Disjoint Hypercyclic Operators

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A Surprising Approximation Theorem

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- That is, for any entire \( g(z), \, \epsilon > 0 \), and compact \( K \subset \mathbb{C} \), we can find an integer \( n \) such that

\[
\sup_{z \in K} |f(z + n) - g(z)| < \epsilon.
\]
Let $T$ be a continuous linear operator on a topological vector space $X$. If there is a vector $f \in X$ such that
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\{ T^n f : n \geq 1 \}
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- $\text{span}\{T^n f : n \geq 1\}$ is dense in $X$, then $T$ is called **cyclic**.

Such a vector $f$ is said to be a hypercyclic, supercyclic, or cyclic vector for $T$, respectively.
Given a linear continuous operator $T : X \rightarrow X$, is it possible to find a non-trivial (not $X$ or $\{0\}$) closed subspace (subset) $F \subset X$ for which $T(F) \subset F$?
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Given a linear continuous operator $T : X \to X$, is it possible to find a non-trivial (not $X$ or $\{0\}$) closed subspace (subset) $F \subset X$ for which $T(F) \subset F$?

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Problems are still open for Hilbert spaces.
(Feldman, 2001) There exists a hypercyclic operator $T$ acting on a separable Hilbert space $\mathcal{H}$ which has the following property. For any compact metrizable space $K$ and any continuous map $f : K \to K$, there exists a $T$-invariant compact set $L \subset \mathcal{H}$ such that $f$ and $T|_L$ are topologically conjugate.
(MacLane, 1952)

Let $H(\mathbb{C})$ denote the space of entire functions endowed with the topology of locally uniform convergence.
Differentiation Operator

(MacLane, 1952)

- Let $H(\mathbb{C})$ denote the space of entire functions endowed with the topology of locally uniform convergence.
- The differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $D(f) = f'$ is hypercyclic.
Let $\ell_p(\mathbb{N}) = \{(x_0, x_1, x_2, \ldots) : \sum_{n=0}^{\infty} |x_n|^p < \infty\}$ for $p \geq 1$. (Rolewicz, 1969)
Backward Shift Operator

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- Let $\ell_p(\mathbb{N}) = \{(x_0, x_1, x_2, \ldots) : \sum_{n=0}^{\infty} |x_n|^p < \infty\}$ for $p \geq 1$.
- Let $B : \ell_p(\mathbb{N}) \to \ell_p(\mathbb{N})$ be the backward shift operator defined by

$$B(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$
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- $\lambda B$ ($\lambda \in \mathbb{C}$) is hypercyclic on $\ell_p(\mathbb{N})$ if and only if $|\lambda| > 1$. 
Composition Operators

Let Ω be a domain in \( \mathbb{C} \) and let \( H(\Omega) \) be the space of holomorphic functions on \( \Omega \), endowed with the topology of locally uniform convergence.
Let $\Omega$ be a domain in $\mathbb{C}$ and let $H(\Omega)$ be the space of holomorphic functions on $\Omega$, endowed with the topology of locally uniform convergence.

For each $\varphi \in H(\Omega)$ with $\varphi(\Omega) \subset \Omega$, let $C_\varphi$ denote the composition operator defined by

$$f \overset{C_\varphi}{\mapsto} f \circ \varphi \quad (f \in H(\Omega)).$$
Hypercyclic Composition Operators

(Birkhoff) Let $\tau$ be the $\mathbb{C}$-automorphism given by $\tau(z) := z + a$ ($a \in \mathbb{C}, a \neq 0$). Then $C_\tau$ is hypercyclic on $H(\mathbb{C})$. 
Hypercyclic Composition Operators

- **(Birkhoff)** Let \( \tau \) be the \( \mathbb{C} \)-automorphism given by \( \tau(z) := z + a \ (a \in \mathbb{C}, a \neq 0) \). Then \( C_\tau \) is hypercyclic on \( H(\mathbb{C}) \).

- **(Seidel and Walsh, 1941)** Let \( \phi \) be the \( \mathbb{D} \)-automorphism given by \( \phi(z) := \frac{z+a}{1+az} \ (a \in \mathbb{D}, a \neq 0) \). Then \( C_\phi \) is hypercyclic on \( H(\mathbb{D}) \).
(J. Bès and A. Peris (2007) also L. Bernal-González (2007))

We say that hypercyclic operators \( T_1, \ldots, T_N \) (\( N \geq 2 \)) are **d-hypercyclic** (**d-supercyclic**) provided that the direct sum operator \( T_1 \oplus \ldots \oplus T_N \) acting on \( X^N \) have a hypercyclic (supercyclic) vector in the form \( (f, \ldots, f) \in X^N \).
Examples of D-Hypercyclic Operators

(Bernal and Bès and Peris) If $a_1, a_2 \in \mathbb{C}$ non-zero with $a_1 \neq a_2$, and $\tau_1(z) := z + a_1$ and $\tau_2(z) := z + a_2$, then $C_{\tau_1}, C_{\tau_2}$ are d-hypercyclic on $H(\mathbb{C})$. 
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- **(Bernal)** If $a_1$ and $a_2$ are non-zero distinct points in $\mathbb{D}$ and $\varphi_1(z) := \frac{z + a_1}{1 + \bar{a}_1 z}$ and $\varphi_2(z) := \frac{z + a_2}{1 + \bar{a}_2 z}$ are their respective non-Euclidean translations, then $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $H(\mathbb{D})$.
Problem 1

- If $\tau_1, \tau_2$ are distinct $\mathbb{C}$-automorphisms such that $C_{\tau_1}$ and $C_{\tau_2}$ are hypercyclic, then $C_{\tau_1}, C_{\tau_2}$ are d-hypercyclic on $H(\mathbb{C})$. 
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- **Problem 1**: (Bernal-González) Let $C_{\varphi_1}$ and $C_{\varphi_2}$ be generated by non-elliptic automorphisms. Must they be d-hypercyclic on $X$, where $X$ is a subspace of $H(\mathbb{D})$? 

Problem 2

- If $T$ is invertible, then $T$ is hypercyclic if and only if $T^{-1}$ is.
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- If \( T_1, T_2 \) are d-hypercyclic, then \( T_1^{-1} \oplus T_2^{-1} \) is hypercyclic
- **Problem 2**: (Bès and Peris) Let \( T_1, T_2 \) be d-hypercyclic and invertible. Must \( T_1^{-1}, T_2^{-1} \) be d-hypercyclic?

Problem 3: When are \( C_{\varphi_1}, C_{\varphi_2} \) d-hypercyclic if \( \varphi_1 \) and \( \varphi_2 \) are self maps of \( \mathbb{D} \)? When are they d-supercyclic?
The group $LFT(\hat{\mathbb{C}})$ of linear fractional transformations consists of bijections of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that are in the form

$$\varphi(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$. 
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The group $\text{LFT}(\hat{\mathbb{C}})$ of linear fractional transformations consists of bijections of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that are in the form

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- $\text{LFT}(\mathbb{D}) = \{\varphi \in \text{LFT}(\hat{\mathbb{C}}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$ is the subgroup consisting of self maps of the unit disc.
- $\text{Aut}(\mathbb{D}) = \{\varphi \in \text{LFT}(\hat{\mathbb{C}}) : \varphi(\mathbb{D}) = \mathbb{D}\}$ is the set of linear transformations that take $\mathbb{D}$ onto itself. These are called automorphisms.
Fixed Points of Linear fractional Transformations

\[ \alpha \in \mathbb{C} \text{ is a fixed point of } \varphi \in LFT(D) \text{ if } \varphi(\alpha) = \alpha. \]
Fixed Points of Linear fractional Transformations

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- A fixed point of $\varphi \in LFT(\mathbb{D})$ is called **attractive** if the iterations
  \[ \varphi[n](z) = (\varphi \circ \ldots \circ \varphi)(z) \to \alpha \]
  as $n \to \infty$ for all $z \in \mathbb{C}$.
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  as $n \to \infty$ for all $z \in \mathbb{C}$.
- A **repulsive** fixed point of $\varphi \in LFT(\mathbb{D})$ is the attractive fixed point of the inverse $\varphi^{-1}$. 
Classification of $LFT(\hat{\mathbb{C}})$

- Let $\varphi \in LFT(\hat{\mathbb{C}})$. $\varphi$ is called **parabolic** if it has one fixed point.
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- If $\varphi$ has two fixed points, then it is conjugate to a mapping in the form $\psi(z) = \lambda z$ ($|\lambda| \geq 1$ and $\lambda \neq 1$).
Classification of $LFT(\hat{\mathbb{C}})$

- Let $\varphi \in LFT(\hat{\mathbb{C}})$. $\varphi$ is called **parabolic** if it has one fixed point.
- If $\varphi$ has two fixed points, then it is conjugate to a mapping in the form $\psi(z) = \lambda z$ ($|\lambda| \geq 1$ and $\lambda \neq 1$).
- Then $\varphi$ is called:
  1. **Elliptic** if $|\lambda| = 1$,
  2. **Hyperbolic** if $\lambda > 1$, and
  3. **Loxodromic** if $\varphi$ is neither elliptic nor parabolic.
Classification of $\text{LFT}(\mathbb{D})$ Depending on the Fixed Points

- **Parabolic** members of $\text{LFT}(\mathbb{D})$ have only one fixed point on $\partial \mathbb{D}$. 
Classification of \( LFT(\mathbb{D}) \) Depending on the Fixed Points

- **Parabolic** members of \( LFT(\mathbb{D}) \) have only one fixed point on \( \partial \mathbb{D} \).

- **Hyperbolic** members of \( LFT(\mathbb{D}) \) have an attractive fixed point in the closure \( \overline{\mathbb{D}} \) and a repulsive fixed point outside of \( \mathbb{D} \). Indeed, both fixed points lie on \( \partial \mathbb{D} \) if and only if the map is a hyperbolic automorphism of \( \mathbb{D} \).
Classification of $LFT(\mathbb{D})$ Depending on the Fixed Points

- **Parabolic** members of $LFT(\mathbb{D})$ have only one fixed point on $\partial \mathbb{D}$.
- **Hyperbolic** members of $LFT(\mathbb{D})$ have an attractive fixed point in the closure $\overline{\mathbb{D}}$ and a repulsive fixed point outside of $\mathbb{D}$. Indeed, both fixed points lie on $\partial \mathbb{D}$ if and only if the map is a hyperbolic automorphism of $\mathbb{D}$.
- **Loxodromic** and **elliptic** members of $LFT(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside of the closed unit disc.
We say that a function analytic on the unit disc (i.e. it is in $H(\mathbb{D})$)

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$$

belongs to the **Hardy space** $H^2(\mathbb{D})$ if its sequence of power series coefficients is square-summable:

$$H^2(\mathbb{D}) = \{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty \}.$$
(Bourdon, Shapiro, Ansari, Gallardo, and Montes) If $C_\varphi$ is a composition operator with $\varphi \in LFT(D)$, then the following are equivalent:

1. $C_\varphi$ is hypercyclic on $H^2(D)$.
2. $C_\varphi$ is supercyclic on $H^2(D)$.
3. $\varphi$ is either a parabolic automorphism or a hyperbolic map without a fixed point in $D$. 
Weighted Dirichlet Spaces

\[ S_\nu = \{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu} < \infty \} \]

where \( \nu \in \mathbb{R} \).
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Weighted Dirichlet Spaces

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- \( S_0 \) is the Hardy space \( H^2(\mathbb{D}) \).
- \( S_{-\frac{1}{2}} \) is the Bergman space.
Weighted Dirichlet Spaces

- $S_{\nu} = \{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu} < \infty \}$
  where $\nu \in \mathbb{R}$.
- $S_0$ is the Hardy space $H^2(\mathbb{D})$.
- $S_{-\frac{1}{2}}$ is the Bergman space.
- $S_{\frac{1}{2}}$ is the Dirichlet space.
(Gallardo and Montes, 2004) Let \( \varphi \in LFT(\mathbb{D}) \). Then

- \( C_\varphi \) is hypercyclic on \( S_\nu \) iff \( \nu < \frac{1}{2} \) and \( C_\varphi \) is hypercyclic on \( H^2(\mathbb{D}) \).
(Gallardo and Montes, 2004) Let \( \varphi \in LFT(\mathbb{D}) \). Then

- \( C_\varphi \) is hypercyclic on \( S_\nu \) iff \( \nu < \frac{1}{2} \) and \( C_\varphi \) is hypercyclic on \( H^2(\mathbb{D}) \).
- If \( \nu < \frac{1}{2} \), then \( C_\varphi \) is supercyclic on \( S_\nu \) iff it is hypercyclic on \( S_\nu \).
Linear Fractional Hypercyclicity on $S_\nu$

(Gallardo and Montes, 2004) Let $\varphi \in LFT(\mathbb{D})$. Then

- $C_\varphi$ is hypercyclic on $S_\nu$ iff $\nu < \frac{1}{2}$ and $C_\varphi$ is hypercyclic on $H^2(\mathbb{D})$.
- If $\nu < \frac{1}{2}$, then $C_\varphi$ is supercyclic on $S_\nu$ iff it is hypercyclic on $S_\nu$.
- $C_\varphi$ is supercyclic on $S_{\frac{1}{2}}$ iff $\varphi$ is a hyperbolic non-automorphism without a fixed point in $\mathbb{D}$. 
Characterization of d-Hypercyclicity

**Theorem:** (J. Bès, Ö. M., and A. Peris) Let $\nu < \frac{1}{2}$ and $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$. The following are equivalent:

1. $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $S_\nu$.
2. $C_{\varphi_1}, C_{\varphi_2}$ are d-supercyclic on $S_\nu$.
3. $\varphi_1$ and $\varphi_2$ are either parabolic automorphisms or hyperbolic maps without fixed points in $\mathbb{D}$ and satisfy that if they have the same attractive fixed point $\alpha$, the expression $\varphi_1'(\alpha) = \varphi_2'(\alpha) < 1$ does not occur.
Problem 1: (Bernal) Let $C_{\varphi_1}$ and $C_{\varphi_2}$ be generated by distinct non-elliptic automorphisms. Must they be d-hypercyclic on $H(\mathbb{D})$?

Problem 2: (Bès and Peris) Let $T_1, T_2$ be d-hypercyclic and invertible. Must $T_1^{-1}, T_2^{-1}$ be d-hypercyclic?
Example

The hyperbolic maps $\varphi_j \in \text{Aut}(\mathbb{D})$ ($j = 1, 2$) given by

$$\varphi_1(z) = \frac{(3 + i)z - 1 - i}{(-1 + i)z + 3 - i} \quad \text{and} \quad \varphi_2(z) = \frac{(3 + 2i)z - 1 - 2i}{(-1 + 2i)z + 3 - 2i}$$

have the attractive fixed points $-i$ and $\frac{3}{5} - \frac{4}{5}i$, respectively, and have the same repellent fixed point 1. Thus, by the main theorem, $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $S_\nu$ ($\nu < \frac{1}{2}$), while $C_{\varphi_1}^{-1} = C_{\varphi_1}^{-1}$ and $C_{\varphi_2}^{-1} = C_{\varphi_2}^{-1}$ are not d-hypercyclic since

$$(\varphi_1^{-1})'(1) = (\varphi_2^{-1})'(1) = \frac{1}{2} < 1.$$
Problem: Let $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$ be hyperbolic non-automorphisms without fixed points in $\mathbb{D}$. When are $C_{\varphi_1}, C_{\varphi_2}$ d-supercyclic on $S_{\frac{1}{2}}$?
Universality

(Birkhoff, 1929) There exists an entire function $f$ and a sequence of $\mathbb{C}$-automorphisms $\tau_n(z) := z + a_n$ ($a_n \in \mathbb{C}$ with $|a_n| \to \infty$) for which the set

$$\{ C_{\tau_n}(f) : n \geq 1 \} = \{ f \circ \tau_n : n \geq 1 \}$$

is dense in $H(\mathbb{C})$. 
Universality

- **(Birkhoff, 1929)** There exists an entire function \( f \) and a sequence of \( \mathbb{C} \)-automorphisms \( \tau_n(z) := z + a_n \) \((a_n \in \mathbb{C} \text{ with } |a_n| \to \infty)\) for which the set

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\{ C_{\tau_n}(f) : n \geq 1 \} = \{ f \circ \tau_n : n \geq 1 \}
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is dense in \( H(\mathbb{C}) \).

- **(Seidel and Walsh, 1941)** There exists a function \( f \) in \( H(\mathbb{D}) \) and a sequence of \( \mathbb{D} \)-automorphisms \( \phi_n(z) := \frac{z+a_n}{1+a_n z} \) \((a_n \in \mathbb{D} \text{ with } |a_n| \to 1)\) so that the set

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\{ C_{\phi_n}(f) : n \geq 1 \} = \{ f \circ \phi_n : n \geq 1 \}
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is dense in \( H(\mathbb{D}) \).
A sequence continuous linear transformations $(T_n)_{n=1}^\infty$ on a topological vector space $X$ is said to be **hypercyclic** (or **universal**) provided there is some $f \in X$ so that the set

$$\{ T_n(f) : n \geq 1 \} = \{ T_1(f), T_2(f), \ldots \}$$

is dense in $X$. 
A sequence continuous linear transformations \((T_n)_{n=1}^\infty\) on a topological vector space \(X\) is said to be **hypercyclic** (or **universal**) provided there is some \(f \in X\) so that the set
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is dense in \(X\).

\((T_n)_{n=1}^\infty\) is said to be **supercyclic** provided there is some \(f \in X\) so that the projective orbit
\[
\{ \lambda T_n(f) : n \geq 0, \lambda \in \mathbb{C} \}
\]
is dense in \(X\).
Characterization of Compositional Hypercyclicity

(Bernal and Montes, 1995)
Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $(\varphi_n) \in Aut(\Omega)^\mathbb{N}$. Then the following are equivalent:

1. $(C_{\varphi_n})$ is hypercyclic on $H(\Omega)$.
2. $(\varphi_n)$ is a run-away sequence: For each compact $K \subset \Omega$, $\exists n \in \mathbb{N}$ so that $\varphi_n(K) \cap K = \emptyset$. 
(Bernal, Bonilla, and Calderón, 2007)

$\varphi_n \in Aut(\Omega)^\mathbb{N}$ where $\Omega \neq \mathbb{C}$ simply connected. Then the following are equivalent:

1. $(C_{\varphi_n})$ is hypercyclic on $H(\Omega)$.
2. $(C_{\varphi_n})$ is supercyclic on $H(\Omega)$.
3. $(\varphi_n)$ is run-away.
(Bernal, Bonilla, and Calderón, 2007)

If \( \{\varphi_n(z) := a_n z + b_n : n \geq 1\} \subset Aut(\mathbb{C}) \) and
\[ 0 < \inf |a_n| \leq \sup |a_n| < \infty, \]
then \( (C_{\varphi_n}) \) is hypercyclic if and only if it is supercyclic.
Disjoint Sequences of Hypercyclic Operators

We say that \( N \geq 2 \) sequences of continuous linear operators \((T_{1,n}), \ldots, (T_{N,n})\) on a topological vector space \( X \) are
\textbf{d-hypercyclic (d-supercyclic)} provided that the sequence of the direct sums \((T_{1,n} \oplus \ldots \oplus T_{N,n})\) has a hypercyclic (supercyclic) vector on the diagonal of \( X^N \).
Compositional $d$-Hypercyclicity Equals $d$-Supercyclicity

**Theorem:** (J. Bès and Ö. M.) Let $(\varphi_{\ell,n}) \in Aut(\Omega)^\mathbb{N} \ (1 \leq \ell \leq N)$, $\Omega$ simply connected. TFAE:

1. $(C_{\varphi_{1,n}}), \ldots, (C_{\varphi_{N,n}})$ are $d$-hypercyclic on $H(\Omega)$.
2. $(C_{\varphi_{1,n}}), \ldots, (C_{\varphi_{N,n}})$ are $d$-supercyclic on $H(\Omega)$.
3. For each $K \in \Omega$ compact, $\exists n \in \mathbb{N}$ such that $K, \varphi_{1,n}(K), \ldots, \varphi_{N,n}(K)$ are pairwise disjoint.
Corollary: If \( \{ \varphi_n(z) := a_n z + b_n : n \geq 1 \} \subset Aut(\mathbb{C}) \), then the following are equivalent:

1. \((C_{\varphi_n})\) is hypercyclic on \( H(\mathbb{C}) \).
2. \((C_{\varphi_n})\) is supercyclic on \( H(\mathbb{C}) \).
3. \((\varphi_n)\) is run-away.
4. \( \sup_n \min\{|b_n|, |b_n/a_n|\} = \infty \).
Non-automorphic Symbols

**Theorem:** (J. Bès and Ö. M.) $\phi_{\ell, n} : \Omega \to \Omega$ holomorphic $(1 \leq \ell \leq N, n \in \mathbb{N})$. TFAE:

1. $(C_{\phi_{1,n}})_{n=1}^{\infty}, \ldots, (C_{\phi_{N,n}})_{n=1}^{\infty}$ are d-hypercyclic.
2. $\forall K \subset \Omega$ compact, $\exists n \in \mathbb{N}$
   - $K, \phi_{1,n}(K), \ldots, \phi_{N,n}(K)$ are pairwise disjoint.
   - Each map $\phi_{\ell,n}|_K : K \to \Omega$ is injective $(1 \leq \ell \leq N)$.

Case $N = 1$: (Grosse-Erdmann, Mortini, 2009).
Corollary: Let $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$. The following are equivalent:

1. $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $H(\mathbb{D})$.
2. $C_{\varphi_1}, C_{\varphi_2}$ are d-supercyclic on $H(\mathbb{D})$.
3. $\varphi_1$ and $\varphi_2$ have no fixed points in $\mathbb{D}$ and satisfy that if they have the same attractive fixed point $\alpha$, then the expression $\varphi'_1(\alpha) = \varphi'_2(\alpha) < 1$ does not occur.
Corollary: Let $\varphi_1, \ldots, \varphi_N \in LFT(\mathbb{D})$ ($N \geq 2$). $C_{\varphi_1}, \ldots, C_{\varphi_N}$ are $d$-hypercyclic on $H(\mathbb{D})$ iff $\mu_1 C_{\varphi_1}, \ldots, \mu_N C_{\varphi_N}$ are $d$-hypercyclic on $H(\mathbb{D})$, for any non-zero scalars.
**Linear Fractional Symbols**

- **Corollary:** Let $\varphi_1, \ldots, \varphi_N \in LFT(\mathbb{D})$ ($N \geq 2$). $C_{\varphi_1}, \ldots, C_{\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$ iff $\mu_1 C_{\varphi_1}, \ldots, \mu_N C_{\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$, for any non-zero scalars.

- **Problem:** Let $\mu_1, \ldots, \mu_N$ be scalars and $\varphi_1, \ldots, \varphi_N \in LFT(\mathbb{D})$ ($N \geq 2$). When are $\mu_1 C_{\varphi_1}, \ldots, \mu_N C_{\varphi_N}$ d-hypercyclic on $S_\nu$?
References

Thanks

Thank you all for attending.