Distortion Properties of Perturbed $m$–valent Janowski Starlike Log-harmonic Functions

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1. Harmonic Univalent Functions

2. Log-Harmonic Functions

3. Main Results
A continuous complex-valued function \( f = u + iv \) defined in a simply connected domain \( \mathbb{D} \) is said to be \textit{harmonic} in \( \mathbb{D} \) if both \( u \) and \( v \) are real harmonic in \( \mathbb{D} \), that is, \( u, v \) satisfy, respectively the \textit{Laplace equations}

\[
\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0
\]
There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions $u$ and $v$ there exist analytic functions $U$ and $V$ so that

$$u = \Re(U) \quad \text{and} \quad v = \Im(V).$$
There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions $u$ and $v$ there exist analytic functions $U$ and $V$ so that

$$u = \Re (U) \quad \text{and} \quad v = \Im (V).$$

Therefore, it has a canonical decomposition

$$f = h + g$$

where $h$ and $g$ are, respectively, the analytic functions

$$h = \frac{U + V}{2} \quad \text{and} \quad g = \frac{U - V}{2}. $$
Example

\[ f(z) = z - \frac{1}{\bar{z}} + 2 \ln |z| \] is a harmonic univalent function form the exterior of the unit disc \( \mathbb{D} \) onto \( \mathbb{C}/\{0\} \), where \( h(z) = z + \log z \) and \( g(z) = \log z - \frac{1}{z} \).
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It is well-known that if \( f = u + iv \) has continuous partial derivatives, then \( f \) is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function.
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It is well-known that if \( f = u + iv \) has continuous partial derivatives, then \( f \) is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function. However, not every complex-valued harmonic function is analytic.
A subject of considerable importance in harmonic mappings is the Jacobian $J_f$ of a function $f = u + iv$, defined by $J_f = u_x v_y - u_y v_x$. Or, in terms of $f_z$ and $f_{\bar{z}}$, we have

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

where $f = h + \bar{g}$ is the harmonic function in the open unit disc.
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An analytic univalent function is a special case of an sense-preserving harmonic univalent function. For analytic function $f$, it is well-known that $J_f \neq 0$ if and only if $f$ is locally univalent at $z$. 
For harmonic functions we have the following useful result due to Lewy

**Theorem**

A harmonic mapping is locally univalent in a neighborhood of a point \( z_0 \) if and only if the Jacobian \( J_f(z) \neq 0 \) at \( z_0 \).

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The first key insight into harmonic univalent mappings came from Clunie and S. Small, who observe that $f = h + \overline{g}$ is locally univalent and sense-preserving if and only if

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \quad (z \in \mathbb{D}).$$

This is equivalent to

$$|g'(z)| < |h'(z)| \quad (z \in \mathbb{D}). \quad (2)$$

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We denote by $S_{\mathcal{H}}$ the family of all harmonic, complex-valued, sense-preserving, normalized and univalent mappings defined on $\mathbb{D}$. Thus a function $f$ in $S_{\mathcal{H}}$ admits the representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic functions in $\mathbb{D}$. 
We denote by $S_H$ the family of all harmonic, complex-valued, sense-preserving, normalized and univalent mappings defined on $\mathbb{D}$. Thus a function $f$ in $S_H$ admits the representation $f = h + \overline{g}$, where

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Definition of a Log-Harmonic Function

Let $\mathcal{H}(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and let $\mathcal{B}$ the set of all functions $w \in \mathcal{H}(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\bar{f}_z = w f_z \left( \frac{\bar{f}}{f} \right),$$

where the second dilation function $w \in \mathcal{B}$ is such that $|w(z)| < 1$ for all $z \in \mathbb{D}$.

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- Observe that nonconstant log-harmonic functions are sense-preserving on \( \mathbb{D} \).

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It has been shown that if $f$ is non-vanishing log-harmonic mapping in $\mathbb{D}$, then $f$ can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where $h$ and $g$ are analytic in $\mathbb{D}$. 

(5)
On the other hand if $f$ vanishes at $z = 0$, but not identically zero then $f$ admits the following representation

$$f(z) = z^m |z|^{2m\beta} h(z) \overline{g(z)}, \quad (6)$$

where

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where

- \( m \) is positive integer,
- \( \text{Re} \beta > -1/2 \),
- \( h(z) \) and \( g(z) \) are analytic in \( \mathbb{D} \), with the normalization \( h(0) \neq 0, \ g(0) = 1 \).

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Example

\[ f(z) = z|z|^{2\beta}, \quad \text{Re}\beta > -\frac{1}{2} \quad \text{and} \quad f(1) = 1 \]

is a solution of the equation

\[ \bar{f}_z = \frac{w f_z}{f} \]

in \( \mathbb{C} \) with \( w \equiv \bar{\beta}/(1 + \beta) \)
Second dilatation of a log-harmonic function is given by:
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f(z) = h(z)g(z) \Rightarrow w(z) = \frac{g'(z)}{h'(z)h(z)}.
\]
Second Dilatation of Log-Harmonic Functions

Second dilatation of a log-harmonic function is given by:

1. \( f(z) = h(z)g(z) \Rightarrow w(z) = \frac{g'(z)}{g(z)} \cdot \frac{h'(z)}{h(z)} \).

2. \( f(z) = z^m|z|^{2\beta m}h(z)g(z) \Rightarrow w(z) = \frac{m\beta + zg'(z)}{m + m\beta + zh'(z)} \).
Indeed: Since given $f$ log-harmonic function is the solution of the non-linear elliptic differential equation

$$\frac{\overline{f_z}}{f} = w \frac{f_z}{f}, \quad (7)$$

we can get from (7)

$$w = \frac{\overline{f_z}}{f} \frac{f}{f_z}. \quad (8)$$

If we take the logarithmic derivative of $f$ with respect to $z$ and $\overline{z}$ respectively, we can obtain results.
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\[ f(z) = h(z)g(z) \Rightarrow J_f(z) = |f(z)|^2 \left[ \left| \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{g'(z)}{g(z)} \right|^2 \right]. \]
Jacobian of Log-Harmonic Functions

Jacobian of a log-harmonic function is given by:

1. \( f(z) = h(z)g(z) \Rightarrow J_f(z) = |f(z)|^2 \left[ \left| \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{g'(z)}{g(z)} \right|^2 \right] \).

2. \( f(z) = z^m |z|^{2\beta m} h(z)g(z) \Rightarrow J_f(z) = |f(z)|^2 \left[ \left| \frac{m(1+\beta)}{z} + \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{m\beta}{z} + \frac{g'(z)}{g(z)} \right|^2 \right] \).
Indeed: Similarly by using the derivative of $f$ with respect to $z$ and $\bar{z}$ in the following statement,

\[
\left| \frac{f_z}{f} \right|^2 - \left| \frac{f_{\bar{z}}}{f} \right|^2 = \frac{1}{|f|^2} \left( |f_z|^2 - |f_{\bar{z}}|^2 \right),
\]  

after the simple calculations we get the Jacobian.
The motivation behind the study of log-harmonic functions comes from the fact that for any sense preserving harmonic function $u = H_1 + \overline{G}_1$, $H_1$ and $\overline{G}_1$ in $\mathcal{H}(\mathbb{D})$, $e^u$ is a non-vanishing function of $\mathcal{H} \cdot \overline{\mathcal{H}}(\mathbb{D})$. Thus, of particular interest are those functions of $\mathcal{H} \cdot \overline{\mathcal{H}}(\mathbb{D})$ that vanish in $\mathbb{D}$, as their zeros correspond to some singularities of harmonic functions.
In this paper, in order to define log-harmonic functions, two set are stated, and with relationships established between these sets, definition of log-harmonic functions are given. Then, some log-harmonic functions’ classes are described, relationships between these classes and analytic functions’ classes are given.
We also note that univalent log-harmonic mappings on the unit disc have been studied extensively in


Let $\mathcal{P}_{LH}$ be the set of all log-harmonic mappings $R$ defined on the unit disc $\mathbb{D}$ which are of the form $R = H \overline{G}$ where $H$ and $G$ are in $\mathcal{H}(\mathbb{D})$, $H(0) = G(0) = 1$ and such that $\text{Re}(R(z)) > 0$ for all $z \in \mathbb{D}$. In particular, the set $\mathcal{P}$ of all analytic functions $p(z)$ in $\mathbb{D}$ with $p(0) = 1$, and $\text{Re}(p(z)) > 0$ in $\mathbb{D}$ is a subset of $\mathcal{P}_{LH}$.

**Theorem**

Let $R = H \overline{G} \in \mathcal{P}_{LH}$. Then $p = H/G \in \mathcal{P}$. Conversely, given $p \in \mathcal{P}$ and $w \in \mathcal{B}$, then there exists nonvanishing functions $H$ and $G$ in $\mathcal{H}(\mathbb{D})$ such that $p = H/G$, $R = H\overline{G} \in \mathcal{P}_{LH}$ and $R$ is a solution of the non-linear elliptic partial differential equation with respect to the given $w$.

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Starlike Log-Harmonic Functions

Let $f(z) = z|z|^{2\beta} h(z) \overline{g(z)}$ be a univalent log-harmonic mapping. We say that $f$ is a starlike log-harmonic mapping if

$$\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) = \text{Re} \left( \frac{zf_z - \overline{zf}_{\overline{z}}}{f} \right) > 0$$

(10)

for every $z \in \mathbb{D}$. The class of all starlike log-harmonic functions is denoted by $S^*_{LH}$. 
Starlike Log-Harmonic Functions

Let \( f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \) be a univalent log-harmonic mapping. We say that \( f \) is a starlike log-harmonic mapping if

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**Theorem**

Let \( f(z) = zh(z)\overline{g(z)} \) be a log-harmonic function on \( \mathbb{D} \), \( 0 \not\in \text{hg}(\mathbb{D}) \).

Then \( f \in S^*_{LH} \) if and only if \( \varphi(z) = \left( z \frac{h(z)}{g(z)} \right) \in S^* \).

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Let \( f(z) = z^m|z|^{2m\beta} h(z)\overline{g(z)} \) be a univalent log-harmonic function, where \( m \) is positive integer. We say that \( f \) is a Janowski starlike log-harmonic function if

\[
\text{Re} \left( \frac{zf_z - \overline{zf}_\overline{z}}{f} \right) > \frac{1 - A}{1 - B}
\]  

(11)

The class of all Janowski starlike log-harmonic functions is denoted by \( S_{PLH}^*(A, B) \). We also note that if \( (z^m h(z)) \) is a \( m \)--valent starlike function, then the Janowski starlike log-harmonic function \( f(z) = z^m|z|^{2m\beta} h(z)\overline{g(z)} \) will be called a perturbed \( m \)--valent Janowski starlike log-harmonic function and the family of such functions will be denoted by \( S_{PLH}^*(A, B) \).
The aim of this paper is to investigate the distortion properties of perturbed $m$–valent Janowski starlike log-harmonic functions.
Let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $s_1(z) = z + a_2 z^2 + \cdots$, $s_2(z) = z + b_2 z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $s_1(z)$ is subordinate to $s_2(z)$ if there exist $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ and it is denoted by $s_1(z) \prec s_2(z)$. 
Let $\Omega$ be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. 
Denote by $\mathcal{P}(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (12)$$

which are analytic in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \quad (13)$$

for some function $\phi(z) \in \Omega$ and all $z \in \mathbb{D}$. 
Class $\mathcal{S}_m^*$

$\mathcal{S}_m^*$ denote the class of functions

$$s(z) = z^m + a_{m+1}z^{m+1} + ...$$

regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}_m^*$ if and only if

$$\Re \left( z \frac{s'(z)}{s(z)} \right) > 0 \Leftrightarrow z \frac{s'(z)}{s(z)} = p(z), p(z) \in \mathcal{P}. \quad (14)$$
Main Results

Lemma (1)

Let $f = z^m |z|^{2m\beta} h(z)g(z)$ be an element of $S_{LH}^*(A, B)$. Then

$$f \in S_{PLH}^*(A, B) \iff \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) < \begin{cases} \frac{(1-m)+(A-Bm)z}{1+Bz}, & B \neq 0; \\ (1 - m) + Az, & B = 0. \end{cases}$$

(15)
Proof.

Consider the Riemann branch for which $1^{2m\beta} = 1$. Let $f = z^m |z|^{2m\beta} h(z)g(z)$ be an element of $S^*_LH(A, B)$. Then

$$\Re \left( \frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) > \frac{1 - A}{1 - B}$$
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$$\text{Re} \left( \frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) > \frac{1 - A}{1 - B}$$

$$f = z^m |z|^{2m\beta} h(z)g(z) \Rightarrow$$

$$\log f(z) = m \log z + m\beta \log z + m\beta \log \bar{z} + \log h(z) + \log g(z) \Rightarrow$$

$$z \frac{f_z}{f} = m + m\beta + z \frac{h'}{h}$$

and

$$\bar{z} \frac{f_{\bar{z}}}{f} = m\beta + \bar{z} \frac{g'}{g} \Rightarrow$$
Proof.

\[ \text{Re} \left( \frac{zf_z - \overline{zf}_{\overline{z}}}{f} \right) = \text{Re} \left( m + z \frac{h'(z)}{h(z)} - \overline{z} \frac{g'(z)}{g(z)} \right) \]

\[ = \text{Re} \left( m + z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) \]

\[ \Leftrightarrow m + z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} = p(z) = \begin{cases} \frac{1 + A\phi(z)}{1 + B\phi(z)}, & B \neq 0; \\ 1 + A\phi(z), & B = 0; \end{cases}, \phi(z) \in \Omega \]
Proof.

\[ z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} = \begin{cases} 
(1 - m) + \frac{(A - Bm)\phi(z)}{1 + B\phi(z)}, & B \neq 0; \\
(1 - m) + A\phi(z), & B = 0; 
\end{cases} \]

\[ z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \preceq \begin{cases} 
\frac{(1-m)+(A-Bm)z}{1+Bz}, & B \neq 0; \\
(1 - m) + Az, & B = 0. 
\end{cases} \]
Lemma (2)

Let $w(z)$ be the second dilatation of 
\[ f = z^m |z|^{2m\beta} h(z)g(z) \in S^*_{PLH_m}(A, B), \]
and let 
\[ w(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D} \]
where $c_n \in \mathbb{C}$, $n = 0, 1, 2, 3, \ldots$. Then
\[ \frac{|\beta| - |\beta + 1| r}{(|\beta + 1| - |\beta| r)} \leq |w(z)| \leq \frac{(|\beta| + |\beta + 1| r)}{(|\beta + 1| + |\beta| r)}. \] (16)
Proof.

\[ f = z^m |z|^{2m\beta} h(z)g(z) \Rightarrow \]

\[ f_z = \left( \frac{m}{z} + \frac{m\beta}{z} + \frac{h'(z)}{h(z)} \right) f, \]

\[ f_{\bar{z}} = \left( \frac{m\beta}{\bar{z}} + \frac{g'(z)}{g(z)} \right) f \Rightarrow f_{\bar{z}} = \left( \frac{m\beta}{z} + \frac{g'(z)}{g(z)} \right) \bar{f} \Rightarrow \]

\[ w(z) = \frac{f_{\bar{z}}}{f_z} = \frac{\left( \frac{m\beta}{z} + \frac{g'(z)}{g(z)} \right) \bar{f}}{\left( \frac{m}{z} + \frac{m\beta}{z} + \frac{h'(z)}{h(z)} \right) f} \Rightarrow \]

\[ w(z) = \frac{m\beta + z \frac{g'(z)}{g(z)}}{m\beta + m + z \frac{h'(z)}{h(z)}} \]
Proof.

Since

\[ h(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_0 = 1), \quad h(0) = 1, \]
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\[ h(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_0 = 1), \quad h(0) = 1, \]

\[ g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (b_0 = 1), \quad g(0) = 1 \]

then

\[ w(0) = \frac{\bar{\beta}}{\beta + 1}, \quad |w(0)| < 1 \]

Therefore we can take \( w(0) = c_0 = \left| \frac{\bar{\beta}}{\beta + 1} \right| e^{i\theta}, \quad \theta \in \mathbb{R}. \)
Proof.

Now consider the function

\[ \phi(z) = e^{i\theta} \frac{e^{-i\theta} w(z) - \frac{\beta}{\beta + 1}}{1 - \frac{\beta}{\beta + 1} e^{i\theta} w(z)}, \quad z \in \mathbb{D}, \]

which satisfies the conditions Schwarz Lemma (\( \phi(z) \) analytic, \( \phi(0) = 0 \) and \( |\phi(z)| < 1 \)) and use the estimate \( |\phi(z)| \leq |z| = r \), to get

\[ |e^{-i\theta} w(z) - \frac{\beta}{\beta + 1}| \leq r \left| \frac{\beta}{\beta + 1} \right| e^{-i\theta} w(z) - 1. \]
Proof.

This is equivalent to

\[
\left| w(z) - \frac{\beta}{\beta+1} \left( 1 - r^2 \right) \right| \leq \frac{r \left( 1 - \left| \frac{\beta}{\beta+1} \right|^2 \right)}{1 - \left| \frac{\beta}{\beta+1} \right|^2 r^2}
\]

with the equality holding only for a function \( w \) of the form

\[
w(z) = e^{i\theta} \frac{e^{i\ell} z + \left| \frac{\beta}{\beta+1} \right|}{1 + \left| \frac{\beta}{\beta+1} \right| e^{i\ell} z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.
\]
Proof.

From the inequality (17) we have following inequalities

\[ |w(z)| = |e^{-i\theta}w(z)| \geq \left| \frac{\beta}{\beta+1} \left( 1 - r^2 \right) \frac{1 - \left| \frac{\beta}{\beta+1} \right|^2}{1 - \left| \frac{\beta}{\beta+1} \right|^2 r^2} - \frac{r \left( 1 - \left| \frac{\beta}{\beta+1} \right|^2 \right)}{1 - \left| \frac{\beta}{\beta+1} \right|^2 r^2} \right| = \left| \frac{\beta}{\beta+1} - r \right| \]

\[ |w(z)| = |e^{-i\theta}w(z)| \leq \left| \frac{\beta}{\beta+1} \left( 1 - r^2 \right) \frac{1 - \left| \frac{\beta}{\beta+1} \right|^2}{1 - \left| \frac{\beta}{\beta+1} \right|^2 r^2} + \frac{r \left( 1 - \left| \frac{\beta}{\beta+1} \right|^2 \right)}{1 - \left| \frac{\beta}{\beta+1} \right|^2 r^2} \right| = \left| \frac{\beta}{\beta+1} + r \right| . \]

By using the simple calculations from the last inequalities we have the result.
Theorem (3)

Let \( f = z^m |z|^{2m\beta} \ h(z)g(z) \in S_{PLH_m}^*(A, B) \). Then

\[
\begin{aligned}
\left\{ \begin{array}{c}
\frac{r^{1-m}}{(1-Br)B} \leq \left| \frac{h(z)}{g(z)} \right| \leq \frac{r^{1-m}}{(1+Br)B}, \quad B \neq 0; \\
e^{-Ar} r^{1-m} \leq \left| \frac{h(z)}{g(z)} \right| \leq e^{Ar} r^{1-m}, \quad B = 0.
\end{array} \right.
\end{aligned}
\]
Proof.

The function

\[
\begin{cases}
\frac{(1-m)+(A-Bm)z}{1+Bz}, & B \neq 0; \\
(1-m) + Az, & B = 0;
\end{cases}
\]  \hspace{1cm} (19)

maps \(|z| = r\) onto the disc with the centre

\[
C(r) = \begin{cases}
\left(-\frac{(m-1)+B(A-Bm)r^2}{1-B^2r^2}, 0\right), & B \neq 0; \\
(1-m, 0), & B = 0;
\end{cases}
\]

and the radius

\[
\rho(r) = \begin{cases}
\frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\
Ar, & B = 0.
\end{cases}
\]

Using Lemma 1 and the image of the (19) function, then we can write
Proof.

\[
\begin{aligned}
\left| \left( \frac{zh'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - \left( -\frac{(m+1)+B(A-Bm)r^2}{1-B^2r^2} \right) \right| &\leq \frac{(A-B)r}{1-B^2r^2}, \quad B \neq 0; \\
\left| \left( \frac{zh'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - (1 - m) \right| &\leq Ar, \\
\left| \left( \frac{zh'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - (1 - m) \right| &\leq Ar, \quad B = 0.
\end{aligned}
\]

(20)
Proof.

Also we have

$$-|z| \leq \Re z \leq |z|$$

if we use this inequality, then inequality (20) can be written in the form

$$\begin{cases} 
\frac{-(A - B)r - (m - 1) - B(A - Bm)r^2}{1 - B^2r^2} \leq \Re \left( \frac{z}{h(z)} \frac{h'(z)}{h(z)} - \frac{z}{g(z)} \frac{g'(z)}{g(z)} \right) \\
\leq \frac{(A - B)r - (m - 1) - B(A - Bm)r^2}{1 - B^2r^2}, B \neq 0; \\
-Ar + (1 - m) \leq \Re \left( \frac{z}{h(z)} \frac{h'(z)}{h(z)} - \frac{z}{g(z)} \frac{g'(z)}{g(z)} \right) \leq Ar + (1 - m), B = 0. 
\end{cases}$$

(21)
Proof.

On the other hand we have

\[
\text{Re}\left(z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)}\right) = r \frac{\partial}{\partial r} (\log |h(z)| - \log |g(z)|).
\]
Proof.

Therefore the inequality (21) can be written in the form

\[
\begin{align*}
-\frac{B(A - Bm)r^2 - (A - B)r - (m - 1)}{r(1 - B^2 r^2)} & \leq \frac{\partial}{\partial r} (\log |h(z)| - \log |g(z)|) \\
& \leq \frac{-B(A - Bm)r^2 + (A - B)r - (m - 1)}{r(1 - B^2 r^2)}, B \neq 0; \\
-\frac{A + \frac{1 - m}{r}}{r} & \leq \frac{\partial}{\partial r} (\log |h(z)| - \log |g(z)|) \leq A + \frac{1 - m}{r}, B = 0;
\end{align*}
\]

(22)

and upon integration of both sides of (22) from 0 to \( r \) we get (18).
Theorem (4)

Let \( f = z^m |z|^{2m\beta} h(z)g(z) \) be an element of \( S^*_{PLH_m}(A, B) \). Then

\[
\begin{align*}
\frac{1}{m(B-A)} \frac{1}{B} &\leq |h(z)| \leq \frac{1}{m(B-A)} \frac{1}{(1+Br) B}, & B \neq 0; \\
e^{-mA_B} &\leq |h(z)| \leq e^{mA_B}, & B = 0.
\end{align*}
\]
Proof.

For $h(z) = 1 + a_1z + a_2z^2 + ..., \text{ we have}$
Proof.

For \( h(z) = 1 + a_1z + a_2z^2 + \ldots \), we have
\[ z^m h(z) = z^m + a_1 z^{m+1} + a_2 z^{m+2} + \ldots \]
Proof.

For $h(z) = 1 + a_1 z + a_2 z^2 + \ldots$, we have

$$z^m h(z) = z^m + a_1 z^{m+1} + a_2 z^{m+2} + \ldots$$

and since

$$f = z^m |z|^{2m\beta} \overline{h(z)g(z)} \in S^*_{PLH_m}(A, B),$$

$s(z) = z^m h(z)$ can be considered as a starlike $m$–valent function. Thus we have
Proof.

For $h(z) = 1 + a_1 z + a_2 z^2 + \ldots$, we have

$$z^m h(z) = z^m + a_1 z^{m+1} + a_2 z^{m+2} + \ldots$$

and since

$$f = z^m |z|^{2m\beta} h(z) g(z) \in S_{PLH_m}^*(A, B), \ s(z) = z^m h(z)$$

can be considered as a starlike $m$-valent function. Thus we have

$$\left| z \frac{s'(z)}{s(z)} - m \left( \frac{1 - ABr^2}{1 - B^2 r^2} \right) \right| \leq \frac{m(A-B)r}{1 - B^2 r^2}, \quad B \neq 0;$$

$$\left| z \frac{s'(z)}{s(z)} - m \right| \leq mAr, \quad B = 0;$$

(24)
Proof.

using the inequality

\[-|z| \leq \text{Re}z \leq |z|\]

we get

\[
\begin{cases}
\frac{m(1-Ar)}{1-Br} \leq \text{Re} \left( z \frac{s'(z)}{s(z)} \right) \leq \frac{m(1+Ar)}{1+Br}, & B \neq 0; \\
\frac{m - mA}{r} \leq \text{Re} \left( z \frac{s'(z)}{s(z)} \right) \leq m + mA, & B = 0;
\end{cases}
\]

which yields

\[
\text{Re} \left( z \frac{s'(z)}{s(z)} \right) = r \frac{\partial}{\partial r} \log |s(z)|.
\]

Now the inequality (25) can be written in the form

\[
\begin{cases}
\frac{m(1-Ar)}{r(1-Br)} \leq \frac{\partial}{\partial r} \log |s(z)| \leq \frac{m(1+Ar)}{r(1+Br)}, & B \neq 0; \\
\frac{m}{r} - mA \leq \frac{\partial}{\partial r} \log |s(z)| \leq \frac{m}{r} + mA, & B = 0.
\end{cases}
\]
Proof.

Integration both sides of (26) from zero to $r$ we get

\[
\begin{cases}
  \frac{r^m}{(1-Br)^{m(B-A)/B}} \leq |s(z)| = |z^m h(z)| \leq \frac{r^m}{(1+Br)^{m(B-A)/B}}, & B \neq 0; \\
  r^m e^{-mA r} \leq |s(z)| = |z^m h(z)| \leq r^m e^{mA r}, & B = 0;
\end{cases}
\]

As a consequence of Theorem 3 and Theorem 4 we have the following corollary:
As a consequence of Theorem 3 and Theorem 4 we have the following corollary:

**Corollary (5)**

Let \( f = z^m |z|^{2m\beta} h(z)\overline{g(z)} \) be an element of \( S^*_{PLH_m}(A, B) \). Then

\[
\begin{align*}
\left( \frac{1+Br}{(1-Br)^m} \right)^{\frac{B-A}{B}} \frac{1}{r^{1-m}} & \leq |g(z)| \leq \left( \frac{1-Br}{(1+Br)^m} \right)^{\frac{B-A}{B}} \frac{1}{r^{1-m}}, & B \neq 0; \\
e^{-\left(m+1\right)Ar} \frac{1}{r^{1-m}} & \leq |g(z)| \leq e^{\left(m+1\right)Ar} \frac{1}{r^{1-m}}, & B = 0.
\end{align*}
\]
As a consequence of Theorem 4 and Corollary 5 we have the following corollary:
As a consequence of Theorem 4 and Corollary 5 we have the following corollary:

Corollary (6)

\[ f = z^m |z|^{2m\beta} h(z) \overline{g(z)} \in S^*_{PLH_m}(A, B), \text{ then} \]

\[
\begin{align*}
& r^{2m-1} e^{2m(\Re \beta \log r - \Im \beta 2k\pi)} \left( \frac{1+Br}{(1-Br)^{2m}} \right)^{\frac{B-A}{B}} \leq |f| \\
& \leq r^{2m-1} e^{2m(\Re \beta \log r - \Im \beta 2k\pi)} \left( \frac{1-Br}{(1+Br)^{2m}} \right)^{\frac{B-A}{B}}, \quad B \neq 0; \quad (27) \\
& r^{2m-1} e^{2m(\Re \beta \log r - \Im \beta 2k\pi)} e^{-(2m+1)Ar} \leq |f| \\
& \leq r^{2m-1} e^{2m(\Re \beta \log r - \Im \beta 2k\pi)} e^{(2m+1)Ar}, \quad B = 0.
\end{align*}
\]
Theorem (7)

Let \( f = z^m |z|^{2m\beta} h(z)g(z) \in S_{PLH_m}^*(A, B) \). Then

\[
\begin{cases}
G_1(m, -Br)F_1(m, -r) \leq |f_z| \leq G_1(m, Br)F_1(m, r), & B \neq 0; \\
G_2(m, -Ar)F_2(m, -r) \leq |f_z| \leq G_2(m, Ar)F_2(m, r), & B = 0;
\end{cases}
\]

\[
\begin{align*}
&\frac{||\beta| - |\beta + 1||r}{(|\beta + 1| - |\beta| r)} G_1(m, -Br)F_1(m, -r) \leq |f_z| \\
&\leq \frac{(|\beta| + |\beta + 1||r)}{(|\beta + 1| + |\beta| r)} G_1(m, Br)F_1(m, r), & B \neq 0; \\
&\frac{||\beta| - |\beta + 1||r}{(|\beta + 1| - |\beta| r)} G_2(m, -Ar)F_2(m, -r) \leq |f_z| \\
&\leq \frac{(|\beta| + |\beta + 1||r)}{(|\beta + 1| + |\beta| r)} G_2(m, Ar)F_2(m, r), & B = 0,
\end{align*}
\]
where

\[
G_1(m, Br) = r^{2m-2} e^{2m(\text{Re}\beta \log r - \text{Im}\beta \cdot 2k\pi)} \left( \frac{1 - Br}{(1 + Br)^{2m}} \right)^{\frac{B-A}{B}},
\]

\[
F_1(m, r) = \frac{|m(1 + \beta) - mB(A + \beta B)r^2| + m(A - B)r}{1 - B^2r^2},
\]

\[
G_2(m, Ar) = r^{2m-2} e^{2m(\text{Re}\beta \log r - \text{Im}\beta \cdot 2k\pi)} e^{(2m+1)Ar},
\]

\[
F_2(m, r) = |m(1 + \beta)| + mAr.
\]
Proof.

Since \( s(z) = z^m h(z) \) is \( m \)-valent starlike, we have

\[
\begin{align*}
&\left| z \frac{s'(z)}{s(z)} - \frac{m(1-ABr^2)}{1-B^2r^2} \right| \leq \frac{m(A-B)r}{1-B^2r^2}, \quad B \neq 0; \\
&\Rightarrow \left| z \frac{s'(z)}{s(z)} - m \right| \leq mAr, \quad B = 0.
\end{align*}
\]

\[
\begin{align*}
&\left| \left( m + z \frac{h'(z)}{h(z)} \right) - \frac{m(1-ABr^2)}{1-B^2r^2} \right| \leq \frac{m(A-B)r}{1-B^2r^2}, \quad B \neq 0; \\
&\left| \left( m + z \frac{h'(z)}{h(z)} \right) - m \right| \leq mAr, \quad B = 0.
\end{align*}
\]
Proof.

\[
\left\{ \begin{aligned}
&\left( m + m\beta + z \frac{h'(z)}{h(z)} \right) - \frac{(m+m\beta) - (mBr^2(\beta B + A))}{1-B^2r^2} \leq \frac{m(A-B)r}{1-B^2r^2}, & B \neq 0; \\
&\left( m + m\beta + z \frac{h'(z)}{h(z)} \right) - m(1 + \beta) \leq mAr, & B = 0.
\end{aligned} \right.
\]  

After simple calculations from (30), we get

\[
\left\{ \begin{aligned}
&F_1(m, -r) \leq \left| m\beta + m + z \frac{h'(z)}{h(z)} \right| \leq F_1(m, r), & B \neq 0; \\
&F_2(m, -r) \leq \left| m\beta + m + z \frac{h'(z)}{h(z)} \right| \leq F_2(m, r), & B = 0.
\end{aligned} \right.
\]  

(31)
Proof.

\[ \left\| \left( m + m\beta + z \frac{h'(z)}{h(z)} \right) - \frac{(m+m\beta) - (mBr^2(\beta B + A))}{1 - B^2 r^2} \right\| \leq \frac{m(A-B)r}{1 - B^2 r^2}, \quad B \neq 0; \]

\[ \left\| \left( m + m\beta + z \frac{h'(z)}{h(z)} \right) - m(1 + \beta) \right\| \leq mA r, \quad B = 0. \]

After simple calculations from (32), we get

\[ \begin{cases} 
F_1(m, -r) \leq \left| m\beta + m + z \frac{h'(z)}{h(z)} \right| \leq F_1(m, r), \quad B \neq 0; \\
F_2(m, -r) \leq \left| m\beta + m + z \frac{h'(z)}{h(z)} \right| \leq F_2(m, r), \quad B = 0. 
\end{cases} \]

(33)
Proof.

\[ w(z) = \frac{m\beta + z \frac{g'(z)}{g(z)}}{m\beta + m + z \frac{h'(z)}{h(z)}} \]
Proof.

Now if we consider the inequality (16) in Lemma 2, we obtain

\[
\left| m\beta + m + z \frac{h'(z)}{h(z)} \right| \frac{|\beta| - |\beta + 1|}{|\beta + 1| - |\beta|} \leq \left| m\beta + z \frac{g'(z)}{g(z)} \right| \leq \left| m\beta + m + z \frac{h'(z)}{h(z)} \right| \frac{|\beta| + |\beta + 1|}{|\beta + 1| + |\beta|}.
\]

(34)

Using the inequality (33) in the inequality (34) we get
Proof.

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{|\beta| - |\beta+1|r}{(|\beta+1|-|\beta|r)} F_1(m, -r) \leq \left| m\beta + z \frac{g'(z)}{g(z)} \right| \leq \frac{(|\beta|+|\beta+1|r)}{(|\beta+1|+|\beta|r)} F_1(m, r), \quad B \neq 0; \\
\frac{|\beta| - |\beta+1|r}{(|\beta+1|-|\beta|r)} F_2(m, -r) \leq \left| m\beta + z \frac{g'(z)}{g(z)} \right| \leq \frac{(|\beta|+|\beta+1|r)}{(|\beta+1|+|\beta|r)} F_2(m, r), \quad B = 0.
\end{array} \right.
\end{aligned}
\]  

(35)
Proof.

On the other hand we have

\[ zf_z = \left( m\beta + m + z \frac{h'(z)}{h(z)} \right) f \]

\[ \overline{zf_z} = \left( m\overline{\beta} + z \frac{g'(z)}{g(z)} \right) \overline{f}. \]  

(36)

Considering Lemma 2, inequalities (33), (35) and the equality (36) together we obtain (28) and (29).
Theorem (8)

Let \( f = z^m |z|^{2m\beta} h(z) \overline{g(z)} \in S_{PLH_m}^*(A, B) \). Then

\[
\begin{cases}
M_1(r) \leq J_f \leq M_2(r), & B \neq 0; \\
M_3(r) \leq J_f \leq M_4(r), & B = 0.
\end{cases}
\]
where

\[ M_1(r) = (G_1(m, -Br))^2 \left( 1 + c_1(\beta) \right) F_1(m, -r) \left[ F_1(m, -r) - c_2(\beta) F_1(m, r) \right] \]
where

\[ M_1(r) = \left( G_1(m, -Br) \right)^2 (1 + c_1(\beta)) F_1(m, -r) \left[ F_1(m, -r) - c_2(\beta) F_1(m, r) \right], \]

\[ M_2(r) = \left( G_1(m, Br) \right)^2 (1 + c_2(\beta)) F_1(m, r) \left[ F_1(m, r) - c_1(\beta) F_1(m, -r) \right], \]
where

\[ M_1(r) = (G_1(m, -Br))^2 \left( 1 + c_1(\beta) \right) F_1(m, -r) \left[ F_1(m, -r) - c_2(\beta)F_1(m, r) \right], \]

\[ M_2(r) = (G_1(m, Br))^2 \left( 1 + c_2(\beta) \right) F_1(m, r) \left[ F_1(m, r) - c_1(\beta)F_1(m, -r) \right], \]

\[ M_3(r) = (G_2(m, -Ar))^2 \left( 1 + c_1(\beta) \right) F_2(m, -r) \left[ F_2(m, -r) - c_2(\beta)F_2(m, r) \right]. \]
where

\[ M_1(r) = (G_1(m, -Br))^2 (1 + c_1(\beta)) F_1(m, -r) [F_1(m, -r) - c_2(\beta) F_1(m, r)] \]

\[ M_2(r) = (G_1(m, Br))^2 (1 + c_2(\beta)) F_1(m, r) [F_1(m, r) - c_1(\beta) F_1(m, -r)] \]

\[ M_3(r) = (G_2(m, -Ar))^2 (1 + c_1(\beta)) F_2(m, -r) [F_2(m, -r) - c_2(\beta) F_2(m, r)] \]

\[ M_4(r) = (G_2(m, Ar))^2 (1 + c_2(\beta)) F_2(m, r) [F_2(m, r) - c_1(\beta) F_2(m, -r)] \]
where

\[ M_1(r) = (G_1(m, -Br))^2 \left(1 + c_1(\beta)\right) F_1(m, -r) \left[F_1(m, -r) - c_2(\beta) F_1(m, r)\right], \]

\[ M_2(r) = (G_1(m, Br))^2 \left(1 + c_2(\beta)\right) F_1(m, r) \left[F_1(m, r) - c_1(\beta) F_1(m, -r)\right], \]

\[ M_3(r) = (G_2(m, -Ar))^2 \left(1 + c_1(\beta)\right) F_2(m, -r) \left[F_2(m, -r) - c_2(\beta) F_2(m, r)\right], \]

\[ M_4(r) = (G_2(m, Ar))^2 \left(1 + c_2(\beta)\right) F_2(m, r) \left[F_2(m, r) - c_1(\beta) F_2(m, -r)\right], \]

\[ c_1(\beta) = \frac{||\beta| - |\beta + 1||r|}{|\beta + 1| - |\beta||r|}, \]
where

\[
M_1(r) = (G_1(m, -Br))^2 (1 + c_1(\beta)) F_1(m, -r) \left[ F_1(m, -r) - c_2(\beta) F_1(m, r) \right],
\]

\[
M_2(r) = (G_1(m, Br))^2 (1 + c_2(\beta)) F_1(m, r) \left[ F_1(m, r) - c_1(\beta) F_1(m, -r) \right],
\]

\[
M_3(r) = (G_2(m, -Ar))^2 (1 + c_1(\beta)) F_2(m, -r) \left[ F_2(m, -r) - c_2(\beta) F_2(m, r) \right],
\]

\[
M_4(r) = (G_2(m, Ar))^2 (1 + c_2(\beta)) F_2(m, r) \left[ F_2(m, r) - c_1(\beta) F_2(m, -r) \right],
\]

\[
c_1(\beta) = \frac{|\beta| - |\beta + 1|r}{|\beta + 1| - |\beta|r},
\]

\[
c_2(\beta) = \frac{|\beta| + |\beta + 1|r}{|\beta + 1| + |\beta|r}.
\]
Using Theorem 3, we have

\[
\left| \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - \left( -\frac{(m-1)+B(A-Bm)r^2}{1-B^2r^2} \right) \right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad B \neq 0; \]

\[
\left| \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - (1 - m) \right| \leq Ar, \quad B = 0.
\]

After straightforward calculations from this inequality we get
Proof.

\[
\begin{cases}
F_1(m, -r)c_1(\beta) \leq \left| m\beta + z\frac{g'(z)}{g(z)} \right| F_1(m, r)c_2(\beta), & B \neq 0; \\
F_2(m, -r)c_1(\beta) \left| m\beta + z\frac{g'(z)}{g(z)} \right| \leq F_2(m, r)c_2(\beta), & B = 0;
\end{cases}
\]

and

\[
\begin{cases}
F_1(m, -r) \leq \left| m\beta + m + z\frac{h'(z)}{h(z)} \right| F_1(m, r), & B \neq 0; \\
F_2(m, -r) \left| m\beta + m + z\frac{h'(z)}{h(z)} \right| \leq F_2(m, r), & B = 0.
\end{cases}
\]
Proof.

On the other hand, we can use the inequalities (38), (39) and Corollary 6 in the following statement and after simple calculations, we can get the result.

\[ |z|^2 J_f = |f|^2 \left( \left| m \beta + m + z \frac{h'(z)}{h(z)} \right|^2 - \left| m \beta + z \frac{g'(z)}{g(z)} \right|^2 \right) \]

\[ = |f|^2 \left( \left| m \beta + m + z \frac{h'(z)}{h(z)} \right| - \left| m \beta + z \frac{g'(z)}{g(z)} \right| \right) \left( \left| m \beta + m + z \frac{h'(z)}{h(z)} \right| + \left| m \beta + z \frac{g'(z)}{g(z)} \right| \right) \quad (40) \]
THANK YOU...