\mathcal{N}(z)\mathcal{N}\left(w\right)\right)^{-\frac{1}{2}}}{\pi m!\left(-1\right)^m\Gamma\left(\sigma-2m\right)\left(1-z\overline{w}\right)^\sigma} \tag{3.4}$$

$$\times\left(\frac{\left(1-z\overline{w}\right)\left(1-\overline{w}z\right)}{\left(1-z\overline{z}\right)\left(1-w\overline{w}\right)}\right)^m\cdot{}_2F_1\left(-m,\sigma-m,\sigma-2m;\frac{\left(1-z\overline{z}\right)\left(1-w\overline{w}\right)}{\left(1-z\overline{w}\right)\left(1-\overline{w}z\right)}\right)$$

where ${}_2F_1$ is a terminating Gauss hypergeometric sum.

Proof. In view of Eq. (3.3), the scalar product of two coherent states $\mid z\ ,\sigma,m>$ and $\mid w\ ,\sigma,m>$ in $\mathcal{H}$ reads

$$< w,\sigma,m \mid z,\sigma,m >_{\mathcal{H}} = \left(\mathcal{N}_{\sigma,m}(z)\mathcal{N}_{\sigma,m}(w)\right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \Phi_k^{\sigma,m}(z)\, \overline{\Phi_k^{\sigma,m}(w)}$$
$$= \frac{(\sigma-2m-1)\left(\mathcal{N}(z)\mathcal{N}(w)\right)^{-\frac{1}{2}}}{\pi\left((1-z\bar{z})(1-w\overline{w})\right)^m} \mathcal{S}_{z,w}^{\sigma,m}, \tag{3.5}$$

where

$$\mathcal{S}_{z,w}^{\sigma,m} = \frac{\Gamma(\sigma-m)(\bar{z}w)^m}{m!\Gamma(\sigma-2m)} \sum_{k=0}^{+\infty} \frac{k!}{(\sigma-2m)_k} \left(\frac{1}{\bar{z}w}\right)^k \tag{3.6}$$
$$\times P_k^{(m-k,\sigma-2m-1)}(1-2z\bar{z})\, P_k^{(m-k,\sigma-2m-1)}(1-2w\overline{w}).$$

Making use of the following identity due to A. Sirvastava and A. B. Rao ([16, p.1329]):

$$\sum_{n=0}^{+\infty} \frac{n!t^n}{(1+\alpha)_n} P_n^{(\gamma-n,\alpha)}(x)\, P_n^{(\gamma-n,\alpha)}(y) = \left(1-\frac{1}{4}(x-1)(y-1)t\right)^{1+\gamma+\alpha} \tag{3.7}$$
$$\times (1-t)^{\gamma}\, .{}_2F_1\left(1+\gamma+\alpha,-\gamma,1+\alpha;\frac{-(x+1)(y+1)t}{(1-t)(4-(x-1)(y-1)t)}\right)$$

for $n=k, t=1/\bar{z}w, \gamma=m, \alpha=\sigma-2m-1, x=1-2z\bar{z}$ and $y=1-2w\overline{w}$, we obtain, after calculations, the expression

$$\mathcal{S}_{z,w}^{\sigma,m} = \frac{\Gamma(\sigma-m)(-1)^m}{m!\Gamma(\sigma-2m)} \frac{\left((1-z\overline{w})(1-\overline{w}z)\right)^m}{(1-z\overline{w})^{\sigma}} \tag{3.8}$$
$$\times_2 F_1\left(-m,\sigma-m,\sigma-2m;\frac{(1-z\bar{z})(1-w\overline{w})}{(1-z\overline{w})(1-\overline{w}z)}\right).$$

Returning back to Eq. (3.5) and inserting the expression (3.8) we arrive at the announced formula. □

Corollary 3.3. *. The normalization factor in* (3.3) *is given by*

$$\mathcal{N}_{\sigma,m}(z) = \frac{(\sigma-2m-1)}{\pi(1-z\bar{z})^{\sigma}}, \tag{3.9}$$

for every $z \in \mathbb{D}$.

Proof. We first make appeal to the relation ([17, p.212]):

$${}_2F_1\left(-n,n+\kappa+\varrho+1,1+\kappa;\frac{1-\tau}{2}\right) = \frac{n!\Gamma(1+\kappa)}{\Gamma(1+\kappa+n)} P_n^{(\kappa,\varrho)}(\tau) \tag{3.10}$$

connecting the ${}_2F_1$−sum with the Jacobi polynomial for the parameters $n=m, \kappa=\sigma-2m-1, \varrho=0$ and the variable

$$\tau = 1-2\frac{(1-z\bar{z})(1-w\overline{w})}{(1-z\overline{w})(1-\overline{w}z)} \tag{3.11}$$

to rewrite Eq. (3.4) as

$$\sqrt{\mathcal{N}_{\sigma,m}(z)\mathcal{N}_{\sigma,m}\left(w\right)}=\frac{\left(-1\right)^{m}\left(\sigma-2m-1\right)\left(1-z\overline{w}\right)^{-\sigma}}{\pi<w;\sigma,m\mid z\,,\sigma,m>_{\mathcal{H}}} \tag{3.12}$$

$$\times P_{m}^{\left(\sigma-2m-1,0\right)}\left(1-2\frac{\left(1-z\overline{z}\right)\left(1-w\overline{w}\right)}{\left(1-z\overline{w}\right)\left(1-\overline{w}z\right)}\right).$$

The factor $\mathcal{N}_{\sigma,m}(z)$ should be such that $< z,\sigma,m\mid z,\sigma,m>_{\mathcal{H}}=1$. So that we put $z=w$ in (3.12) and we use the well known symmetry identity satisfied by the Jacobi polynomials $P_{m}^{(\gamma,\varrho)}\left(\xi\right)=\left(-1\right)^{m}P_{m}^{(\varrho,\gamma)}\left(-\xi\right)$ to obtain the expression

$$\mathcal{N}_{\sigma,m}(z)=\frac{\left(\sigma-2m-1\right)}{\pi\left(1-z\overline{z}\right)^{\sigma}}P_{m}^{\left(0,\sigma-2m-1\right)}\left(1\right) \tag{3.13}$$

Finally, we apply the fact that ([17, p.209]):

$$P_{n}^{(\alpha,\varrho)}\left(1\right)=\frac{\Gamma\left(n+\alpha+1\right)}{n!\Gamma\left(\alpha+1\right)} \tag{3.14}$$

in the case of $\alpha=0,n=m$ and $\varrho=\sigma-2m-1$. This ends the proof. □

Proposition 3.4. *Let $\sigma>1$ and $m=0,1,\cdots,\left[\left(\sigma-1\right)/2\right]$. Then, the states in* (3.3) *satisfy the following resolution of the identity*

$$\mathbf{1}_{\mathcal{H}}=\int_{\mathbb{D}}\mid z,\sigma,m><z,\sigma,m\mid d\mu_{\sigma,m}(z) \tag{3.15}$$

where $\mathbf{1}_{\mathcal{H}}$ is the identity operator, $d\mu_{\sigma,m}(z)$ is a measure given by

$$d\mu_{\sigma,m}(z):=\pi^{-1}(\sigma-2m-1)\left(1-z\overline{z}\right)^{-2}d\nu(z), \tag{3.16}$$

and $d\nu(z)$ being the Lebesgue measure on $\mathbb{D}$.

Proof. Let us assume that the measure takes the form $d\mu_{\sigma,m}(z)=\mathcal{N}_{\sigma,m}(z)\Omega\left(z\right)d\nu(z)$ where $\Omega(z)$ is an auxiliary density to be determined. Let $\varphi\in\mathcal{H}$ and let us start by writing the following action

$$\mathcal{O}\left[\varphi\right]:=\left(\int_{\mathbb{D}}\mid z,\sigma,m><z,\sigma,m\mid d\mu_{\sigma,m}(z)\right)\left[\varphi\right] \tag{3.17}$$

$$=\int_{\mathbb{D}}<\varphi\mid z,\sigma,m><z,\sigma,m\mid d\mu_{\sigma,m}(z). \tag{3.18}$$

Making use Eq. (3.3), we obtain successively

$$\mathcal{O}\left[\varphi\right]=\int_{\mathbb{D}}<\varphi\mid\left(\mathcal{N}_{\sigma,m}(z)\right)^{-\frac{1}{2}}\sum_{k=0}^{+\infty}\Phi_{k}^{\sigma,m}(z)\mid\psi_{k}>><z,\sigma,m\mid d\mu_{\sigma,m}(z) \tag{3.19}$$

$$=\left(\sum_{j,k=0}^{+\infty}\int_{\mathbb{D}}\overline{\Phi_{j}^{\sigma,m}(z)}\Phi_{k}^{\sigma,m}(z)\mid\psi_{k}><\psi_{j}\mid\left(\mathcal{N}_{\sigma,m}(z)\right)^{-1}d\mu_{\sigma,m}(z)\right)\left[\varphi\right]. \tag{3.20}$$

We replace the measure $d\mu_{\sigma,m}(z)$ by the expression $\mathcal{N}_{\sigma,m}(z)\Omega(z)d\nu(z)$, then Eq. (3.20) can be written without φ as follows

$$\mathcal{O}=\sum_{j,k=0}^{+\infty}\left[\int_{\mathbb{D}}\overline{\Phi_{j}^{\sigma,m}(z)}\Phi_{k}^{\sigma,m}(z)\Omega(z)d\nu(z)\right]\mid\psi_{j}><\psi_{k}\mid. \tag{3.21}$$

Therefore, we need to have

$$\int_{\mathbb{D}} \overline{\Phi_j^{\sigma,m}(z)} \Phi_k^{\sigma,m}(z) \Omega(z) d\nu(z) = \delta_{jk}. \tag{3.22}$$

For this we recall the orthogonality relation of the $\Phi_k^{\sigma,m}(z)$ in the Hilbert space $L^{2,\sigma}(\mathbb{D})$, which reads

$$\int_{\mathbb{D}} \overline{\Phi_j^{\sigma,m}(z)} \Phi_k^{\sigma,m}(z) \left(1 - z\bar{z}\right)^{\sigma-2} d\nu(z) = \delta_{jk}. \tag{3.23}$$

This suggests us to set $\Omega(z) := (1 - z\bar{z})^{\sigma-2}$. Therefore, we get that

$$d\mu_{\sigma,m}(z) = \frac{(\sigma - 2m - 1)}{\pi \left(1 - z\bar{z}\right)^2} d\nu(z). \tag{3.24}$$

we arrive at the announced expression of the measure. Therefore, Eq. (3.21) reduces to

$$\mathcal{O} = \sum_{j,k=0}^{+\infty} \delta_{jk} \mid \psi_j >< \psi_k \mid = \mathbf{1}_{\mathcal{H}}. \tag{3.25}$$

The proof is finished. □

Proposition 3.5. *Let $\sigma > 1$ and $m = 0, 1, \cdots, [(\sigma - 1)/2]$. Then, the states $\mid z, \sigma, m >$ satisfy the continuity property with respect to the label $z \in \mathbb{D}$. That is, the norm of the difference of two states*

$$d_{\sigma,m}(z, w) := \|(\mid z, \sigma, m > - \mid w, \sigma, m >)\|_{\mathcal{H}} \tag{3.26}$$

goes to zero whenever $z \to w$.

Proof. By using the fact that any state $\mid z, \sigma, m >$ is normalized by the factor given in (3.9), direct calculations enable us to write the square of the quantity in (3.26) as

$$d^2_{\sigma,m}(z, w) = 2\left(1 - \Re(< z, \sigma, m \mid w, \sigma, m >)\right). \tag{3.27}$$

Next, we use of the expression of the scalar product in (3.9) form which it is clear that the overlap takes the value 1 as $z \to w$ and consequently $d_{\sigma,m}(z, w) \to 0$. □

We have verified that the basic minimum properties for the constructed states to be considered as coherent states are satisfied. Namely, the conditions which have been formulated by Klauder [18]: (a) the continuity of labeling, (b) the fact that these states are normalizable but not orthogonal and (c) these states fulfilled the resolution of the identity with a positive weight function. As we can see, these coherent states are independent of the basis $\mid \psi_k >$ we use and the only condition which is implicitly fulfilled is the orthonormality of the basis vectors of $\mathcal{H}$. But if we want to attach these coherent states to a concrete quantum system then a Hamiltonian operator should be specified together with a corresponding explicit eigenstates basis. This will be the goal of the next section.

4. A relativistic pseudoharmonic oscillator

In this section, we recall some needed results which have been developed in [5], where the authors considered a model for the relativistic pseudo-harmonic oscillator with the following interaction potential

$$U_{m_*,\omega,\lambda,g}(x) := \left(\frac{1}{2} m_* \omega^2 x (x + i\lambda) + \frac{g}{x(x + i\lambda)}\right) e^{i\beta\partial_x} \tag{4.1}$$

where ω is a frequency, $g \geq 0$ is a real quantity and $\lambda = \hbar/m_* c$ denotes the Compton wavelength defined by the ratio of Planck's constant $\hbar$ by the mass m_* times the speed of light c. The corresponding stationary Schrödinger equation is described by the finite-difference equation

$$\left(mc^2 \cosh i\beta\partial_\xi + U_{m_*,\omega,\beta,g}(x)\right)\varphi(\xi) = E\varphi(\xi) \tag{4.2}$$

with the boundary conditions for the wave function $\varphi(0) = 0$ and $\varphi(\infty) = 0$. As in [5], we will restrict ourself to the interval $0 \leq x < \infty$ and in terms of dimensionless variable $\xi = x/\lambda$ and parameters $\omega_0 = \hbar\omega/m_*c^2, g_0 = m_*g/\hbar^2$ the equation (4.2) takes the form

$$\left(\cosh i\partial_\xi + \frac{1}{2}\omega_0^2\xi(\xi+i)e^{i\partial_\xi} + \frac{g_0}{\xi(\xi+i)}\right)\varphi(\xi) = \frac{E}{m_*c^2}\varphi(\xi) \tag{4.3}$$

The authors in [5] have obtained the energy spectrum of the Schrödinger operator in (4.2) as

$$E_k := \hbar\omega(2k + \alpha_+ + \alpha_-); \quad k = 0, 1, 2, \cdots, \tag{4.4}$$

where

$$2\alpha_\pm - 1 = \sqrt{1 + \frac{2}{\omega_0^2}\left(1 \pm \sqrt{1 - 8g_0\omega_0^2}\right)}, \tag{4.5}$$

For our purpose to ensure that E_k are real we choose $1 - 8g_0\omega_0^2 = 0$ which means, in system of units $\hbar = m_* = \omega = 1$, the choice $g = c^4/8$. In this case $\alpha_+ = \alpha_- = \gamma_c = (1 + \sqrt{1 + 2c^4})/2 \equiv \gamma$. So that we will be concerned with eigenstates of the form ([5]):

$$\varphi_k^\gamma(\xi) := \frac{\sqrt{2}i^\gamma \left(c^{-4}\right)^{i\xi}\Gamma^2(\gamma + i\xi)(\Gamma(i\xi))^{-1}}{\Gamma\left(k + \gamma + \frac{1}{2}\right)\sqrt{k!\Gamma(k + 2\gamma)}} S_k\left(\xi^2; \gamma, \gamma, \frac{1}{2}\right) \tag{4.6}$$

where $S_k\left(\xi^2, a, b, c\right)$ denotes the continuous dual Hahn polynomial ([4, p.331]), which can be defined in terms of the ${}_3F_2$-sum as

$$S_n\left(\xi^2; a, b, c\right) := (a+b)_n (a+c)_n \cdot {}_3F_2\left(\begin{array}{c} -n, a+i\xi, a-i\xi \\ a+b, a+c \end{array}; 1\right). \tag{4.7}$$

Finally, we note that the wave functions in (4.6) satisfy the relations:

$$\int_0^{+\infty} \varphi_k^\gamma(\xi)\overline{\varphi_j^\gamma(\xi)}d\xi = \delta_{k,j} \tag{4.8}$$

which means that they constitute an orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+, d\xi)$.

5. A FAMILY OF COHERENT STATES FOR THE RELATIVISTIC PSEUDOHARMONIC OSCILLATOR

We now adopt the Hilbertian probabilistic schem discussed above in Section 3 to define a class of coherent states as follows.

Definition 5.1. *For $\sigma > 1$ and $m = 0, 1, \cdots, \left[\frac{\sigma-1}{2}\right]$ a class of coherent states for the relativistic pseudoharmonic oscillator* (4.2) *are defined by*

$$\varphi_{z,\sigma,\gamma,m}(\cdot) := \left(\mathcal{N}_{\sigma,m}(z)\right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \overline{\Phi_k^{\sigma,m}(z)} \mid \varphi_k^\gamma(\cdot) > \tag{5.1}$$

where $\mathcal{N}_{\sigma,m}(z)$ is the factor in (3.9) *, $\Phi_k^{\sigma,m}(z)$ are given by* (2.5) *and $\varphi_k^\gamma(\cdot)$ are the eigenstates in* (4.6).

Proposition 5.2. *Let $\sigma = 2(m+\gamma)$ and $z \in \mathbb{D}$ be a fixed labeling point. Then, the wave functions of the states* (5.1) *are of the form*

$$\varphi_{z,\gamma,m}(\xi) = \frac{\sqrt{2\Gamma(m+2\gamma)}i^{\gamma}\left(c^{-4}\right)^{i\xi}}{\Gamma(2\gamma)\Gamma\left(\gamma+\frac{1}{2}\right)\sqrt{m!}}\frac{\Gamma^2(\gamma+i\xi)}{\Gamma(i\xi)} \tag{5.2}$$

$$\times\frac{(1-z\bar{z})^{\gamma}}{(1-z)^{2\gamma}}\left(\frac{\bar{z}-1}{1-z}\right)^m F_5\left(\begin{array}{ccc}\gamma+i\xi, & \gamma-i\xi: & 2\gamma+m\\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array} \mid \frac{1-\bar{z}z}{(1-\bar{z})(z-1)}, \frac{1}{1-\bar{z}}\right)$$

for any $\xi \in \mathbb{R}_+$, where F_5 is a special Appel-Kampé de Fériet hypergeometric function.

Proof. We start form Eq. (5.1) by replacing the coefficients $\overline{\Phi_k^{\sigma,m}(z)}$ by their expressions taking into account Eq. (2.5). This leads to the expression

$$\varphi_{z,\gamma,m}(\xi) = \left(\frac{(\sigma-2m-1)}{\pi(1-z\bar{z})^{\sigma}}\right)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sqrt{\frac{(\sigma-2m-1)\,k!\Gamma(\sigma-m)}{\pi m!\Gamma(\sigma-2m+k)}} \tag{5.3}$$

$$\times(1-z\bar{z})^{-m}(-1)^k z^{m-k}P_k^{(m-k,\sigma-2m-1)}(1-2z\bar{z})\,\varphi_k^{\gamma}(\xi).$$

Next, introducing the variable $u := 2z\bar{z}-1$ and inserting the expression of $\varphi_k^{\gamma}(\xi)$ in Eq. (5.3), we obtain that

$$\varphi_{z,\gamma,m}(\xi) = z^m(1-z\bar{z})^{\frac{1}{2}\sigma-m}\sqrt{\frac{2\Gamma(\sigma-m)}{m!}}i^{\gamma}\left(c^{-4}\right)^{i\xi}\frac{\Gamma^2(\gamma+i\xi)}{\Gamma(i\xi)} \tag{5.4}$$

$$\times\sum_{k=0}^{\infty}\sqrt{\frac{k!}{\Gamma(\sigma-2m+k)}}\frac{P_k^{(\sigma-2m-1,m-k)}(u)\,S_k\left(\xi^2,\gamma,\gamma,1/2\right)}{z^k\Gamma\left(k+\gamma+\frac{1}{2}\right)\sqrt{k!\Gamma(k+2\gamma)}}.$$

We use the notation $\sigma-2m=2\gamma$ and we focus on the sum in (5.4):

$$\mathfrak{S} := \sum_{k=0}^{\infty}\sqrt{\frac{k!}{\Gamma(2\gamma+k)}}\frac{P_k^{(2\gamma-1,m-k)}(u)\,S_k\left(\xi^2,\gamma,\gamma,1/2\right)}{z^k\Gamma\left(k+\gamma+\frac{1}{2}\right)\sqrt{k!\Gamma(k+2\gamma)}}. \tag{5.5}$$

Next, we set $t := 1/z$ and we rewrite (5.5) in a simple form as

$$\mathfrak{S} = \sum_{k=0}^{\infty}t^k\frac{P_k^{(2\gamma-1,m-k)}(u)\,S_k\left(\xi^2,\gamma,\gamma,1/2\right)}{\Gamma\left(k+\gamma+\frac{1}{2}\right)\Gamma(2\gamma+k)}. \tag{5.6}$$

Now, we need the definition of the continuous dual Hahn polynomials by the hypergeometric terminating ${}_3F_2$-sum ([4, p.331]):

$$S_k\left(\xi^2,\gamma,\gamma,1/2\right) = \frac{\Gamma(2\gamma+k)\Gamma\left(\gamma+\frac{1}{2}+k\right)}{\Gamma(2\gamma)\Gamma\left(\gamma+\frac{1}{2}\right)}.{}_3F_2\left(\begin{array}{c}-k,\gamma+i\xi,\gamma-i\xi\\ 2\gamma,\gamma+\frac{1}{2}\end{array}\mid 1\right). \tag{5.7}$$

So that the sum (5.6) becomes

$$\mathfrak{S} = \frac{1}{\Gamma(2\gamma)\Gamma\left(\gamma+\frac{1}{2}\right)}\sum_{k=0}^{\infty}t^kP_k^{(2\gamma-1,m-k)}(u)\,.{}_3F_2\left(\begin{array}{c}-k,\gamma+i\xi,\gamma-i\xi\\ 2\gamma,\gamma+\frac{1}{2}\end{array}\mid 1\right). \tag{5.8}$$

Now, we make use of the integral representation ([19, p.84]):

$${}_3F_2\left(\begin{array}{c}\alpha,\beta,\rho\\ \tau,\rho+\omega\end{array}\mid 1\right) = \frac{\Gamma(\rho+\omega)}{\Gamma(\rho)\Gamma(\omega)}\int_0^1 x^{\rho-1}(1-x)^{\omega-1}.{}_2F_1\left(\begin{array}{c}\alpha,\beta\\ \tau\end{array}\mid x\right)dx \tag{5.9}$$

provided that $\Re(\rho) > 0$, $\Re(\omega) > 0$ and $\Re(\tau + \omega - \alpha - \beta) > 0$ for the parameters $\alpha = -k, \beta = \gamma + i\xi, \rho = \gamma - i\xi, \tau = 2\gamma, \omega = \frac{1}{2} + i\xi$ to rewrite the ${}_3F_2$-sum in (5.8) as

$$ {}_3F_2\left(\begin{matrix} -k, \gamma + i\xi, \gamma - i\xi \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix} \mid 1 \right) = \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma\left(\gamma - i\xi\right)\Gamma\left(\frac{1}{2} + i\xi\right)} \times \int_0^1 x^{\gamma - i\xi - 1}\left(1 - x\right)^{-\frac{1}{2} + i\xi} . {}_2F_1\left(\begin{matrix} -k, \gamma + i\xi \\ 2\gamma \end{matrix} \mid x \right) dx. \tag{5.10} $$

Therefore, Eq. (5.8) transforms successively as

$$ \mathfrak{S} = \frac{1}{\Gamma\left(2\gamma\right)\Gamma\left(\gamma + \frac{1}{2}\right)} \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma\left(\gamma - i\xi\right)\Gamma\left(\frac{1}{2} + i\xi\right)} \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}\left(u\right) \times \int_0^1 x^{\gamma - i\xi - 1}\left(1 - x\right)^{-\frac{1}{2} + i\xi} . {}_2F_1\left(\begin{matrix} -k, \gamma + i\xi \\ 2\gamma \end{matrix} \mid x \right) dx \tag{5.11} $$

$$ = \frac{1}{\Gamma\left(2\gamma\right)} \int_0^1 \Phi\left(x, \xi\right) \left(\sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}\left(u\right) . {}_2F_1\left(\begin{matrix} -k, \gamma + i\xi \\ 2\gamma \end{matrix} \mid x \right) \right) dx, \tag{5.12} $$

where

$$ \Phi\left(x, \xi\right) := \frac{x^{\gamma - i\xi - 1}\left(1 - x\right)^{-\frac{1}{2} + i\xi}}{\Gamma\left(\gamma - i\xi\right)\Gamma\left(\frac{1}{2} + i\xi\right)}. \tag{5.13} $$

Now, we look closely at the sum in (5.12):

$$ \Xi\left(x\right) := \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}\left(u\right) . {}_2F_1\left(\begin{matrix} -k, \gamma + i\xi \\ 2\gamma \end{matrix} \mid x \right). \tag{5.14} $$

We exploit the connection formula ([20, p.63]):

$$ P_n^{(\alpha,\beta)}\left(u\right) = \left(\frac{1 - u}{2}\right)^n P_n^{(-2n-\alpha-\beta-1, \beta)}\left(\frac{u + 3}{u - 1}\right) \tag{5.15} $$

for the parameters $\alpha = 2\gamma - 1, \beta = m - k$ and $n = k$ to rewrite the Jacobi polynomial in (5.14) as

$$ P_k^{(2\gamma-1, m-k)}\left(u\right) = \left(1 - z\bar{z}\right)^k P_k^{(-2\gamma-m-k, m-k)}\left(\frac{z\bar{z} + 1}{z\bar{z} - 1}\right). \tag{5.16} $$

Therefore Eq. (5.14) can be rewritten as

$$ \Xi = \sum_{k=0}^{\infty} \left(\frac{1 - z\bar{z}}{z}\right)^k P_k^{(-2\gamma-m-k, m-k)}\left(\frac{z\bar{z} + 1}{z\bar{z} - 1}\right) . {}_2F_1\left(\begin{matrix} -k, \gamma + i\xi \\ 2\gamma \end{matrix} \mid x \right). \tag{5.17} $$

We introduce the variables

$$ \theta := \frac{1 - z\bar{z}}{z}, V := \frac{z\bar{z} + 1}{z\bar{z} - 1} \tag{5.18} $$

in terms of which Eq. (5.17) also reads

$$ \Xi = \sum_{k=0}^{\infty} \frac{(2\gamma)_k}{(2\gamma)_k} \theta^k P_k^{(-2\gamma-m-k, m-k)}\left(V\right) . {}_2F_1\left(\begin{matrix} -k, \gamma + i\xi \\ 2\gamma \end{matrix} \mid x \right). \tag{5.19} $$

We are now in position to apply the bilinear generating formula due to S. Saran ([21, p.14]):

$$\sum_{k=0}^{\infty}\frac{(b)_k}{(d)_k}\theta^k P_k^{(\alpha-k,\beta-k)}(V)\,.{}_2F_1\left(\begin{array}{c}-k,c\\ b\end{array}\mid y\right) \tag{5.20}$$

$$=(1-y)^{-c}F_8\left(b,b,b,c,-\alpha,-\beta;b,d,d;\frac{y}{y-1},-\frac{1}{2}(V+1)\theta,-\frac{1}{2}(V-1)\theta\right).$$

In our context $b=d=2\gamma$, this implies that the Lauricella triple hypergeometric series F_8, which is denoted F_G by S. Saran, reduces to the expression

$$\left[1+\frac{(V+1)\theta}{2}\right]^{\alpha}\left[1+\frac{(V-1)\theta}{2}\right]^{\beta}(1-y)^{c} \tag{5.21}$$

$$\times F_1\left(c,-\alpha,-\beta;b;\frac{y(V+1)\theta}{2+(V+1)\theta},\frac{y(V-1)\theta}{2+(V-1)\theta}\right)$$

in terms of the first F_1 Appell's hypergeometric function $([19],\text{p.}265)$. Therefore, in terms of our parameters, the sum (5.19) also has the following expression

$$\Xi=\left[1+\frac{(V+1)\theta}{2}\right]^{-2\gamma-m}\left[1+\frac{(V-1)\theta}{2}\right]^{m} \tag{5.22}$$

$$\times F_1\left(\gamma+i\xi,2\gamma+m,-m;2\gamma;\frac{x(V+1)\theta}{2+(V+1)\theta},\frac{x(V-1)\theta}{2+(V-1)\theta}\right).$$

Now if we denote the parameters and arguments occurring the last Appell F_1-sum respectively by $a=\gamma+i\xi, b=2\gamma+m, c=-m, d=2\gamma$,

$$X=\frac{x(V+1)\theta}{2+(V+1)\theta}=\mu_z x, Y=\frac{x(V-1)\theta}{2+(V-1)\theta}=\nu_z x \tag{5.23}$$

with

$$\mu_z:=\frac{(V+1)\theta}{2+(V+1)\theta}=\frac{\overline{z}}{\overline{z}-1},\nu_z:=\frac{(V-1)\theta}{2+(V-1)\theta}=\frac{1}{1-z}, \tag{5.24}$$

then this F_1-sum can be presented as $F_1(a,b,c;d;X,Y)$ with a particularity here consisting on the fact that the parameters b,c and d satisfy $d=b+c$ so that it can be reduced to a Gauss hypergeometric function according to the transformation ([22]):

$$F_1(a,b,c,b+c;X,Y)=(1-Y)^{-a}\,.{}_2F_1\left(\begin{array}{c}a,b\\ b+c\end{array}\mid\frac{X-Y}{1-Y}\right). \tag{5.25}$$

Therefore, using (5.25), we obtain from (5.22) the following fact

$$F_1(\gamma+i\xi,2\gamma+m,-m;2\gamma;\mu_z x,\nu_z x)$$

$$=(1-\nu_z x)^{-a}\,.{}_2F_1\left(\begin{array}{c}\gamma+i\xi,2\gamma+m\\ 2\gamma\end{array}\mid\frac{(\mu_z-\nu_z)x}{1-\nu_z x}\right). \tag{5.26}$$

We set

$$\tau_z:=\mu_z-\nu_z=\frac{\overline{z}}{\overline{z}-1}-\frac{1}{1-z}=\frac{1-\overline{z}z}{(1-z)(\overline{z}-1)}. \tag{5.27}$$

So that the sum in (5.22) becomes

$$\Xi(x)=\left[1+\frac{(V+1)\theta}{2}\right]^{-2\gamma-m}\left[1+\frac{(V-1)\theta}{2}\right]^{m} \tag{5.28}$$

$$\times(1-\nu_z x)^{-\gamma-i\xi}\,.{}_2F_1\left(\begin{array}{c}\gamma+i\xi,2\gamma+m\\ 2\gamma\end{array}\mid\frac{(\mu_z-\nu_z)x}{1-\nu_z x}\right).$$

Next, to compute the prefactor in the last Eq. (5.28), in terms of the fixed labeling point $z \in \mathbb{D}$, we are of need of the equalities

$$1-\overline{z}=1+\frac{(V+1)\,\theta}{2} \text{ and } \frac{z-1}{z}=1+\frac{(V-1)\,\theta}{2}. \tag{5.29}$$

in order to rewrite (5.28) as

$$\Xi\,(x)=(1-\overline{z})^{-2\gamma}\,z^{-m}\left(\frac{z-1}{1-\overline{z}}\right)^{m} \tag{5.30}$$
$$\times\,(1-\nu_z x)^{-\gamma-i\xi}\;{}_2F_1\left(\begin{array}{c}\gamma+i\xi,2\gamma+m\\ 2\gamma\end{array}\mid\frac{\tau_z x}{1-\nu_z x}\right).$$

Returning back to the sum in (5.12) and inserting $\Xi(t)$, we get that

$$\mathfrak{S}=\frac{1}{\Gamma\,(2\gamma)\,\Gamma\,(\gamma-i\xi)\,\Gamma\left(\frac{1}{2}+i\xi\right)}\int_0^1 x^{\gamma-i\xi-1}\,(1-x)^{-\frac{1}{2}+i\xi}\,(\Xi\,(x))\,dx. \tag{5.31}$$

Explicitly, this last quantity reads

$$\mathfrak{S}=\frac{1}{\Gamma\,(2\gamma)\,\Gamma\,(\gamma-i\xi)\,\Gamma\left(\frac{1}{2}+i\xi\right)}\,(1-\overline{z})^{-2\gamma}\,z^{-m}\left(\frac{z-1}{1-\overline{z}}\right)^{m} \tag{5.32}$$

$$\times\int_0^1 x^{\gamma-i\xi-1}\,(1-x)^{-\frac{1}{2}+i\xi}\,(1-\nu_z x)^{-\gamma-i\xi}\;{}_2F_1\left(\begin{array}{c}2\gamma+m,\gamma+i\xi\\ 2\gamma\end{array}\mid\frac{\tau_z x}{1-\nu_z x}\right)dx.$$

A this stade we can make use of the integral representation due to S.K. Kulshreshtha ([8, p.137, Eq. (2.2)]), of a very special case of the Appel-Kampé de Fériet's hypergeometric function of two variables of higher order [23] as follows

$$F\left[\begin{array}{ccc|c} 2 & c & d & \\ 1 & a & b & \chi \\ 1 & e & . & \zeta \\ 1 & a' & b & \end{array}\right]\equiv F_5\left(\begin{array}{ccc}c, & d: & a\\ . & e: & a'\end{array}\mid\chi,\zeta\right) \tag{5.33}$$

$$=\frac{\Gamma\,(e)}{\Gamma\,(d)\,\Gamma\,(e-d)}\int_0^1 t^{d-1}\,(1-t)^{e-d-1}\,(1-\zeta t)^{-c}\;{}_2F_1\left(\begin{array}{c}a,c\\ a'\end{array}\mid\frac{\chi t}{1-\zeta t}\right)dt \tag{5.34}$$

for the parameters $d=\gamma-i\xi, e=\frac{1}{2}+\gamma, c=\gamma+i\xi, a=2\gamma+m, a'=2\gamma, \zeta=\nu_z, \chi=\tau_z$ and $t=x$. So that the integral occurring in (5.32) reads

$$\frac{\Gamma\,(\gamma-i\xi)\,\Gamma\left(\frac{1}{2}+i\xi\right)}{\Gamma\left(\gamma+\frac{1}{2}\right)}F_5\left(\begin{array}{ccc}\gamma+i\xi, & \gamma-i\xi: & 2\gamma+m\\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array}\mid\tau_z,\nu_z\right), \tag{5.35}$$

and therefore the sum in (5.32) takes the form

$$\mathfrak{S}=\frac{(1-\overline{z})^{-2\gamma}}{\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\,(2\gamma)\,z^{m}}\left(\frac{z-1}{1-\overline{z}}\right)^{m}F_5\left(\begin{array}{ccc}\gamma+i\xi, & \gamma-i\xi: & 2\gamma+m\\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array}\mid\tau_z,\nu_z\right). \tag{5.36}$$

Summarizing the above calculations we arrive at the expression of the wave functions:

$$\varphi_{z,\gamma,m}(\xi) = (1-z\bar{z})^{\gamma}\sqrt{\frac{2\Gamma\left(m+2\gamma\right)}{m!}}i^{\gamma}\left(c^{-4}\right)^{i\xi}\Gamma^{2}\left(\gamma+i\xi\right)\left(\Gamma\left(i\xi\right)\right)^{-1} \tag{5.37}$$
$$\times\frac{(1-\bar{z})^{-2\gamma}}{\Gamma\left(2\gamma\right)\Gamma\left(\gamma+\frac{1}{2}\right)}\left(\frac{z-1}{1-\bar{z}}\right)^{m}F_{5}\left(\begin{array}{ccc}\gamma+i\xi, & \gamma-i\xi: & 2\gamma+m\\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array}\mid\tau_{z},\nu_{z}\right)$$

as announced in the proposition. Replacing the arguments ν_z and τ_z by their expressions in (5.24) and (5.27), we end the proof. □

6. Relativistic Bargmann-type transforms

Now, since we have obtained the expression of the wave functions (5.2), we can apply the coherent states transform formalism [9] to obtain an integral transform $\mathcal{B}_{c,m}^{\mathrm{II}}:L^{2}\left(\mathbb{R}_{+}\right)\rightarrow\mathcal{A}^{2(\gamma+m)}\left(\mathbb{D}\right)$ defined by

$$\mathcal{B}_{c,m}^{\mathrm{II}}\left[f\right](z):=\left(\mathcal{N}_{2(\gamma+m),m}(z)\right)^{\frac{1}{2}}\left\langle f,\varphi_{z,\gamma,m}\right\rangle_{L^{2}(\mathbb{R}_{+})}. \tag{6.1}$$

We precisely state the following precise result.

Theorem 6.1. *The coherent state transform associated with the wave functions* (5.2) *is the isometry* $\mathcal{B}_{c,m}^{\mathrm{II}}:L^{2}\left(\mathbb{R}_{+}\right)\rightarrow\mathcal{A}^{2(\gamma+m)}\left(\mathbb{D}\right)$ *defined by*

$$\mathcal{B}_{c,m}^{\mathrm{II}}\left[f\right](z):=\frac{(2\gamma-1)^{\frac{1}{2}}}{\sqrt{\pi m!}\,(1-z)^{2\gamma}}\frac{\sqrt{2\Gamma\left(m+2\gamma\right)}\,(-i)^{\gamma}}{\Gamma\left(2\gamma\right)\Gamma\left(\gamma+\frac{1}{2}\right)}\left(\frac{\bar{z}-1}{(1-z)\left(1-z\bar{z}\right)}\right)^{m} \tag{6.2}$$
$$\times\int_{0}^{+\infty}\frac{\Gamma^{2}\left(\gamma-i\xi\right)\left(c^{-4}\right)^{-i\xi}}{\Gamma\left(-i\xi\right)}F_{5}\left(\begin{array}{ccc}\gamma-i\xi, & \gamma+i\xi: & 2\gamma+m\\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array}\mid\frac{1-\bar{z}z}{(1-\bar{z})\left(z-1\right)},\frac{1}{1-\bar{z}}\right)f(\xi)d\xi$$

Definition 6.2. *Let* $\gamma=(1+\sqrt{1+2c^{4}})/2$, *then the coherent state transform* (6.2) *will be called a relativistic Bargmann-type transform attached to the hyperbolic Landau level* $\epsilon_{m}^{\gamma}:=4m\left(m+2\gamma-1\right)$ *on the Poincaré disk.*

Corollary 6.3. *For* $m=0$*,the coherent state transform associated with the wave functions* (5.2) *is the isometry* $\mathcal{B}_{c,0}^{\mathrm{II}}$ *mapping the Hilbert* $L^{2}\left(\mathbb{R}_{+}\right)$ *onto the Bergman space* $\mathcal{A}^{2\gamma}\left(\mathbb{D}\right)$ *of holomorphic functions* $\phi:\mathbb{D}\rightarrow\mathbb{C}$ *with* $\int_{\mathbb{D}}\left|\phi(z)\right|^{2}\left(1-z\bar{z}\right)^{2\gamma-2}d\nu(z)<+\infty$ *by*

$$\mathcal{B}_{c,0}^{\mathrm{II}}\left[f\right](z)=\left(\frac{2\gamma-1}{\pi}\right)^{\frac{1}{2}}\frac{(-i)^{\gamma}}{(1-z)^{\gamma}}\frac{\sqrt{2}}{\Gamma\left(\gamma+\frac{1}{2}\right)\sqrt{\Gamma\left(2\gamma\right)}} \tag{6.3}$$
$$\times\int_{0}^{+\infty}\left(c^{-4}\right)^{-i\xi}\frac{\Gamma^{2}\left(\gamma-i\xi\right)}{(1-z)^{i\xi}\,\Gamma\left(-i\xi\right)}\cdot{}_{2}F_{1}\left(\begin{array}{c}\gamma-i\xi,\frac{1}{2}-i\xi\\ \gamma+\frac{1}{2}\end{array}\mid z\right)f(\xi)d\xi.$$

Proof. We start by putting $m=0$ in the expression (5.2). This gives

$$\varphi_{z,\gamma,0}(\xi)=\frac{\sqrt{2}i^{\gamma}\left(c^{-4}\right)^{i\xi}}{\sqrt{\Gamma\left(2\gamma\right)}\Gamma\left(\gamma+\frac{1}{2}\right)}\frac{\Gamma^{2}\left(\gamma+i\xi\right)}{\Gamma\left(i\xi\right)} \tag{6.4}$$
$$\times\frac{(1-z\bar{z})^{\gamma}}{(1-\bar{z})^{2\gamma}}F_{5}\left(\begin{array}{ccc}\gamma+i\xi, & \gamma-i\xi: & 2\gamma\\ . & \gamma+\frac{1}{2}: & 2\gamma\end{array}\mid\frac{1-\bar{z}z}{(1-z)\left(\bar{z}-1\right)},\frac{1}{1-z}\right).$$

We now observe that the two parameters a and a' in the F_5-sum as denoted in (5.33) are equal in the case of Eq. (6.4). In this situation the reduction ([8, p.136, Eq. (1.7)]):

$$F_5\left(\begin{array}{ccc} c, & d: & a \\ . & e: & a \end{array} \mid \chi,\zeta\right) = {}_2F_1\left(\begin{array}{c} c,d \\ e \end{array} \mid \chi+\zeta\right) \tag{6.5}$$

can be applied and enables us to write

$$F_5\left(\begin{array}{ccc} \gamma+i\xi, & \gamma-i\xi: & 2\gamma \\ . & \gamma+\frac{1}{2}: & 2\gamma \end{array} \mid \tau_z,\nu_z\right) = {}_2F_1\left(\begin{array}{c} \gamma+i\xi,\gamma-i\xi \\ \gamma+\frac{1}{2} \end{array} \mid \frac{z}{z-1}\right), \tag{6.6}$$

where we have replaced of the quantities ν_z and τ_z by their above expression respectively. Next, we make use appeal to the Pffaf transformation ([4, p.68]):

$${}_2F_1\left(\begin{array}{c} a,b \\ c \end{array} \mid x\right) = (1-x)^{-a}\,.{}_2F_1\left(\begin{array}{c} a,c-b \\ c \end{array} \mid \frac{x}{x-1}\right) \tag{6.7}$$

to present the Gauss hypergeometric function in right hand side of Eq. (6.6) as

$$(1-\bar{z})^{\gamma+i\xi}\,.{}_2F_1\left(\begin{array}{c} \gamma+i\xi,\frac{1}{2}+i\xi \\ \gamma+\frac{1}{2} \end{array} \mid z\right). \tag{6.8}$$

Returning back to (6.4), and inserting (6.8), we arrive at the expression

$$\varphi_{z,\gamma,0}(\xi) = (1-z\bar{z})^{\gamma}(1-\bar{z})^{-\gamma+i\xi}\, i^{\gamma}\left(c^{-4}\right)^{i\xi}\frac{\Gamma^2(\gamma+i\xi)}{\Gamma(i\xi)} \tag{6.9}$$

$$\times\frac{\sqrt{2}}{\Gamma\left(\gamma+\frac{1}{2}\right)\sqrt{\Gamma(2\gamma)}}\,.{}_2F_1\left(\begin{array}{c} \gamma+i\xi,\frac{1}{2}+i\xi \\ \gamma+\frac{1}{2} \end{array} \mid \bar{z}\right).$$

Finally, we write the quantity $\left(\mathcal{N}_{2\gamma,0}(z)\right)^{\frac{1}{2}}\left\langle f,\varphi_{z,\gamma,0}\right\rangle_{L^2(\mathbb{R}_+)}$ for an arbitrary function f in $L^2(\mathbb{R}_+)$. □

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P.O. BOX.123, BÉNI MELLAL, MOROCCO