Distortion Properties of Perturbed $m-$valent Janowski Starlike Log-harmonic Functions

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December 19, 2011

Harmonic Functions

Definition

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathfrak{D}$ is said to be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$, that is, u, v satisfy, respectively the *Laplace equations*

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0$$

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$$u = \Re\mathfrak{e}(U) \text{ and } v = \Im\mathfrak{m}(V).$$

Therefore, it has a canonical decomposition

$$f = h + \overline{g} \tag{1}$$

where h and g are, respectively, the analytic functions

$$h = \frac{U+V}{2} \text{ and } g = \frac{U-V}{2}.$$

Example

$f(z) = z - 1/\bar{z} + 2\ln|z|$ is a harmonic univalent function form the exterior of the unit disc $\mathbb{D}$ onto $\mathbb{C}/\{0\}$, where $h(z) = z + \log z$ and $g(z) = \log z - 1/z$.

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It is well-known that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function.

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It is well-known that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function. However, not every complex-valued harmonic function is analytic.

- A subject of considerable importance in harmonic mappings is the Jacobian J_f of a function $f = u + iv$, defined by $J_f = u_x v_y - u_y v_x$. Or, in terms of f_z and $f_{\bar{z}}$, we have

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

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- When J_f is positive in $\mathcal{D}$, the harmonic function f is called orientation-preserving or sense-preserving in $\mathcal{D}$.
- An analytic univalent function is a special case of an sense-preserving harmonic univalent function. For analytic function f, it is well-known that $J_f \neq 0$ if and only if f is locally univalent at z.

For harmonic functions we have the following useful result due to Lewy

Theorem

A harmonic mapping is locally univalent in a neighborhood of a point z_0 if and only if the Jacobian $J_f(z) \neq 0$ at z_0.

Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc., 42(1936), 689-692.

- The first key insight into harmonic univalent mappings came from Clunie and S. Small, who observe that $f = h + \overline{g}$ is locally univalent and sense-preserving if and only if $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$|g'(z)| < |h'(z)| \quad (z \in \mathbb{D}). \tag{2}$$

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Class $\mathcal{S}_{\mathcal{H}}$

- We denote by $\mathcal{S}_{\mathcal{H}}$ the family of all harmonic, complex-valued, sense-preserving, normalized and univalent mappings defined on $\mathbb{D}$. Thus a function f in $\mathcal{S}_{\mathcal{H}}$ admits the representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{3}$$

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Definition of a Log-Harmonic Function

Let $\mathcal{H}(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and let $\mathcal{B}$ the set of all functions $w \in \mathcal{H}(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w f_z \left(\frac{\overline{f}}{f} \right), \tag{4}$$

where the second dilation function $w \in \mathcal{B}$ is such that $|w(z)| < 1$ for all $z \in \mathbb{D}$.

[0] Z. Abdulhadi and D. Bshouty, Univalent functions in $H \cdot \overline{H}(\mathbb{D})$, *Tran. Amer. Math. Soc.*, **305**(2) (1988), 841-849.

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- Observe that nonconstant log-harmonic functions are sense-preserving on $\mathbb{D}$.

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It has been shown that if f is non-vanishing log-harmonic mapping in $\mathbb{D}$, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{5}$$

where h and g are analytic in $\mathbb{D}$.

On the other hand if f vanishes at $z = 0$, but not identically zero then f admits the following representation

$$f(z) = z^m |z|^{2m\beta} h(z)\overline{g(z)}, \tag{6}$$

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- m is positive integer,
- $\mathrm{Re}\beta > -1/2$,
- $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$, with the normalization $h(0) \neq 0$, $g(0) = 1$.

[0] Z. Abdulhadi and Abu Muhanna, Starlike log-harmonic mappings of order α, *JIPAM*, **123**(7) (2006).

Example

$f(z) = z|z|^{2\beta}$, $\mathrm{Re}\beta > -\frac{1}{2}$ and $f(1) = 1$ is a solution of the equation

$$\overline{f_{\bar{z}}} = w f_z \left(\frac{\overline{f}}{f} \right)$$

in $\mathbb{C}$ with $w \equiv \bar{\beta}/(1+\beta)$

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- $f(z) = z^m|z|^{2\beta m}h(z)\overline{g(z)} \Rightarrow w(z) = \frac{m\overline{\beta}+\frac{zg'(z)}{g(z)}}{m+m\beta+\frac{zh'(z)}{h(z)}}.$

Indeed: Since given f log-harmonic function is the solution of the non-linear elliptic differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = w\frac{f_z}{f}, \tag{7}$$

we can get from (7)

$$w = \frac{\overline{f_{\bar{z}}}}{\overline{f}}\frac{f}{f_z}. \tag{8}$$

If we take the logarithmic derivative of f with respect to z and $\overline{z}$ respectively, we can obtain results.

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- $f(z) = h(z)\overline{g(z)} \Rightarrow J_f(z) = |f(z)|^2 \left[\left| \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{g'(z)}{g(z)} \right|^2 \right].$

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- $f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)} \Rightarrow J_f(z) =$
  $|f(z)|^2 \left[\left| \frac{m(1+\beta)}{z} + \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{m\beta}{z} + \frac{g'(z)}{g(z)} \right|^2 \right]$

Indeed: Similarly by using the derivative of f with respect to z and $\overline{z}$ in the folowing statement,

$$\left|\frac{f_z}{f}\right|^2 - \left|\frac{f_{\overline{z}}}{f}\right|^2 = \frac{1}{|f|^2}\left(|f_z|^2 - |f_{\overline{z}}|^2\right), \tag{9}$$

after the simple calculations we get the Jacobian.

Harmonic and Log-Harmonic Functions

The motivation behind the study of log-harmonic functions comes from the fact that for any sense preserving harmonic function $u = H_1 + \overline{G}_1$, H_1 and $\overline{G}_1$ in $\mathcal{H}(\mathbb{D})$, e^u is a non-vanishing function of $\mathcal{H} \cdot \overline{\mathcal{H}}(\mathbb{D})$. Thus, of particular interest are those functions of $H \cdot \overline{H}(\mathbb{D})$ that vanish in $\mathbb{D}$, as their zeros correspond to some singularities of harmonic functions.

Theory of log-harmonic functions are studied in 1984 and first study by Z. Abdulhadi with his paper:
Z. Abdulhadi, D. Bshouty, *Univalent functions in* $\mathcal{H}.\overline{\mathcal{H}}(D)$, Trans. Amer. Math. Soc., 305(1988), 841-849.
In this paper, in order to define log-harmonic functions, two set are stated, and with relationships established between these sets, definition of log-harmonic functions are given. Then, some log-harmonic functions' classes are described, relationships between these classes and analytic functions' classes are given.

We also note that univalent log-harmonic mappings on the unit disc have been studied extensively in

1. Z. Abdulhadi, Close-to-starlike log-harmonic mappings, *Internat. J. Math. and Math. Sci.*, **19**(3) (1996), 563-574.
2. Z. Abdulhadi, Typically real logharmonic mappings, *Internat. J. Math. and Math. Sci.*, **31**(1) (2002), 1-9.
3. Z. Abdulhadi and Y. Abu Muhanna, Starlike Log-harmonic mappings of Order α, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Article 123.
4. Z. Abdulhadi and W. Hengartner, Spirallike logharmonic mappings, *Complex Variables Theory Appl.*, **9**(2-3) (1987), 121-130.
5. Z. Abdulhadi and W. Hengartner, One pointed univalent logharmonic mappings, *J. Math. Anal. Appl.*, **203**(2) (1996), 333-351.

1. Y. Polatoglu, E. Deniz, Janowski starlike log-harmonic mappings, *Hacettepe J. of Math. and Statistics*, **38**(1) (2009), 45-49.
2. Z. Abdulhadi, W. Hengartner, J. Szynal, Univalent log-harmonic ring mappings, *Amer. Math. Soc.*, **119**(1) (1993), 735-745.
3. M. Darus, Maisarah Haji Mohd, Generalized Janowski starlike and close-to-starlike log-harmonic mappings, *Hindawi Publishing Corporation, Int. J. of Math. and Math. Sci.*, (2011), Article ID 356915.
4. H. E. Ozkan, Log-harmonic univalent functions for which analytic part is Janowski starlike functions, *GFTA, 2008*, 222-226, Malaysia.
5. Y. Polatoglu, H. E. Ozkan, E. Duman, Growth theorems for perturbated starlike log-harmonic mappings of complex order, *General Mathematics*, **17**(4) (2009), 185-193.

Class $\mathcal{P}_{LH}$

Let $\mathcal{P}_{LH}$ be the set of all log-harmonic mappings R defined on the unit disc $\mathbb{D}$ which are of the form $R = H.\overline{G}$ where H and G are in $\mathcal{H}(\mathbb{D})$, $H(0) = G(0) = 1$ and such that $\operatorname{Re}(R(z)) > 0$ for all $z \in \mathbb{D}$. In particular, the set $\mathcal{P}$ of all analytic functions $p(z)$ in $\mathbb{D}$ with $p(0) = 1$, and $\operatorname{Re}(p(z)) > 0$ in $\mathbb{D}$ is a subset of $\mathcal{P}_{LH}$.

Theorem

Let $R = H.\overline{G} \in \mathcal{P}_{LH}$. Then $p = H/G \in \mathcal{P}$. Conversely, given $p \in \mathcal{P}$ and $w \in \mathcal{B}$, then there exists nonvanishing functions H and G in $\mathcal{H}(\mathbb{D})$ such that $p = H/G$, $R = H\overline{G} \in \mathcal{P}_{LH}$ and R is a solution of the non-linear elliptic partial diffrential equation with respect to the given w.

[0] Z. Abdulhadi, Close-to-Starlike Logharmonic Mappings, *Internat. J. Math. & Math. Sci.*, **19**(3), 563-574, 1996.

Starlike Log-Harmonic Functions

Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \mathrm{Re}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) > 0 \qquad (10)$$

for every $z \in \mathbb{D}$. The class of all starlike log-harmonic functions is denoted by $\mathcal{S}^*_{LH}$.

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Theorem

$f(z) = zh(z)\overline{g(z)}$ be a log-harmonic function on $\mathbb{D}$, $0 \notin hg(\mathbb{D})$. Then $f \in \mathcal{S}^*_{LH}$ if and only if $\varphi(z) = \left(z\frac{h(z)}{g(z)}\right) \in \mathcal{S}^*$.

Z. Abdulhadi and Y. Abu Muhanna, Starlike Log-harmonic Mappings of Order α, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Article 123.

Class $\mathcal{S}^*_{PLH}(A, B)$

Let $f(z) = z^m|z|^{2m\beta}h(z)\overline{g(z)}$ be a univalent log-harmonic function, where m is positive integer. We say that f is a Janowski starlike log-harmonic function if

$$\operatorname{Re}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) > \frac{1-A}{1-B} \tag{11}$$

. The class of all Janowski starlike log-harmonic functons is denoted by $\mathcal{S}^*_{LH}(A, B)$. We also note that if $(z^m h(z))$ is a $m-$valent starlike function, then the Janowski starlike log-harmonic function $f(z) = z^m|z|^{2m\beta}h(z)\overline{g(z)}$ will be called a perturbed $m-$valent Janowski starlike log-harmonic function and the family of such functions will be denoted by $\mathcal{S}^*_{PLH}(A, B)$.

The aim of this paper is to investigate the distortion properties of perturbed $m-$valent Janowski starlike log-harmonic functions.

Subordination Principle

Let Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $s_1(z) = z + a_2z^2 + \cdots$, $s_2(z) = z + b_2z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $s_1(z)$ is subordinate to $s_2(z)$ if there exist $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ and it is denoted by $s_1(z) \prec s_2(z)$.

Class Ω

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.

Class $\mathcal{P}(A, B)$

Denote by $\mathcal{P}(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{12}$$

which are analytic in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{13}$$

for some function $\phi(z) \in \Omega$ and all $z \in \mathbb{D}$.

Class $\mathcal{S}_m^*$

$\mathcal{S}_m^*$ denote the class of functions

$$s(z) = z^m + a_{m+1}z^{m+1} + \ldots$$

regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}_m^*$ if and only if

$$Re\left(z\frac{s'(z)}{s(z)}\right) > 0 \Leftrightarrow z\frac{s'(z)}{s(z)} = p(z), p(z) \in \mathcal{P}. \qquad (14)$$

Main Results

Lemma (1)

Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{LH}(A,B)$. *Then*

$$f \in \mathcal{S}^*_{PLH_m}(A,B) \Leftrightarrow \left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \right) \prec \begin{cases} \frac{(1-m)+(A-Bm)z}{1+Bz}, & B \neq 0; \\ (1-m)+Az, & B = 0. \end{cases} \quad (15)$$

Proof.

Consider the Riemann branch for which $1^{2m\beta} = 1$. Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{LH}(A, B)$. Then

$$\mathrm{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > \frac{1-A}{1-B}$$

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$$f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \Rightarrow$$

$$\log f(z) = m\log z + m\beta\log z + m\beta\log\overline{z} + \log h(z) + \log\overline{g(z)} \Rightarrow$$

$$z\frac{f_z}{f} = m + m\beta + z\frac{h'}{h}$$

and

$$\overline{z}\frac{f_{\overline{z}}}{f} = m\beta + \overline{z}\frac{\overline{g'}}{\overline{g}} \Rightarrow$$

Proof.

$$\mathrm{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) = \mathrm{Re}\left(m + z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right)$$

$$= \mathrm{Re}\left(m + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right)$$

$$\Leftrightarrow m + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = p(z) = \begin{cases} \frac{1+A\phi(z)}{1+B\phi(z)}, & B \neq 0; \\ 1 + A\phi(z), & B = 0; \end{cases}, \phi(z) \in \Omega$$

Proof.

$$\Leftrightarrow z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} = \begin{cases} \frac{(1-m)+(A-Bm)\phi(z)}{1+B\phi(z)}, & B \neq 0; \\ (1-m) + A\phi(z), & B = 0; \end{cases}$$

$$\Leftrightarrow z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \prec \begin{cases} \frac{(1-m)+(A-Bm)z}{1+Bz}, & B \neq 0; \\ (1-m) + Az, & B = 0. \end{cases}$$

Lemma (2)

Let $w(z)$ be the second dilatation of $f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A, B)$, and let

$$w(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D}$$

where $c_n \in \mathbb{C}$, $n = 0, 1, 2, 3, \ldots, .$ Then

$$\frac{||\beta| - |\beta + 1|\, r|}{(|\beta + 1| - |\beta|\, r)} \leq |w(z)| \leq \frac{(|\beta| + |\beta + 1|\, r)}{(|\beta + 1| + |\beta|\, r)}. \qquad (16)$$

Proof.

$$f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \Rightarrow$$

$$f_z = \left(\frac{m}{z} + \frac{m\beta}{z} + \frac{h'(z)}{h(z)}\right) f,$$

$$f_{\overline{z}} = \left(\frac{m\beta}{\overline{z}} + \frac{\overline{g'(z)}}{\overline{g(z)}}\right) f \Rightarrow \overline{f_{\overline{z}}} = \left(\frac{m\overline{\beta}}{z} + \frac{g'(z)}{g(z)}\right) \overline{f} \Rightarrow$$

$$w(z) = \frac{\overline{f_{\overline{z}}}}{f_z} . \frac{f}{\overline{f}} = \frac{\left(\frac{m\overline{\beta}}{z} + \frac{g'(z)}{g(z)}\right) \overline{f}}{\left(\frac{m}{z} + \frac{m\beta}{z} + \frac{h'(z)}{h(z)}\right) f} . \frac{f}{\overline{f}} = \frac{m\overline{\beta} + z\frac{g'(z)}{g(z)}}{m\beta + m + z\frac{h'(z)}{h(z)}}$$

Proof.

Since

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_0 = 1), h(0) = 1,$$

Proof.

Since

$$h(z)=\sum_{n=0}^{\infty} a_n z^n \quad (a_0=1), h(0)=1,$$

$$g(z)=\sum_{n=0}^{\infty} b_n z^n \quad (b_0=1), g(0)=1$$

then

$$w(0)=\frac{\overline{\beta}}{\beta+1}, \quad |w(0)|<1$$

Therefore we can take $w(0)=c_0=\left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\theta}, \quad \theta\in\mathbb{R}.$

Proof.

Now consider the function

$$\phi(z) = e^{i\theta}\frac{e^{-i\theta}w(z) - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\theta}w(z)}, \quad z \in \mathbb{D},$$

which satisfies the conditions Schwarz Lemma ($\phi(z)$ analytic, $\phi(0) = 0$ and $|\phi(z)| < 1$)and use the estimate $|\phi(z)| \leq |z| = r$, to get

$$\left|e^{-i\theta}w(z) - \left|\frac{\overline{\beta}}{\beta+1}\right|\right| \leq r\left|\left|\frac{\overline{\beta}}{\beta+1}\right| e^{-i\theta}w(z) - 1\right|.$$

Proof.

This is equivalent to

$$\left| w(z) - \frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} \right| \leq \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} \tag{17}$$

with the equality holding only for a function w of the form

$$w(z) = e^{i\theta}\frac{e^{i\ell}z+\left|\frac{\overline{\beta}}{\beta+1}\right|}{1+\left|\frac{\overline{\beta}}{\beta+1}\right|e^{i\ell}z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.$$

Proof.

From the inequality (17) we have following inequalities

$$|w(z)| = \left|e^{-i\theta}w(z)\right| \geq \left| \frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} - \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} \right| = \frac{\left|\left|\frac{\overline{\beta}}{\beta+1}\right| - r\right|}{1-\left|\frac{\overline{\beta}}{\beta+1}\right| r}$$

$$|w(z)| = \left|e^{-i\theta}w(z)\right| \leq \frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} + \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} = \frac{\left|\frac{\overline{\beta}}{\beta+1}\right| + r}{1+\left|\frac{\overline{\beta}}{\beta+1}\right| r}.$$

By using the simple calculations from the last inequalities we have the result.

Proof.

This lemma based on the Zdzislaw Lewandowski, Starlike majorants and subordination, Annales Universitatis Marie-Curie Sklodowska, Sectio A, Vol. XV, (1961), 79-84.

Theorem (3)

Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A,B)$. *Then*

$$\begin{cases} \dfrac{r^{1-m}}{(1-Br)^{\frac{B-A}{B}}} \leq \left|\dfrac{h(z)}{g(z)}\right| \leq \dfrac{r^{1-m}}{(1+Br)^{\frac{B-A}{B}}}, & B \neq 0; \\ e^{-Ar} r^{1-m} \leq \left|\dfrac{h(z)}{g(z)}\right| \leq e^{Ar} r^{1-m}, & B = 0. \end{cases} \tag{18}$$

Proof.

The function

$$\begin{cases} \frac{(1-m)+(A-Bm)z}{1+Bz}, & B \neq 0; \\ (1-m)+Az, & B = 0; \end{cases} \qquad (19)$$

maps $|z| = r$ onto the disc with the centre

$$C(r) = \begin{cases} \left(-\frac{(m-1)+B(A-Bm)r^2}{1-B^2r^2}, 0\right), & B \neq 0; \\ (1-m, 0), & B = 0; \end{cases}$$

and the radius

$$\rho(r) = \begin{cases} \frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\ Ar, & B = 0. \end{cases}$$

Using Lemma 1 and the image of the (19) function, then we can write

Proof.

$$\begin{cases} \left| \left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \right) - \left(-\frac{(m+1)+B(A-Bm)r^2}{1-B^2r^2} \right) \right| \leq \frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left| \left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \right) - (1-m) \right| \leq Ar, & B = 0. \end{cases} \tag{20}$$

Proof.

Also we have

$$-|z| \leq \text{Re}z \leq |z|$$

if we use this inequality, then inequality (20) can be written in the form

$$\begin{cases} \dfrac{-(A-B)r-(m-1)-B(A-Bm)r^2}{1-B^2r^2} \leq \text{Re}\left(z\dfrac{h'(z)}{h(z)} - z\dfrac{g'(z)}{g(z)}\right) \\ \qquad \leq \dfrac{(A-B)r-(m-1)-B(A-Bm)r^2}{1-B^2r^2}, B \neq 0; \\ -Ar+(1-m) \leq \text{Re}\left(z\dfrac{h'(z)}{h(z)} - z\dfrac{g'(z)}{g(z)}\right) \leq Ar+(1-m), B=0. \end{cases} \tag{21}$$

Proof.

On the other hand we have

$$\mathrm{Re}\left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) = r\frac{\partial}{\partial r}\left(\log|h(z)| - \log|g(z)|\right).$$

Proof.

Therefore the inequality (21) can be written in the form

$$\begin{cases} \dfrac{-B(A-Bm)r^2-(A-B)r-(m-1)}{r(1-B^2r^2)} \leq \dfrac{\partial}{\partial r}\left(\log|h(z)|-\log|g(z)|\right) \\ \qquad \leq \dfrac{-B(A-Bm)r^2+(A-B)r-(m-1)}{r(1-B^2r^2)}, B \neq 0; \\ \\ -A+\dfrac{1-m}{r} \leq \dfrac{\partial}{\partial r}\left(\log|h(z)|-\log|g(z)|\right) \leq A+\dfrac{1-m}{r}, B=0; \end{cases} \tag{22}$$

and upon integration of both sides of (22) from 0 to r we get (18).

Theorem (4)

Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{PLH_m}(A, B)$*. Then*

$$\begin{cases} \dfrac{1}{(1-Br)^{\frac{m(B-A)}{B}}} \leq |h(z)| \leq \dfrac{1}{(1+Br)^{\frac{m(B-A)}{B}}}, & B \neq 0; \\ e^{-mAr} \leq |h(z)| \leq e^{mAr}, & B = 0. \end{cases} \tag{23}$$

Proof.

For $h(z) = 1 + a_1 z + a_2 z^2 + \ldots$, we have

Proof.

For $h(z) = 1 + a_1 z + a_2 z^2 + \ldots$, we have
$z^m h(z) = z^m + a_1 z^{m+1} + a_2 z^{m+2} + \ldots$

Proof.

For $h(z) = 1 + a_1 z + a_2 z^2 + \ldots$, we have $z^m h(z) = z^m + a_1 z^{m+1} + a_2 z^{m+2} + \ldots$ and since $f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A, B)$, $s(z) = z^m h(z)$ can be considered as a starlike $m-$valent function. Thus we have

Proof.

For $h(z) = 1 + a_1 z + a_2 z^2 + \dots$, we have
$z^m h(z) = z^m + a_1 z^{m+1} + a_2 z^{m+2} + \dots$ and since
$f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A, B)$, $s(z) = z^m h(z)$ can be considered as a starlike $m-$valent function. Thus we have

$$\begin{cases} \left| z\frac{s'(z)}{s(z)} - \frac{m(1-ABr^2)}{1-B^2r^2} \right| \leq \frac{m(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left| z\frac{s'(z)}{s(z)} - m \right| \leq mAr, & B = 0; \end{cases} \tag{24}$$

Proof.

using the inequality

$$-|z| \leq \mathrm{Re} z \leq |z|$$

we get

$$\begin{cases} \frac{m(1-Ar)}{1-Br} \leq \mathrm{Re}\left(z\frac{s'(z)}{s(z)}\right) \leq \frac{m(1+Ar)}{1+Br}, & B \neq 0; \\ m - mAr \leq \mathrm{Re}\left(z\frac{s'(z)}{s(z)}\right) \leq m + mAr, & B = 0; \end{cases} \tag{25}$$

which yields

$$\mathrm{Re}\left(z\frac{s'(z)}{s(z)}\right) = r\frac{\partial}{\partial r}\log|s(z)|.$$

Now the inequality (25) can be written in the form

$$\begin{cases} \frac{m(1-Ar)}{r(1-Br)} \leq \frac{\partial}{\partial r}\log|s(z)| \leq \frac{m(1+Ar)}{r(1+Br)}, & B \neq 0; \\ \frac{m}{r} - mA \leq \frac{\partial}{\partial r}\log|s(z)| \leq \frac{m}{r} + mA, & B = 0. \end{cases} \tag{26}$$

Proof.

Integration both sides of (26) from zero to r we get

$$\begin{cases} \dfrac{r^m}{(1-Br)^{\frac{m(B-A)}{B}}} \leq |s(z)| = |z^m h(z)| \leq \dfrac{r^m}{(1+Br)^{\frac{m(B-A)}{B}}}, & B \neq 0; \\ r^m e^{-mAr} \leq |s(z)| = |z^m h(z)| \leq r^m e^{mAr}, & B = 0; \end{cases}$$

which gives (23) (See: I. I. Barvin, Functions Star and Convex Univalent of Order α with Weight, Doklady. Math., Vol 76. Issue 3 (2007), 848-850 for details).

As a consequence of Theorem 3 and Theorem 4 we have the following corollary:

As a consequence of Theorem 3 and Theorem 4 we have the following corollary:

Corollary (5)

Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{PLH_m}(A, B)$. *Then*

$$\begin{cases} \left(\frac{1+Br}{(1-Br)^m}\right)^{\frac{B-A}{B}} \frac{1}{r^{1-m}} \le |g(z)| \le \left(\frac{1-Br}{(1+Br)^m}\right)^{\frac{B-A}{B}} \frac{1}{r^{1-m}}, & B \neq 0; \\ \frac{e^{-(m+1)Ar}}{r^{1-m}} \le |g(z)| \le \frac{e^{(m+1)Ar}}{r^{1-m}}, & B = 0. \end{cases}$$

As a consequence of Theorem 4 and Corollary 5 we have the following corollary:

As a consequence of Theorem 4 and Corollary 5 we have the following corollary:

Corollary (6)

$f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A, B)$, *then*

$$\begin{cases} r^{2m-1} e^{2m(Re\beta \log r - Im\beta 2k\pi)} \left(\frac{1+Br}{(1-Br)^{2m}}\right)^{\frac{B-A}{B}} \leq |f| \\ \leq r^{2m-1} e^{2m(Re\beta \log r - Im\beta 2k\pi)} \left(\frac{1-Br}{(1+Br)^{2m}}\right)^{\frac{B-A}{B}}, \quad B \neq 0; \\ r^{2m-1} e^{2m(Re\beta \log r - Im\beta 2k\pi)} e^{-(2m+1)Ar} \leq |f| \\ \leq r^{2m-1} e^{2m(Re\beta \log r - Im\beta 2k\pi)} e^{(2m+1)Ar}, \quad B = 0. \end{cases} \tag{27}$$

Theorem (7)

Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A,B)$. *Then*

$$\begin{cases} G_1(m,-Br)F_1(m,-r) \le |f_z| \le G_1(m,Br)F_1(m,r), & B \neq 0; \\ G_2(m,-Ar)F_2(m,-r) \le |f_z| \le G_2(m,Ar)F_2(m,r), & B = 0; \end{cases} \tag{28}$$

$$\begin{cases} \dfrac{||\beta| - |\beta+1|\, r|}{(|\beta+1| - |\beta|\, r)} G_1(m,-Br)F_1(m,-r) \le |f_{\bar{z}}| \\ \quad \le \dfrac{(|\beta| + |\beta+1|\, r)}{(|\beta+1| + |\beta|\, r)} G_1(m,Br)F_1(m,r), \qquad B \neq 0; \\ \dfrac{||\beta| - |\beta+1|\, r|}{(|\beta+1| - |\beta|\, r)} G_2(m,-Ar)F_2(m,-r) \le |f_{\bar{z}}| \\ \quad \le \dfrac{(|\beta| + |\beta+1|\, r)}{(|\beta+1| + |\beta|\, r)} G_2(m,Ar)F_2(m,r), \qquad B = 0, \end{cases} \tag{29}$$

where

$$G_1(m, Br) = r^{2m-2} e^{2m(\mathrm{Re}\beta \log r - Im\beta.2k\pi)} \left(\frac{1 - Br}{(1 + Br)^{2m}} \right)^{\frac{B-A}{B}},$$

$$F_1(m, r) = \frac{|m(1 + \beta) - mB(A + \beta B)r^2| + m(A - B)r}{1 - B^2 r^2},$$

$$G_2(m, Ar) = r^{2m-2} e^{2m(\mathrm{Re}\beta \log r - Im\beta 2k\pi)} e^{(2m+1)Ar},$$

$$F_2(m, r) = |m(1 + \beta)| + mAr.$$

Proof.

Since $s(z) = z^m h(z)$ is $m-$valent starlike, we have

$$\begin{cases} \left| z\frac{s'(z)}{s(z)} - \frac{m(1-ABr^2)}{1-B^2r^2} \right| \leq \frac{m(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left| z\frac{s'(z)}{s(z)} - m \right| \leq mAr, & B = 0. \end{cases} \Rightarrow$$

$$\begin{cases} \left| \left(m + z\frac{h'(z)}{h(z)} \right) - \frac{m(1-ABr^2)}{1-B^2r^2} \right| \leq \frac{m(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left| \left(m + z\frac{h'(z)}{h(z)} \right) - m \right| \leq mAr, & B = 0. \end{cases}$$

Proof.

$$\begin{cases} \left| \left(m + m\beta + z\frac{h'(z)}{h(z)} \right) - \frac{(m+m\beta)-(mBr^2(\beta B+A))}{1-B^2r^2} \right| \leq \frac{m(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left| \left(m + m\beta + z\frac{h'(z)}{h(z)} \right) - m(1+\beta) \right| \leq mAr, & B = 0. \end{cases} \tag{30}$$

After simple calculations from (30), we get

$$\begin{cases} F_1(m,-r) \leq \left| m\beta + m + z\frac{h'(z)}{h(z)} \right| \leq F_1(m,r), & B \neq 0; \\ F_2(m,-r) \leq \left| m\beta + m + z\frac{h'(z)}{h(z)} \right| \leq F_2(m,r), & B = 0. \end{cases} \tag{31}$$

Proof.

$$\begin{cases} \left|\left(m + m\beta + z\frac{h'(z)}{h(z)}\right) - \frac{(m+m\beta)-(mBr^2(\beta B+A))}{1-B^2r^2}\right| \leq \frac{m(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left|\left(m + m\beta + z\frac{h'(z)}{h(z)}\right) - m(1+\beta)\right| \leq mAr, & B = 0. \end{cases} \tag{32}$$

After simple calculations from (32), we get

$$\begin{cases} F_1(m,-r) \leq \left|m\beta + m + z\frac{h'(z)}{h(z)}\right| \leq F_1(m,r), & B \neq 0; \\ F_2(m,-r) \leq \left|m\beta + m + z\frac{h'(z)}{h(z)}\right| \leq F_2(m,r), & B = 0. \end{cases} \tag{33}$$

Proof.

$$w(z) = \frac{m\overline{\beta} + z\frac{g'(z)}{g(z)}}{m\beta + m + z\frac{h'(z)}{h(z)}}$$

Proof.

Now if we consider the inequality (16) in Lemma 2, we obtain

$$\left|m\beta+m+z\frac{h'(z)}{h(z)}\right|\frac{||\beta|-|\beta+1|r|}{(|\beta+1|-|\beta|r)}\leq\left|m\overline{\beta}+z\frac{g'(z)}{g(z)}\right| \leq\left|m\beta+m+z\frac{h'(z)}{h(z)}\right|\frac{(|\beta|+|\beta+1|r)}{(|\beta+1|+|\beta|r)}. \tag{34}$$

Using the inequality (33) in the inequality (34) we get

Proof.

$$\begin{cases} \frac{||\beta|-|\beta+1|r|}{(|\beta+1|-|\beta|r)} F_1(m,-r) \leq \left| m\overline{\beta} + z\frac{g'(z)}{g(z)} \right| \leq \frac{(|\beta|+|\beta+1|r)}{(|\beta+1|+|\beta|r)} F_1(m,r), & B \neq 0; \\ \frac{||\beta|-|\beta+1|r|}{(|\beta+1|-|\beta|r)} F_2(m,-r) \leq \left| m\overline{\beta} + z\frac{g'(z)}{g(z)} \right| \leq \frac{(|\beta|+|\beta+1|r)}{(|\beta+1|+|\beta|r)} F_2(m,r), & B = 0. \end{cases} \tag{35}$$

Proof.

On the other hand we have

$$\begin{aligned} zf_z &= \left(m\beta + m + z\frac{h'(z)}{h(z)}\right) f \\ \overline{z}\overline{f_{\overline{z}}} &= \left(m\overline{\beta} + z\frac{g'(z)}{g(z)}\right) \overline{f}. \end{aligned} \qquad (36)$$

Considering Lemma 2, inequalities (33), (35) and the equality (36) together we obtain (28) and (29).

Theorem (8)

Let $f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \in \mathcal{S}^*_{PLH_m}(A, B)$. *Then*

$$\begin{cases} M_1(r) \le J_f \le M_2(r), & B \neq 0; \\ M_3(r) \le J_f \le M_4(r), & B = 0. \end{cases} \tag{37}$$

where

$$M_1(r) = (G_1(m,-Br))^2\,(1+c_1(\beta))\,F_1(m,-r)\,[F_1(m,-r) - c_2(\beta)F_1(m, r$$

where

$$M_1(r) = (G_1(m, -Br))^2 (1 + c_1(\beta)) F_1(m, -r) [F_1(m, -r) - c_2(\beta) F_1(m, r$$

$$M_2(r) = (G_1(m, Br))^2 (1 + c_2(\beta)) F_1(m, r) [F_1(m, r) - c_1(\beta) F_1(m, -r)] ,$$

where

$$M_1(r) = (G_1(m, -Br))^2 (1 + c_1(\beta)) F_1(m, -r) [F_1(m, -r) - c_2(\beta) F_1(m, r$$

$$M_2(r) = (G_1(m, Br))^2 (1 + c_2(\beta)) F_1(m, r) [F_1(m, r) - c_1(\beta) F_1(m, -r)],$$

$$M_3(r) = (G_2(m, -Ar))^2 (1 + c_1(\beta)) F_2(m, -r) [F_2(m, -r) - c_2(\beta) F_2(m, r$$

where

$$M_1(r) = \left(G_1(m, -Br)\right)^2 \left(1 + c_1(\beta)\right) F_1(m, -r) \left[F_1(m, -r) - c_2(\beta) F_1(m, r\right.$$

$$M_2(r) = \left(G_1(m, Br)\right)^2 \left(1 + c_2(\beta)\right) F_1(m, r) \left[F_1(m, r) - c_1(\beta) F_1(m, -r)\right],$$

$$M_3(r) = \left(G_2(m, -Ar)\right)^2 \left(1 + c_1(\beta)\right) F_2(m, -r) \left[F_2(m, -r) - c_2(\beta) F_2(m, r\right.$$

$$M_4(r) = \left(G_2(m, Ar)\right)^2 \left(1 + c_2(\beta)\right) F_2(m, r) \left[F_2(m, r) - c_1(\beta) F_2(m, -r)\right],$$

where

$$M_1(r) = (G_1(m, -Br))^2 (1 + c_1(\beta)) F_1(m, -r) [F_1(m, -r) - c_2(\beta) F_1(m, r$$

$$M_2(r) = (G_1(m, Br))^2 (1 + c_2(\beta)) F_1(m, r) [F_1(m, r) - c_1(\beta) F_1(m, -r)],$$

$$M_3(r) = (G_2(m, -Ar))^2 (1 + c_1(\beta)) F_2(m, -r) [F_2(m, -r) - c_2(\beta) F_2(m, r$$

$$M_4(r) = (G_2(m, Ar))^2 (1 + c_2(\beta)) F_2(m, r) [F_2(m, r) - c_1(\beta) F_2(m, -r)],$$

$$c_1(\beta) = \frac{||\beta| - |\beta + 1| r|}{|\beta + 1| - |\beta| r},$$

where

$$M_1(r) = (G_1(m, -Br))^2 (1 + c_1(\beta)) F_1(m, -r) [F_1(m, -r) - c_2(\beta) F_1(m, r$$

$$M_2(r) = (G_1(m, Br))^2 (1 + c_2(\beta)) F_1(m, r) [F_1(m, r) - c_1(\beta) F_1(m, -r)],$$

$$M_3(r) = (G_2(m, -Ar))^2 (1 + c_1(\beta)) F_2(m, -r) [F_2(m, -r) - c_2(\beta) F_2(m, r$$

$$M_4(r) = (G_2(m, Ar))^2 (1 + c_2(\beta)) F_2(m, r) [F_2(m, r) - c_1(\beta) F_2(m, -r)],$$

$$c_1(\beta) = \frac{||\beta| - |\beta + 1| r|}{|\beta + 1| - |\beta| r},$$

$$c_2(\beta) = \frac{|\beta| + |\beta + 1| r}{|\beta + 1| + |\beta| r}.$$

Proof.

Using Theorem 3, we have

$$\begin{cases} \left| \left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \right) - \left(-\frac{(m-1)+B(A-Bm)r^2}{1-B^2r^2} \right) \right| \leq \frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left| \left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)} \right) - (1-m) \right| \leq Ar, & B = 0. \end{cases}$$

After straightforward calculations from this inequality we get

Proof.

$$\begin{cases} F_1(m,-r)c_1(\beta) \leq \left| m\beta + z\frac{g'(z)}{g(z)} \right| F_1(m,r)c_2(\beta), & B \neq 0; \\ \\ F_2(m,-r)c_1(\beta) \left| m\beta + z\frac{g'(z)}{g(z)} \right| \leq F_2(m,r)c_2(\beta), & B = 0; \end{cases} \tag{38}$$

and

$$\begin{cases} F_1(m,-r) \leq \left| m\beta + m + z\frac{h'(z)}{h(z)} \right| F_1(m,r), & B \neq 0; \\ \\ F_2(m,-r) \left| m\beta + m + z\frac{h'(z)}{h(z)} \right| \leq F_2(m,r), & B = 0. \end{cases} \tag{39}$$

Proof.

On the other hand, we can use the inequalities (38), (39) and Corollary 6 in the following statement and after simple calculations, we can get the result.

$$
\begin{aligned}
|z|^2 J_f &= |f|^2 \left(\left| m\beta + m + z\frac{h'(z)}{h(z)} \right|^2 - \left| m\beta + z\frac{g'(z)}{g(z)} \right|^2 \right) \\
&= |f|^2 \left(\left| m\beta + m + z\frac{h'(z)}{h(z)} \right| - \left| m\beta + z\frac{g'(z)}{g(z)} \right| \right) . \\
&\quad \left(\left| m\beta + m + z\frac{h'(z)}{h(z)} \right| + \left| m\beta + z\frac{g'(z)}{g(z)} \right| \right)
\end{aligned} \tag{40}
$$

THANK YOU...