Istanbul. MAY, 29

Capacity dimension of the Brjuno set.

A. Sadullaev, K. Rakhimov

AMS MSC 2010: 31A15, 37F10

Accepted for publication by Indiana University Mathematics Journal

1. Introduction. The set $\mathcal{B}$ of Brjuno numbers arises in connection with the problem of linearization of holomorphic germ $f(z) = \lambda z + a_2 z^2 + ...$ in a neighborhood of point $z = 0$:

$$\varphi(z) \circ f(z) \circ \varphi^{-1}(z) = \lambda z, \tag{1}$$

where φ is a germ of holomorphic function in the neighborhood of $z = 0$, $\varphi'(0) \neq 0$. This problem is important in Complex Dynamics, to describe the dynamics $f^n = f \circ ... \circ f$ in a neighborhood of fixed point. For $|\lambda| \neq 0, 1$ the linearization (1) always is possible, according to the theorem G. Königs [3]. However, for $\lambda = e^{2\pi\alpha i}$ with irrational $\alpha \in \mathbb{R}$, the question is answered in terms of Brjuno numbers.

Theorem 1 (Brjuno [4]). *Any convergent germ $f(z) = e^{2\pi\alpha i} z + a_2 z^2 + ...$ is linearizable if $\alpha \in \mathcal{B}$ is a Brjuno number.*

Theorem 2 (Yoccoz [5]). *If the polynomial $f(z) = e^{2\pi i\alpha} z + z^2$ is linearisable then $\alpha \in \mathcal{B}$.*

The Brjuno set $\mathcal{B}$ consists of irrational numbers whose partial fraction expansions satisfy the condition given below. Specifically, we start with the Gauss translation $A(\alpha) : (0, 1] \to [0, 1]$ defined by $A(\alpha) = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right]$, where $[x]$ is the whole part x. For an irrational $\alpha \in \mathbb{R}$, we define $\alpha_0 = \alpha - [\alpha]$, $a_0 = [\alpha]$, and then we continue by induction: $\alpha_{n+1} = A(\alpha_n)$, $a_{n+1} = \left[\frac{1}{\alpha_n}\right]$. Since α is irrational, we obtain an infinite continued fraction expansion

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + ...}} \quad .$$

The fraction at the nth stage

$$\frac{P_n}{Q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + ... + \frac{1}{a_n}}} \tag{2}$$

is called the nth convergent of α. We say that α is a *Brjuno number* if

$$\sum_{n=1}^{\infty} \frac{\ln Q_{n+1}}{Q_n} < \infty.$$

The main result of the paper involves σ-capacity, the definition of which will be recalled below:

Theorem 3. *The complement* $\mathbb{R}\backslash\mathcal{B}$ *of the Brjuno set has zero* C_σ *-capacity with respect to the kernel* $k_\sigma(z,\xi) = |\ln|z-\xi||^\sigma$ *,* $\sigma > 2$*.*

An irrational number α is said to be *Liouville* if for every positive integer n there are integers p and q with $q > 1$ and $0 < |\alpha - \frac{p}{q}| < \frac{1}{q^n}$. We let $\mathbb{L}$ denote the set of Liouville numbers, so it follows that $\mathbb{R}\setminus\mathcal{B} \subset \mathbb{L}\cup\mathbb{Q}$ (see [8]). The size of the $\mathbb{L}\cup\mathbb{Q}$ was studied by several authors (see [6,9,10]). In particular, $\mathbb{L}$ is known to have zero Hausdorff δ-dimensional measure for all $\delta > 0$.

To estimate the size of a set of Hausdorff dimension 0, we may consider the Hausdorff measure with respect to a gauge function $h(t)$ with $h(t) \gg t^\delta$ for all $\delta > 0$. Olsen [11] (see also [12]) has given a rather precise criterion: if $t \mapsto h(t)$, $t > 0$, is a gauge function for which the function

$$r \mapsto \inf_{0<t<r} r\frac{h(t)}{t} \tag{3}$$

increases slower than some power function r^δ, $\delta > 0$, then $H^h(\mathbb{L}) = 0$. Conversely, if $t \mapsto h(t)$ is a gauge function for which the function (3) increases faster than any power function, then $H^h(\mathbb{L}) = \infty$.

If we apply Olsen's Theorem to the case $h_\sigma(t) := |\ln t|^{-\sigma}$, then we find that $H^{h_\sigma}(\mathbb{L}) = \infty$ for all $\sigma > 0$. In particular, it follows from Theorem 3 that $\mathbb{L}$ has a larger gauge dimension than $\mathbb{R}\backslash\mathcal{B}$:

Corollary 4. *If* h_σ *is as above, and* $\sigma > 2$*, then* $H^{h_\sigma}(\mathbb{R}\setminus\mathcal{B}) = 0$*.*

2. σ-Capacity. For details regarding the following definitions, we refer the reader to [1,2]. We fix a compact $K \subset \mathbb{C}$ and kernel $k_\sigma(z,\xi) = |\ln|z-\xi||^\sigma$, $\sigma > 0$. Let M_K^+ is a set all positive Borel measure, supported on K. Subset of the probability measures $|\mu| = 1$, we denote as $\overset{o}{M}{}_K^+$. Then the integral

$$U^\mu(z) = \int\limits_K k_\sigma(z,\xi)d\mu(\xi),\ \mu \in \overset{o}{M}{}_K^+$$

is called as potential of the measure μ.

Let

$$I(\mu) = \int\limits_K U^\mu(z)d\mu(z) = \int\int\limits_{K\times K} k_\sigma(z,\xi)d\mu(z)d\mu(\xi),$$

and let $W(K) = \inf\left\{I(\mu) :\ \mu \in \overset{o}{M}{}_K^+\right\}$. The *$\sigma$-capacity of K* is defined as

$$C_\sigma(K) = W^{-1}(K)$$

As usual, we define the inner and outer capasities, putting

$$C_{*\sigma} = \sup\left\{C_\sigma(K) : K \subset E,\ K - \text{compact}\right\},$$

$$C^*_\sigma = \inf\left\{C_{*\sigma}(U) : E \subset U,\ U - \text{open}\right\}.$$

We note three classical properties of capacities, which follows from the general theory of capasities (see [1, 2]):

1). Every Borel set $E \subset \mathbb{C}$ is σ-capacitible: its inner and outer capacities coincide: $C^*_\sigma(E) = C_{*\sigma}(E) = C_\sigma(E)$;

2). The capacity of a Borel set is zero, $C_\sigma(E) = 0$, if and only if there exist a finite Borel measure $\mu \in M^+_{\overline{E}}$: $U^\mu(z) \equiv +\infty\ \forall z \in E$;

3). If $C_\sigma(E) = 0$, then the Hausdorff measure H^h with respect to the gauge function $h(t) = |\ln t|^{-\delta}$ is zero for $\delta > \sigma$. Conversely, if $H^h(E) < \infty$ for $h(t) = |\ln t|^{-\sigma}$, then $C_\sigma(K) = 0$.

Next we note three classical properties of continued fractions (see, an example, [6]):

a). The nth convergent $\frac{P_n}{Q_n}$ gives the best approximation of α by rational fractions, i.e. $|Q_n\alpha - P_n| > |Q_{n+1}\alpha - P_{n+1}|$, $n = 1, 2, \ldots$, and $|Q\alpha - P| > |Q_n\alpha - P_n|$ for any $0 < Q < Q_{n+1}$, $P \in \mathbb{Z}$, $(P, Q) \neq (P_n, Q_n)$;

b). For any $n \geq 1$ we have

$$Q_n \geq \frac{1}{2} \cdot \left(\frac{\sqrt{5}+1}{2}\right)^{n-1} > \left[\sqrt{2}\right]^{n-3};$$

c). For any $n \geq 1$ we have

$$\frac{1}{2Q_n \cdot Q_{n+1}} < \left|\alpha - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n \cdot Q_{n+1}}.$$

3. Proof of Theorem 3. The Brjuno set $\mathcal{B} \subset \mathbb{R}$ is Borel set. Therefore, by the properti 1) of capacity, it is enough, that $C_\sigma(K) = 0$ for any compact $K \subset \mathbb{R} \backslash \mathcal{B}$ and for any fixed $\sigma = 2 + \varepsilon,\ \varepsilon > 0$. Without lost a generality, we can assume, that $K \subset [0, 1]$.

We consider the following Borel measure

$$\mu = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{\delta_{\frac{p}{q}}}{q^{2+\varepsilon/4}}, \tag{4}$$

where, $\delta_{\frac{p}{q}}$ is the point mass supported at $\frac{p}{q}$.

Then $\operatorname{supp}\mu \subset [0, 1]$, $|\mu| \leq \sum\limits_{q=1}^{\infty} \frac{1}{q^{1+\varepsilon/4}} < \infty$. The potential of μ is

$$U^\mu(z) = \int\limits_K k_\sigma(z, \xi) d\mu(\xi) = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{\left|\ln\left|z - \frac{p}{q}\right|\right|^{2+\varepsilon}}{q^{2+\varepsilon/4}}.$$

We show, that $U^\mu(z)=\infty,\ \forall z\in K$. Indeed, for a rational number $z=\frac{p}{q}\in K$ it is clear. For irrational number $z=\alpha\notin\mathcal{B}$ according to (2) we construct the sequence $\{Q_n\}$. Let $\delta>0$ so small, that $(2+\varepsilon)(1-\delta)\geq 2+\varepsilon/4$. Then by the Hölder inequality we have

$$\infty=\sum_{n=1}^{\infty}\frac{\ln Q_{n+1}}{Q_n}\leq\left(\sum_{n=1}^{\infty}\frac{\ln^{2+\varepsilon}Q_{n+1}}{Q_n^{(2+\varepsilon)(1-\delta)}}\right)^{\frac{1}{2+\varepsilon}}\left(\sum_{n=1}^{\infty}\frac{1}{Q_n^{\frac{2+\varepsilon}{1+\varepsilon}\cdot\delta}}\right)^{\frac{1+\varepsilon}{2+\varepsilon}}=(S_1)^{\frac{1}{2+\varepsilon}}\cdot(S_2)^{\frac{1+\varepsilon}{2+\varepsilon}}. \quad (5)$$

It follows, by the property b) of continued fraction, that the series S_2 converges. Since, $(2+\varepsilon)(1-\delta)\geq 2+\varepsilon/4$ then

$$S_1\leq\sum_{n=1}^{\infty}\frac{\ln^{2+\varepsilon}Q_{n+1}}{Q_n^{2+\varepsilon/4}}$$

and by (5) we have

$$\sum_{n=1}^{\infty}\frac{\ln^{2+\varepsilon}Q_{n+1}}{Q_n^{2+\varepsilon/4}}=\infty.$$

Therefore, by the property c) of continued fraction,

$$\sum_{n=1}^{\infty}\frac{\left|\ln\left|\alpha-\frac{P_n}{Q_n}\right|\right|^{2+\varepsilon}}{Q_n^{2+\varepsilon/4}}\geq\sum_{n=1}^{\infty}\frac{\left|\ln^{2+\varepsilon}Q_nQ_{n+1}\right|}{Q_n^{2+\varepsilon/4}}\geq\sum_{n=1}^{\infty}\frac{\left|\ln^{2+\varepsilon}Q_{n+1}\right|}{Q_n^{2+\varepsilon/4}}=\infty.$$

It follows, that

$$U^\mu(\alpha)=\sum_{q=1}^{\infty}\sum_{p=1}^{q-1}\frac{\left|\ln\left|\alpha-\frac{p}{q}\right|\right|^{2+\varepsilon}}{q^{2+\varepsilon/4}}=\sum_{n=1}^{\infty}\frac{\left|\ln\left|\alpha-\frac{P_n}{Q_n}\right|\right|^{2+\varepsilon}}{Q_n^{2+\varepsilon/4}}+\sum_{n=1}^{\infty}\sum_{p\neq P_n,p=1}^{q-1}\frac{\left|\ln\left|\alpha-\frac{p}{Q_n}\right|\right|^{2+\varepsilon}}{Q_n^{2+\varepsilon/4}}+$$

$$+\sum_{q=1,q\neq Q_n}^{\infty}\sum_{p=1}^{q-1}\frac{\left|\ln\left|\alpha-\frac{p}{q}\right|\right|^{2+\varepsilon}}{q^{2+\varepsilon/4}}=\infty,$$

and $U^\mu(\alpha)=\infty$ for any $\alpha\in K$. Consequently, $C_{2+\varepsilon}(\mathbb{R}\backslash\mathcal{B})=0$. *The theorem is proved.*

Remark 5. If $\sigma=2$, then the potential of the finite measure (4) will be

$$U^\mu(z)=\int_K k_2(z,\xi)d\mu(\xi)=\sum_{q=1}^{\infty}\sum_{p=1}^{q-1}\frac{\left|\ln\left|\alpha-\frac{p}{q}\right|\right|^{2}}{q^{2+\varepsilon/4}}=$$

$$=\sum_{n=1}^{\infty}\frac{\left|\ln\left|\alpha-\frac{P_n}{Q_n}\right|\right|^{2}}{Q_n^{2+\varepsilon/4}}+\sum_{n=1}^{\infty}\sum_{p\neq P_n,p=1}^{q-1}\frac{\left|\ln\left|\alpha-\frac{p}{Q_n}\right|\right|^{2}}{Q_n^{2+\varepsilon/4}}+\sum_{n=1}^{\infty}\sum_{Q_n<q<Q_{n+1}}\sum_{p=1}^{q-1}\frac{\left|\ln\left|\alpha-\frac{p}{q}\right|\right|^{2}}{q^{2+\varepsilon/4}},$$

where the dominant part of the sums

$$\sum_{n=1}^{\infty}\frac{\left|\ln\left|\alpha-\frac{P_n}{Q_n}\right|\right|^2}{Q_n^{2+\varepsilon/4}}\leq\sum_{n=1}^{\infty}\frac{\ln^2 2Q_nQ_{n+1}}{Q_n^{2+\varepsilon/4}}\leq const+2\sum_{n=1}^{\infty}\frac{\ln^2 Q_{n+1}}{Q_n^{2+\varepsilon/4}}. \tag{6}$$

For non-Brjuno number $\alpha\in\mathbb{R}\backslash\mathcal{B}$, $\frac{\ln Q_{n+1}}{Q_n}\leq C<\infty$, $n=1,2,...$, the right part of (6) is not equal $+\infty$, so that $U^{\mu}(z)=\infty$ not for any $\alpha\in\mathbb{R}\backslash\mathcal{B}$. It seems, that may be $C_2(\mathbb{R}\backslash\mathcal{B})>0$. However, we do not know any proof of this conjecture, which means also that the size of the $\mathbb{R}\backslash\mathcal{B}$ is exactly as $h(t)=|\ln t|^{-2}$.

Acknowledgement. Authors express their sincere thanks to Professor Eric Bedford for their great help in the work. In particular, thanks to him we became known research of N. Olsen and D.L. Renfro, which is directly related to this article.

References.

1. Ландкоф Н. С., *Основы современной теории потенциалов.* М., “Наука”, 1966.

2. Carleson L., *Selected problems on exceptional sets. Van Nostrand, 1967.*

3. Königs G., *Recherchessur les integrals de certains equations fonctionelles.* Ann. Sci. Éc. Norm. Sup., (3-e ser.) 1, (1884), pp. 1-41.

4. Brjuno A. D., Analytical form of differential equations. Trans. Moscow Math. Soc., 25, (1971), pp.131 - 288; 26, (1972), pp.199 - 239.

5. Yoccoz J.-C., *Theoreme de Siegel, polynomes quadratiques et nombres de Brjuno.* Asterisque, 231, (1995), pp. 3–88.

6. Marmi S., *An introduction to small divisors problems.* Dottorato di ricerca in matematica. Universita Di Pisa Dipartimento Di Matematica,1999.

7. Marmi S., MoussaP.,YoccozJ.-C.,*Complex Brjuno functions.* Journal of AMS., 14: 4, (2001), pp.265-293.

8. Milnor J., *Dynamics in one complex variables. Introductory lectures,* Stony Brook, 1999.

9. Oxtoby J., *Measure and category,* Springer Verlage, 1980.

10. Beker R. C., *On approximations with algebraic numbers of bounded degree.* Mathematica, 23, (1976), pp. 18-31.

11. Olsen N., *On the exact Hausdorff dimension of the set of Liouville Numbers.* Manuscripta Math., 116, (2005), pp. 157-172.

12. Olsen N., Renfro D. L., *On the exact Hausdorff dimension of the set of Liouville Numbers*, II.Manuscripta Math., 119, (2006), pp. 217-224.

Sadullaev, National University of Uzbekistan, sadullaev@mail.ru
Rakhimov, National University of Uzbekistan, karimjon1705@gmail.com