Hypercyclicity versus disjoint hypercyclicity

J. Bès[1], Ö. Martin[2], A. Peris[3], R. Sanders[4], and S. Shkarin[5]

Istanbul Analysis Seminars
November 1, 2013

[1]Bowling Green State University
[2]Miami University
[3]Universidad Politécnica de Valencia
[4]University of Milwakee
[5]Queen's University Belfast

Throughout, X is a separable, infinite dimensional Fréchet space.

Definition

$T : X \to X$ is *hypercyclic* provided $\exists f \in X$ so that

$$\overline{\{f, Tf, T^2 f, \dots\}} = X.$$

We call f a *hypercyclic vector* for T.

$HC(T) = \{\text{hyp. vectors for } T\}$

Definition (Bernal '07, Bès, Peris, '07)

$T_1, T_2 : X \to X$ are *disjoint-hypercyclic* (d-hypercyclic) provided $\exists f \in X$:

$$\overline{\{(T_1^n f, T_2^n f) : \ n = 0, 1, \dots\}} = X \times X.$$

(i.e., $T_1 \oplus T_2 : X \times X \to X \times X$ has a hypercyclic vector of the form (f, f)).

We call f a *d-hyp. vector* for T_1, T_2.

$d\text{-HC}(T_1, T_2) = \{d\text{-hyp. vectors for } T_1, T_2\}$

Goal: To compare these two notions; look for properties they "share" or not.

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The three simplest conditions to show an operator is hypercyclic are:

1. Topological transitivity.
2. The Hypercyclicity Criterion.
3. The Blow-up/collapse property.

Each such condition has a "disjoint" version, with d-hypercyclic consequences:

1. d-topological transitivity: Translation operators on $H(\mathbb{C})$
2. The d-Hypercyclicity Criterion:

$$\begin{cases} \text{Composition operators on } H(\mathbb{D}) \text{ or } H^2(\mathbb{D}) \\ \text{Powers of weighted shift operators on } \ell^p \ (1 \leq p < \infty) \end{cases}$$

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3. The d-Blow-up/collapse property: Weighted shift operators.

Definition

$T : X \to X$ is *topologically transitive* provided for each non-empty open subsets U, V of X there exits n so that

$$T^n(U) \cap V \neq \emptyset. \qquad (1)$$

T is *mixing* if (1) always holds for all but finitely many positive integers n.

$T_1, T_2 : X \to X$ are *d-topologically transitive* provided for each V_0, V_1, V_2 there exists n so that

$$V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \neq \emptyset. \qquad (2)$$

T_1, T_2 *d-mixing* if (2) always holds for all but finitely many positive integers n.

Facts:

- TFAE:
  1. T is top. transitive.
  2. $HC(T)$ is residual.
  3. $HC(T)$ is dense.
- T mixing $\Rightarrow T^r$ top. transitive.

- TFAE:
  1. T_1, T_2 d-top. transitive.
  2. d-HC(T_1, T_2) is residual.
  3. d-HC(T_1, T_2) is dense.
- T_1, T_2 d-mixing $\Rightarrow T_1^r, T_2^r$ d-top. transitive

Let $H(\mathbb{C}) = \{\text{entire functions}\}$, with compact-open topology.

$$\tau_a : H(\mathbb{C}) \to H(\mathbb{C}), \quad \tau_a(f)(z) = f(z+a) \quad (z, a \in \mathbb{C}).$$

Hypercyclic translations on $H(\mathbb{C})$

- (Birkhoff, '29) τ_a is hypercyclic $\Leftrightarrow$ $a \in \mathbb{C} \setminus \{0\}$.
- (Dujos-Ruiz '83) τ_1 has hypercyclic vectors of arbitrary small order (> 0).
- (Chan, Shapiro '91) τ_1 is hypercyclic on Hilbert subspaces $E_{2,\gamma} \subset H(\mathbb{C})$ of entire functions growth order one and exponential type zero.

d-Hypercyclic translations on $H(\mathbb{C})$

- (Bernal '07 / Bès, Peris, '07)

$$\tau_{a_1}, \tau_{a_2} \text{ d-hypercyclic } \Leftrightarrow \quad a_1, a_2 \in \mathbb{C} \setminus \{0\} \text{ are distinct.}$$

- (Bès, M., Peris, Shkarin, '12)
  If $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ are distinct, then τ_{a_1}, τ_{a_2} are d-hypercyclic on $E_{2,\gamma}$.

Powers and rotations of hypercyclic and of mixing operators

- (Ansari '95) $HC(T) = HC(T^r)$ $(r \in \mathbb{N})$.
- (León-Müller '04) $HC(T) = HC(e^{i\theta}T)$.
- If T is mixing, so is T^r $(r \in \mathbb{N})$.
- If T is mixing, so is $e^{i\theta}T$ $(\theta \in \mathbb{R})$.

Powers and rotations of d-hypercyclic and of d-mixing operators

- (Ansari '95) $d\text{-HC}(T_1, \dots, T_N) = d\text{-HC}(T_1^r, \dots, T_N^r)$ $(r \in \mathbb{N})$.
- $d\text{-HC}(T_1, \dots, T_N) = d\text{-HC}(e^{i\theta_1}T_1, \dots, e^{i\theta_N}T_N)$ $(\theta_j \in [0, 2\pi))$. Indeed,

$$HC(T_1 \oplus \cdots \oplus T_N) = HC(e^{i\theta_1}T_1 \oplus \cdots \oplus e^{i\theta_N}T_N) \text{ (Shkarin '08)}$$

- $T_1, \dots, T_N$ d-mixing $\not\Rightarrow$ $T_1^{r_1}, \dots, T_N^{r_N}$ d-hypercyclic.
- $T_1, \dots, T_N$ d-mixing $\Rightarrow$ $e^{i\theta_1}T_1, \dots, e^{i\theta_N}T_N$ d-mixing.

Hypercyclicity Criterion (Kitai, Gethner-Shapiro)

Let $T : X \to X$ so that there exist integers $1 < n_1 < n_2 < \dots$ and dense subsets X_0, Y_0 of X and mappings $S_k : Y_0 \to X$ $(k \in \mathbb{N})$ so that

1. $T^{n_k}x \to 0 \quad (x \in X_0)$
2. $S_k x \to 0 \quad (x \in Y_0)$, and
3. $T^{n_k}S_k x \to x \quad (x \in Y_0)$.

Then T is hypercyclic.

D-Hypercyclicity Criterion. (Bès, Peris, '07)

Let $T_1, T_2 : X \to X$ so that there exist $1 < n_1 < n_2 < \dots$, dense subsets X_0, X_1, X_2 of X, and mappings $S_{\ell,k} : X_\ell \to X$ $(k \in \mathbb{N}, \ell = 1, 2)$ satisfying

1. $T_\ell^{n_k} x \to 0 \quad (x \in X_0)$,
2. $S_{\ell,k} x \to 0 \quad (x \in X_\ell)$, and
3. $(T_i^{n_k} S_{\ell,k} - \delta_{i,\ell} Id_{X_\ell})x \to 0 \quad (x \in X_\ell)$, $(1 \le i \le N)$.

Then T_1, T_2 are d-hypercyclic.

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Then T_1, T_2 are d-hypercyclic.

Bès, Peris '99

TFAE:

1. T satisfies the Hyp. Criterion.
2. $\forall r$: $\bigoplus_{j=1}^{r} T$ is hypercyclic on X^r.
3. T is hereditarily topologically transitive.

In particular,

mixing $\Rightarrow$ Hyp. Criterion.

Bès, Peris, '07

TFAE:

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2. $\forall r$: $\bigoplus_{j=1}^{r} T_1$, $\bigoplus_{j=1}^{r} T_2$ are d-hypercyclic on X^r.
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Notation:

$$\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$$
$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
$$LFT(\widehat{\mathbb{C}}) := \{\varphi(z) = \frac{az+b}{cz+d} : \quad a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0\}$$
$$LFT(\mathbb{D}) := \{\varphi \in LFT(\widehat{\mathbb{C}}) : \ \varphi(\mathbb{D}) \subseteq \mathbb{D}\}$$
$$Aut(\mathbb{D}) := \{\varphi \in LFT(\widehat{\mathbb{C}}) : \ \varphi(\mathbb{D}) = \mathbb{D}\}$$

Classification of $LFT(\widehat{\mathbb{C}})$

- $\varphi \in LFT(\widehat{\mathbb{C}})$ is called **parabolic** if it has (exactly) one fixed point α. If so, $\psi = \sigma \circ \varphi \circ \sigma^{-1} = z + b$, for some $0 \neq b \in \mathbb{C}$, where $\sigma(z) = \frac{1}{z-\alpha}$.
- If φ has (exactly) two fixed points α and β, then taking $\sigma(z) = \frac{z-\beta}{z-\alpha}$ it is conjugate to $\psi = \sigma \circ \varphi \circ \sigma^{-1} = \lambda z$, for some $\lambda \neq 1$. Then φ is called:
  1. **Elliptic** if $|\lambda| = 1$,
  2. **Hyperbolic** if $\lambda > 0$, and
  3. **Loxodromic** if φ is neither elliptic nor hyperbolic.

Fixed point configuration of members of $LFT(\mathbb{D})$

(a) Parabolic members of $LFT(\mathbb{D})$ have their fixed point on $\partial\mathbb{D}$.

(b) Hyperbolic members of $LFT(\mathbb{D})$ must have their attractive fixed point in $\overline{\mathbb{D}}$, with the other fixed point outside $\mathbb{D}$, and lying on $\partial\mathbb{D}$ if and only if the map is an automorphism.

(c) Loxodromic and elliptic members of $LFT(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside $\overline{\mathbb{D}}$. The elliptic ones are precisely the automorphisms in $LFT(\mathbb{D})$ with this fixed point configuration.

(Shapiro '01)

Let $\Omega = \mathbb{D}$ or $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ holomorphic. TFAE

1. C_φ hypercyclic on $H(\Omega)$
2. C_φ mixing on $H(\Omega)$.
3. φ univalent, without fixed points on Ω.

(Bourdon, Shapiro, '97)

Let $\varphi \in LFT(\mathbb{D})$, $H^2 = \{f = \sum_{n\geq 0} a_n z^n \in H(\mathbb{D}) : \ \|f\|^2 = \sum_{n\geq 0} |a_n|^2 < \infty\}$. TFAE:

1. C_φ is hypercyclic on H^2
2. C_φ is mixing on H^2
3. φ is a parabolic automorphism or hyperbolic without fixed points on $\mathbb{D}$.

Linear fractional d-hypercyclicity

For $v \in \mathbb{R}$, let

$$S_v := \{f(z) = \sum_n a_n z^n \in H(\mathbb{D}) : \ \|f\|_v^2 := \sum_n |a_n|^2 (n+1)^{2v} < \infty\}$$

If $v_1 < v_2$, then $S_{v_1} \supset S_{v_2}$.

S_v is the Bergman, Hardy, and Dirichlet space, when $v = -\frac{1}{2}, 0$ and $\frac{1}{2}$, respectively.

Gallardo, Montes, '05

Let $\varphi \in LFT(\mathbb{D})$. Then

1. C_φ hypercyclic on $S_v \Leftrightarrow C_\varphi$ hypercyclic on $S_0 = H^2(\mathbb{D})$, and $v < \frac{1}{2}$.
2. On S_v with $v < \frac{1}{2}$, C_φ hypercyclic $\Leftrightarrow C_\varphi$ supercyclic.

($T : X \to X$ is supercyclic provided there is some $f \in X$ for which

$$\overline{\{\lambda T^n f : \ \lambda \in \mathbb{C}, n = 0, 1, \dots\}} = X)$$

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(Bès, M., Peris, '12)

Theorem 1.
Let $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$ without fixed points in $\mathbb{D}$. Then T.F.A.E.:

1. $C_{\varphi_1}, C_{\varphi_2}$ are d-supercyclic on $H(\mathbb{D})$.
2. $\lambda_1 C_{\varphi_1}, \lambda_2 C_{\varphi_2}$ are d-hypercyclic on $H(\mathbb{D})$, for any $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$.
3. If φ_1 and φ_2 have the same attractive fixed point α, then

$$\varphi_1'(\alpha) = \varphi_2'(\alpha) < 1 \quad \text{does not occur.}$$

Theorem 2.
Let $v < \frac{1}{2}$. For $j = 1, 2$, let $\varphi_j \in LFT(\mathbb{D})$ be either a parabolic automorphism or hyperbolic, with no fixed point in $\mathbb{D}$. Then T.F.A.E.:

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Fact: For T invertible, T hypercyclic $\Leftrightarrow$ T^{-1} hypercyclic.

In contrast, T_1, T_2 d-mixing and invertible $\not\Rightarrow$ T_1^{-1}, T_2^{-1} d-supercyclic.

The hyperbolic maps $\varphi_j \in \text{Aut}(\mathbb{D})$ $(j = 1, 2)$ given by

$$\varphi_1(z) = \frac{(3+i)z - 1 - i}{(-1+i)z + 3 - i} \quad \text{and} \quad \varphi_2 = \frac{(3+2i)z - 1 - 2i}{(-1+2i)z + 3 - 2i}$$

have the attractive fixed points $-i$ and $\frac{3}{5} - \frac{4}{5}i$, respectively, and have the same repellent fixed point 1. So $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $H(\mathbb{D})$, by Theorem 1. But $C_{\varphi_1}^{-1} = C_{\varphi_1^{-1}}$, $C_{\varphi_2}^{-1} = C_{\varphi_2^{-1}}$ satisfy that $\varphi_1^{-1}, \varphi_2^{-1}$ have the same attractive fixed point at 1, with

$$(\varphi_1^{-1})'(1) = (\varphi_2^{-1})'(1) = \frac{1}{2} < 1.$$

By Theorem 1, $C_{\varphi_1}^{-1}, C_{\varphi_2}^{-1}$ $(= C_{\varphi_1^{-1}}, C_{\varphi_2^{-1}})$ are not d-supercyclic. $\square$

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By Theorem 1, $C_{\varphi_1}^{-1}, C_{\varphi_2}^{-1}$ $(= C_{\varphi_1^{-1}}, C_{\varphi_2^{-1}})$ are not d-supercyclic. $\square$

Fact: For T invertible, T hypercyclic $\Leftrightarrow$ T^{-1} hypercyclic.

In contrast, T_1, T_2 d-mixing and invertible $\not\Rightarrow$ T_1^{-1}, T_2^{-1} d-supercyclic.

The hyperbolic maps $\varphi_j \in \text{Aut}(\mathbb{D})$ ($j = 1, 2$) given by

$$\varphi_1(z) = \frac{(3+i)z - 1 - i}{(-1+i)z + 3 - i} \quad \text{and} \quad \varphi_2 = \frac{(3+2i)z - 1 - 2i}{(-1+2i)z + 3 - 2i}$$

have the attractive fixed points $-i$ and $\frac{3}{5} - \frac{4}{5}i$, respectively, and have the same repellent fixed point 1. So $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $H(\mathbb{D})$, by Theorem 1. But $C_{\varphi_1}^{-1} = C_{\varphi_1^{-1}}$, $C_{\varphi_2}^{-1} = C_{\varphi_2^{-1}}$ satisfy that $\varphi_1^{-1}, \varphi_2^{-1}$ have the same attractive fixed point at 1, with

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Bès, Peris, '07

Let $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, and let $1 \leq r_1 \leq r_2$ be integers. Then

1. $\lambda_1 D^{r_1}$, $\lambda_2 D^{r_2}$ are d-hypercyclic on $H(\mathbb{D}) \Leftrightarrow r_1 < r_2$.
2. Let $B : \ell^2 \to \ell^2$, $(x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$.
   Then $\lambda_1 B^{r_1}$, $\lambda_2 B^{r_2}$ are d-hypercyclic on $\ell^2 \Leftrightarrow r_1 < r_2$ and $1 < |\lambda_1| < |\lambda_2|$.
   (e.g., $2B$, $3B^2$ are d-hypercyclic on ℓ^2, while $3B$, $2B^2$ are not)
3. Let B_1, B_2 unilateral (or bilateral) weighted backward shifts on ℓ_2 ($\ell_2(\mathbb{Z})$).
   If $r_1 < r_2$, then

$$B_1^{r_1}, B_2^{r_2} \text{ d-hypercyclic} \Leftrightarrow B_1^{r_1}, B_2^{r_2} \text{ satisfy the d-Hyp. Criterion.}$$

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Corollary of d-hypercyclic LFT theorem:

Let $T = C_\varphi$ be hypercyclic on $X = H(\mathbb{D})$ or S_v, where $\varphi \in LFT(\mathbb{D})$ and $1 \leq r_1 < r_2$.

1. Then T^{r_1}, T^{r_2} are d-hypercyclic on X.
2. If in addition $\varphi \in Aut(\mathbb{D})$, then T^{r_1}, T^{r_2} are d-hypercyclic on X for any non-zero integers $r_1 < r_2$.

Corollary of d-hypercyclic powers of shifts theorem:

Let T be a hypercyclic weighted shift on ℓ^2, and let $1 \leq r_1 < r_2$ be integers. Then T^{r_1}, T^{r_2} are d-hypercyclic on $\ell^2 \Leftrightarrow$ the direct sum operator $T^{r_1} \oplus T^{r_2} \oplus T^{r_2 - r_1}$ is hypercyclic on $\ell^2 \times \ell^2 \times \ell^2$.

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Let $\mathcal{U}(X) = \{\emptyset \neq V \subset X \text{ open}\}$, and $\mathcal{U}_0(X) = \{W \in \mathcal{U}(X) : 0 \in W\}$.

$T : X \to X \ldots$

- is *weakly mixing* provided $T \oplus T : X \times X \to X \times X$ is top. transitive.
- satisfies the *Blow-up/Collapse* property provided $\forall U, V \in \mathcal{U}(X)$ and $W \in \mathcal{U}_0(X)$, $\exists n$ so that $W \cap T^{-n}(V) \neq \emptyset \neq U \cap T^{-n}(W)$.

Definition.
$T_1, T_2 : X \to X \ldots$

- are *d-weakly mixing* when $T_1 \oplus T_1$, $T_2 \oplus T_2$ are d-topologically transitive on $X \times X$.
- satisfy the *d-Blow up/Collapse* property provided for each $V_0, V_1, V_2 \in \mathcal{U}$ and $W \in \mathcal{U}_0$, $\exists n$:

$$W \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \neq \emptyset \neq V_0 \cap T_1^{-n}(W) \cap T_2^{-n}(W).$$

Bès, Peris '07

- d-Blow-up/collapse $\Rightarrow$ d-topologically transitive.
- For $N = 2, 3, \ldots$, there exist unilateral weighted backward shifts $B_1, \ldots, B_N$ satisfying the d-Blow-up/collapse property on ℓ_2.

Problem: When are $B_1, \ldots, B_N$ given weighted shifts d-hypercyclic?
If so, do they satisfy the d-Hypercyclicity Criterion?

Recall:

- (Bès, Peris '99, Bernal, Grosse-Erdmann '03) For $T : X \to X$,

  Hyp. Criterion $\Leftrightarrow$ Hered. top. transitive $\Leftrightarrow$ Weak mixing $\Leftrightarrow$ Blow-up/Collapse.

- (León, Montes '97) Every hypercyclic shift on ℓ_2 is weakly mixing.
- (De la Rosa, Read '08) $\exists$ T hypercyclic, *not* weakly mixing (Hard!).

Theorem 1. Bès, M., Sanders (JOT +)

Let $B_\ell : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \;\; B_\ell e_j = w_j^{(\ell)} e_{j-1}, \;\; \ell = 1, 2, \ldots N$. Then TFAE:

(1) $B_1, B_2, \ldots, B_N$ are d-hypercyclic.

(2) $B_1, B_2, \ldots, B_N$ satisfy the d-Blow-up/Collapse property.

(3) There exist integers $1 < n_1 < n_2 < \ldots$ so that

(a) for each $|i| \leq k$:

$$\begin{cases} |w_{i+1}^{(1)} w_{i+2}^{(1)} \cdots w_{i+n_k}^{(1)}| \underset{k\to\infty}{\to} \infty \\ |w_{i-n_k+1}^{(\ell)} \cdots w_{i-1}^{(\ell)} w_i^{(\ell)}| \underset{k\to\infty}{\to} 0 \quad \ell = 1, \ldots, N. \end{cases}$$

(b) the set

$$A = \{(\ldots, \alpha_{-1,n_k}^{(2)}, \alpha_{-1,n_k}^{(3)}, \ldots, \alpha_{-1,n_k}^{(N)}, \alpha_{0,n_k}^{(2)}, \ldots, \alpha_{0,n_k}^{(N)}, \ldots) : \; k = 0, 1, \ldots\}$$

is dense in $(\mathbb{K}^{\mathbb{Z}}$, product topology), where $\alpha_{i,n}^{(\ell)} = \dfrac{\prod_{j=1}^{n} w_{i+j}^{(\ell)}}{\prod_{j=1}^{n} w_{i+j}^{(1)}}$

Fact: If B_1, B_2 are bilateral shifts on $\ell_2(\mathbb{Z})$, they are **not** d-weakly mixing!

Proof: Let $f \oplus g \in \ell_2(\mathbb{Z}) \times \ell_2(\mathbb{Z})$ be d-hypercyclic for $B_1 \oplus B_1$, $B_2 \oplus B_2$. Get n so that

$$\begin{aligned} \|(B_1 \oplus B_1)^n (f \oplus g) - (-e_0 \oplus e_0)\| &< \frac{1}{4} \\ \|(B_2 \oplus B_2)^n (f \oplus g) - (e_0 \oplus e_0)\| &< \frac{1}{4}. \end{aligned} \tag{3}$$

Then there exist $|\epsilon_j| < \frac{1}{4}$ $(j = 1, \dots, 4)$ so that

$$\begin{aligned} \langle f, e_n \rangle \prod_{j=1}^{n} w_j^{(1)} &= \langle B_1^n f, e_0 \rangle = -1 + \epsilon_1 \\ \langle g, e_n \rangle \prod_{j=1}^{n} w_j^{(1)} &= \langle B_1^n g, e_0 \rangle = \ \ 1 + \epsilon_2 \\ \langle f, e_n \rangle \prod_{j=1}^{n} w_j^{(2)} &= \langle B_2^n f, e_0 \rangle = \ \ 1 + \epsilon_3 \\ \langle g, e_n \rangle \prod_{j=1}^{n} w_j^{(2)} &= \langle B_2^n g, e_0 \rangle = \ \ 1 + \epsilon_4, \end{aligned} \tag{4}$$

a contradiction. □

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Bès, M., Sanders

- d-weakly-mixing $\Rightarrow$ d-Blow-up/collapse.
- For $N = 2, 3, \ldots,$ there exist bilateral weighted backward shifts $B_1, \ldots, B_N$ satisfying the d-Blow-up/collapse property.

In consequence,

hereditarily d-topologically transitive	$\Leftrightarrow$	d-Hypercyclicity Criterion
$\Downarrow$		$\Downarrow$ $\not\Uparrow$
d-weakly mixing	$\not\Leftarrow$ / $\Rightarrow$	d-Blow-up/collapse

Problem 1: T_1, T_2 d-weakly mixing $\overset{?}{\Rightarrow}$ T_1, T_2 satisfy the d-Hyp. Criterion?

Problem 2: T_1, T_2 d-hypercyclic $\overset{?}{\Rightarrow}$ T_1, T_2 d-topologically transitive?

- Equivalently, must $d\text{-}HC(T_1, T_2)$ be either empty or dense in X?
- True for weighted shifts and their powers, linear fractional composition operators, certain differentiation operators, operators satisfying the d-Blow-up/collapse property. . . .

Problem 3: T_1, T_2 d-hypercyclic $\overset{?}{\Rightarrow}$ T_1, T_2 support a d-hypercyclic manifold?

- True whenever T_1, T_2 satisfy the d-Hypercyclicity Criterion (Bès, Peris '07).
- True in some other cases (Salas +)
- What if T_1, T_2 are d-topologically transitive?

Problem 4: $T \oplus T^2$ hypercyclic on $X \times X \overset{?}{\Rightarrow}$ T, T^2 are d-hypercyclic?

- True if $T = B_w$ unilateral/bilateral , or if $T = C_\varphi$ with $\varphi \in LFT(\mathbb{D})$.
- $T \oplus T^2$ mixing on $X \times X \not\Rightarrow$ T, T^2 d-mixing. (Shkarin '12)

Let X be a separable, infinite dimensional Fréchet space.

Existence of hypercyclic operators and c_0-semigroups

1. X supports a mixing operator with a hypercyclic subspace. (Grivaux '05, Bonet, Peris '98, León, Montes '01, Bernal '99, Ansari '97)
2. If X is Banach with separable dual X^*, it supports a dual hypercyclic operator T (Salas '07).
3. If $X \neq \omega$ supports a mixing c_0-semigroup (Bermúdez, Bonilla, Conejero, Peris '05).

Existence of d-hypercyclic operators and c_0-semigroups

1. For each $k \geq 2$, $\exists T_1, \ldots, T_k$ d-mixing on X, commuting, with a d-hypercyclic subspace (Salas '11, Shkarin '10, Bès, M., Peris, Shkarin '12, Bès, M., Sanders +).
2. If X Banach with X^* separable and $k \geq 2$, $\exists T_1, \ldots, T_k$ dual-hypercyclic and commuting. (Salas '11, Bès, M., Peris, Shkarin '12)
3. If $X \neq \omega$ and $k \geq 2$, $\exists \{T_t^{(1)}\}_{t\geq 0}, \ldots, \{T_t^{(k)}\}_{t\geq 0}$ d-mixing and commuting c_0-semigroups. (Bès, M., Peris, Shkarin '12)