DENSE IDEALS IN TOPOLOGICAL ALGEBRAS

by W. Żelazko, Mathematical Institute

of the Polish Academy of Sciences

A **TOPOLOGICAL ALGEBRA** A is a complete topological vector space provided with a jointly continuous (associative) multiplication. We shall be considering only unital algebras with the unity denoted by e. We shall assume also commutativity unless otherwise is stated.

A **LOCALLY CONVEX ALGBBRA** (LC) is a topological algebra which is a locally convex space. Its topology can be given by the means of a familly $(\| \cdot \|_\alpha)$ of seminorms. It shall be assumed that this family is saturated, i.e. it contains maxima of all its finite subfamilies. In this case the seminorms can be chosen so that for each α there is a β with

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta \quad \text{and} \quad \|e\|_\alpha = 1, x, y \in A.$$

An F-**ALGEBRA** is a topological algebra which is an F-space, i.e. a complete metric t.v.s.

$B_0 = F \cap LC$. The topology of a B_0-algebra can be given by a sequence of seminorms satisfying

$$\|x\|_1 \leq \|x\|_2 \leq \dots \quad \text{and} \quad \|xy\|_i \leq \|x\|_{i+1} \|y\|_{i+1}.$$

For Banach algebras every maximal ideal is closed and it is of codimension one as the kernel of a multiplicative linear functional (character). This is due to the openess of the set of invertible elements and to the Gelfand-Mazur Theorem. This is not true in more general situations.

EXAMPLE 1 Denote by $\mathcal{E}$ the algebra of entire functions of one complex variable with the topology of uniform convergence on compact subsets of the complex plane (the compact-open topology). It is given by seminorms (norms)

$$\|x\|_n = \max\{|x(\zeta)| : |\zeta| \leq n\}, n = 1, 2, \ldots,$$

so it is a B_0-algebra). Moreover the seminorms satisfy a stronger condition

$$\|xy\|_n \leq \|x\|_n \|y\|_n.$$

LC-algebras with this property are called multiplicatively-convex (shortly m-convex). Take $(\zeta_n) \subset \mathbb{C}$ tending to an infinity. Define

$$I = \{x \in \mathcal{E} : x(\zeta_i) = 0 \quad \text{for} \quad i \geq c(x)\}.$$

Clearly it is an ideal in $\mathcal{E}$. Let M be any maximal ideal in $\mathcal{E}$ containing I. We shall show that

(i) it is dense and

(ii) has an infinite codimension.

For proving (i) we show that for each element $y \in \mathcal{E}$, each k and each $\varepsilon > 0$ there is an element x in

I with $\|x-y\|_k < \varepsilon$ which means that I and so M is dense. Find an element u in I which does not vanish on the disc $\{\zeta : \|\zeta| \le k\}$ and so u^{-1} is holomorphic in a neighbourhood of this disc and can be there arbitrarily approximated by polynomials. So there is a polynomial p with

$$\|xpu - x\|_k \le \varepsilon,$$

and (i) follows. To see (ii) observe that by the Frobenius Theorem every field of the form $\mathcal{E}/M$ is either of dimension 1 or ∞. If it is of dimension 1, it is the kernel of a discontinuous character. We have to show that all characters of $\mathcal{E}$ are continuous. For such a character f put $\zeta_0 = f(z)$, where $z(\zeta) = \zeta$. Let $y \in \mathcal{E}$ and put $y_0(\zeta) = y(\zeta) - y(\zeta_0)e(\zeta)$, where

$e(\zeta) = 1$ is the unity of $\mathcal{E}$. Since $y_0(\zeta_0) = 0$, we can write $y_0(\zeta) = (\zeta - \zeta_0)y_1(\zeta)$ for some y_1 in $\mathcal{E}$. Since

$$y(\zeta) = y_0(\zeta) + y_0(\zeta_0) = (\zeta - \zeta_0)y_1(\zeta) + y(\zeta_0)$$

and $f(e) = 1$, we obntain

$$f(y) = f(z - \zeta_0 e)f(y_1) + y(\zeta_0)f(e) = y(\zeta_O).$$

so $|f(y)| \leq \|y\|_n$ for $n \geq |\zeta_0|$ and for all y in $\mathcal{E}$. Thus f is continuous and (ii) follows. ALL DENSE IDEALS THERE ARE OF AN INFINITE CODIMENSION.

Example 2. Here $A = C[0, \infty)$ with seminorms $\|x\|_n = \max\{|x(t)| : o \leq t \leq n\}$. Similarly as in Example 1 we show that every maximal ideal in

A containing $I = \{x \in A : x(t) = 0 \quad \text{for} \quad t \geq c(x)\}$ is dense and of infinite codimension.

A topological algebra is said a Q-algebra if the set of all its invertible elements is open.

The following result characterizes the situation when an m-convex algebra has all maximal ideals closed ([5])

Theorem 1. Let A be an m-convex algebra, then the following are equivalent.

(i) All maximal ideals in A are of codimension one;

(ii) Each element of A has a bounded spectrum;

(iii) Each element of A has a compact spectrum;

(iv) A is a (complete) m-convex Q-algebra under some topology stronger than the original one;

(v) A is a (complete) m-convex Q-algebra under some topology;

If, moreover, A is a barreled space (in particular a B_0-space), then the above are equivalent to

(vi) Every maximal ideal in A is closed;

(vii) A is a Q-algebra.

In m-convex algebras the Gelfand-Mazur Theorem holds true, and since their quotients are also m-convex, all closed maximal ideals there are of codimension one.

Example 3. Denote by ω_1 the first uncountable ordinal. The set of ordinals $[1, \omega_1]$ is a compact space under the interval topology (the neighbourhoods are open intervals). The half-open interval $[1, \omega_{)} 1)$ is not compact but all continuous functions on it must be constant beginning from some point on. The algebra $A = C[1, \omega_1)$ is a non-metrizable m-convex algebra under seminorms

$$\|x\|_\alpha = \max\{|x(t)| : 1 \leq t \leq \alpha\}, 1 \leq \alpha < \omega_1.$$

As an algebra it is isomorphic to the Banach algebra

$C[1, \omega_1]$, and so all its maximal ideals are of codimension one. However the maximal ideal in A corresponding to the ordinal ω_1 is dense, and it is the kernel of a discontinuous character. Thus the situations (iv) and (v) in the above theorem can occur.

The very famous open problem (Michael Problem, or Michael-Mazur Problem) is as follow.

Problem 1. Let A be an m-convex B_0-algebra. Are all its characters continuous ?

The above Problem is open also for B_0-algebras and for F-algebras.

Since m-convex algebras have continuous submultiplicative seminorms they must also have some characters, even some continuous characters. This is not the case for B_0-algebras.

Example 4. The Arens algebra ([1]) is $A = L^\omega[0,1] = \bigcap_{1\le p<\infty} L_p[0,1]$ provided with L_p norms. The same topology is given by any sequence $1 \le p_1 < p_2, \ldots$, with $\lim_i p_i = \infty$. It is a B_0-space (it can be shown that it is complete). We have

$$\int_0^1 |xy|^p dt \le \Big(\int_0^1 |x|^{2p} dt\Big)^{1/2} \Big(\int_0^1 |y|^{2p} dt\Big)^{1/2}.$$

Thus

$$\|xy\|_p \le \|x\|_{2p}\|y\|_{2p},$$

and so A is a B_0-algebra. It has the following properties:

(1) A has no characters – if f is such a charac-

ter, then its restriction to $C(0,1)$ is $f(x) = x(t_0), 0 \leq t_0 \leq 1$. But we can find a continuous function x in A with $x(t_0) = 0$ which is invertible in A. This implies that all maximal ideals in A are of infinite codimension, and they must be dense, since the Gelfand-Mazur theorem hold true for B_0-algebrtas.

(2) A has a discontinuous inverse: it is possible to find a sequence (x_n) of continuous invertible functions in A with $x_n \to e$ and with divergent (x_n^{-1}).

(3) A has a dense singly generated ideal xA. That means $\lim_n z_n x = e$ for some sequence $(z_n) \subset A$.

Definition. An element x of a topological algebra A is said topologically invertible, if for some net $(x_\alpha) \subset A$ we have $\lim_\alpha z_\alpha x = e$. It is proper, if it is non-invertible. Thus the Arens algebra has proper topologically invertible elements.

Theorem 2 ([4]). An F-algebra has a continuous inverse if and only if the group $G(A)$ of its invertible elements is a G_δ-set.

Theorem 3 ([7]). Let A be an F-algebra with a discontinuous inverse, then A has proper topologically invertible elements. The proof follows from the above theorem and from the fact that the set of all topologically invertible elements of A is a G_δ-set.

Problem 2. Is the converse to the Theorem 3 true?

The Arens algebra has some proper closed ideals. Aharon Atzmon ([2]) constructed a non-metrizable LC-algebra in which all proper ideals are dense, in particular all its non-zero elements are topologically invertible.

Problem 3. Does there exist a B_o-algebra or an F-algebra with all proper ideals dense ?

It is possible to have an infinite-dimensional m-convex B_0-algebra with all ideals closed. It is the algebra $A = (s)$ of all power series $x = \sum_{n=0}^{\infty} \mathcal{A}_i(x)t^i$, with seminorms $\|x\|_n = \sum_0^{n-1} |a_i(x)|$. Every ideal in this algebra is of the form $I_n = t^n A$ and such ideals are evidently closed.

We pass now the the non-commutative case.

Theorem 4 ([8]). There exists a non-commutative m-convex $B_)$-algebra with all left ideals closed and with some right ideals non-closed.

Theorem 5 ([9]). Lat A be an F-algebra. Then the following are equivalent.

(1) All maximal left ideals in A are closed;

(ii) The set $G_l(A)$ of all left-invertible elements in A is open;

(iii) A is a Q-algebra;

(iv) The set $G_r(A)$ of all right invertible elements is open;

(v) All maximal right ideals in A are closed.

The above means that some dense ideals must exist in every F-algebra which is not a Q-algebra. This is not true for arbitrary topological algebras, even in the locally convex situation.

Theorem 6 ([6]). Let A be a free algebra in variables (t_α) (all polynomials in these non-commuting variables). Provide it with the maximal locally convex topology (given by means of all seminorms on A). Then it is a topological algebra if and only if the number of variables is at most countable.

Under the above topology all linear subspaces, in particular all ideals are closed, but the only invertible elements are scalar multiples of the unit element.

The same result is true if the variables commute.

Finally note, that every commutative non-m-convex B_0-algebra must have a dense maximal ideal This follows from the fact that if a commutative $B_)$-algebra is a Q-algebra, then it must be m-convex.

References

[1] R. Arens, Topological space L^ω and convex topologiocal rings, Bull. Amer. Math. Soc. 52 (1946), 931-935.

[2] A. Atzmon, An operator without invariant subspaces on a nuklear Fréchet space, Ann. of Math.

117 (1983), 669-694.

[3] E. Michael, Locally Multiplicatively-Convex Topological Algebras, Mem. Amer. math. Soc. 11. 1955

[4] W. Żelazko, Metric Generalizations of Banach Algebras, Dissert. Math. (Rozprawy Mat.) 47, 1965.,

[5] — , On maximal ideals in commutative m-convex algebras, Studia Math. 58 (1976), 291-298.

[6] — , On topologization of countably generated algebras, ibidem 112 (1994), 83-88.

[7] — , When a commutative F-algebra has a dense principal ideal, in Contemporary Mathematics

vol. 341, Providence 2004, 133-137.

[8] — , An m-convex $-algebraB_0$-algebra with all left but not all right ideals closed, Coll. Math.1 104 (2006), 317-324.

[9] — , When a unital F-algebra has all maximal left (right) ideals closed? Studia Math. 175 (2006), 279-284.