A NOTE ON A PROBLEM OF ABRAMOVICH, ALIPRANTIS AND BURKINSHAW

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Abstract. Let B and T be two positive operators on a Banach lattice such that B is compact-friendly and T is locally quasi-nilpotent. Introducing the concept of positive quasi-similarity, we prove that T has a non-trivial closed invariant subspace provided B is positively quasi-similar to T. This gives an affirmative answer to a problem of Abramovich, Aliprantis and Burkinshaw with the commutativity condition replaced by the positive quasi-similarity of the corresponding operators. The notion of strong compact-friendliness is also introduced and relevant facts about it are discussed.

1. introduction

A result due to Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [2] asserts that a positive operator on a Banach lattice which is compact-friendly and locally quasi-nilpotent has a non-trivial closed invariant subspace. Accordingly, it was stated in [2] as an open problem whether both properties, namely being compact-friendly and locally quasi-nilpotent, can be distributed between two commuting positive operators B and T on a Banach lattice in order to ensure the existence of a B-invariant subspace, or a T-invariant subspace, or a common invariant subspace for B and T. In this note, we prove that the answer to this problem is affirmative if the commutativity condition is replaced by the positive quasi-similarity of the corresponding operators. Moreover, if the notion of positive quasi-similarity is used to modify the concept of compact-friendliness, a class of positive operators lying properly between the classes of compact and compact-friendly operators is obtained.

Throughout the paper the letters X and Y will denote infinite-dimensional Banach spaces while E and F will be fixed infinite-dimensional Banach lattices. As usual, $\mathcal{L}(X, Y)$ stands for the algebra of all bounded linear operators between X and Y, and $\mathcal{L}(X) := \mathcal{L}(X, X)$.

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A subspace V of a Banach space X is called ***non-trivial*** if $\{0\} \neq V \neq X$. If V is a subspace of a Banach lattice and if $v \in V$ and $|u| \leq |v|$ imply $u \in V$, then V is called an ***ideal***.

An operator $Q \in \mathcal{L}(X,Y)$ is a ***quasi-affinity*** if Q is one-to-one and has dense range. An operator $T \in \mathcal{L}(X)$ is said to be a ***quasi-affine transform*** of an operator $S \in \mathcal{L}(Y)$ if there exists a quasi-affinity $Q \in \mathcal{L}(X,Y)$ such that $QT = SQ$. If both T and S are quasi-affine transforms of each other, the operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are called ***quasi-similar*** and this is denoted by $T \overset{qs}{\sim} S$. The notion of quasi-similarity, which can be easily verified to be an equivalence relation on the class of all operators, was first introduced by B. Sz.-Nagy and C. Foiaş in connection with their work on the harmonic analysis of operators on Hilbert space [12]. For more about this concept, see [4, 6, 8, 9, 12].

An operator T on E is said to be ***dominated*** by a positive operator B on E, denoted by $T \prec B$, provided $|Tx| \leq B|x|$ for each $x \in E$. A positive operator B on E is said to be ***compact-friendly*** [1] if there is a positive operator that commutes with B and dominates a non-zero operator which is dominated by a compact positive operator. It is worth mentioning that the notion of compact-friendliness is of substance only on infinite-dimensional Banach lattices, since every positive operator on a finite-dimensional Banach lattice is compact. Also, if B is compact, letting the three operators appearing in the above definition equal to B, it is seen that compact operators are compact-friendly whilst the converse is not true as the identity operator on an infinite-dimensional space shows. Lastly, it is straightforward to observe that every power (even every polynomial with non-negative coefficients) of a compact-friendly operator is also compact-friendly. A fairly complete treatment of compact-friendly operators is given in [1, 2]. For all unexplained notation and terminology, we refer to [1, 3, 10].

2. POSITIVE QUASI-SIMILARITY

In analogy with the notion of quasi-similarity, we define the concept of positive quasi-similarity.

Definition 2.1. *Two positive operators $S \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F)$ are **positively quasi-similar**, denoted by $S \overset{pqs}{\sim} T$, if there exist positive quasi-affinities $P \in \mathcal{L}(E,F)$ and $Q \in \mathcal{L}(F,E)$ such that $TP = PS$ and $QT = SQ$.*

The following auxiliary result, which is also of interest in itself, states that compact-friendliness is well-behaved under the positive quasi-similarity relation. Before proceeding, let us point out that since positive operators between Banach lattices are continuous, each operator dominated by a positive operator is automatically continuous. This guarantees the non-triviality of an operator of the form QTP if P and Q are positive quasi-affinities on E, whenever T is a non-trivial operator dominated by a positive operator on E.

Lemma 2.2. *Let B and T be two positive operators on E. If B is compact-friendly and T is positively quasi-similar to B, then T is also compact-friendly.*

Proof. Since $T \overset{pqs}{\sim} B$, there exist quasi-affinities P and Q such that $BP = PT$ and $QB = TQ$. As B is compact-friendly, there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $RB = BR$, $C \prec R$, and $C \prec K$. Therefore, it follows from $RB = BR$ that $BRP = RBP = RPT$, which implies $QBRP = QRPT$, which in turn implies that $T(QRP) = (QRP)T$. On the other hand, the dominations $C \prec R$ and $C \prec K$ yield $QCP \prec QRP$ and $QCP \prec QKP$, respectively. It is, thus, enough to take $R_1 := QRP$, $K_1 := QKP$, and $C_1 := QCP$ as the required three operators for the compact-friendliness of T. □

Remark 2.3. Lemma 2.2 shows that, although quasi-similarity need not preserve compactness of an operator as shown by T.B. Hoover in [8, p. 683], positive quasi-similarity does preserve compact-friendliness.

By means of the preceding lemma, we can now easily prove that the notion of positive quasi-similarity is strong enough to guarantee an affirmative answer to the problem mentioned in the title in the following form.

Theorem 2.4. *Let B and T be two positive operators on E such that B is compact-friendly and T is locally quasi-nilpotent at a non-zero positive element of E. If $B \overset{pqs}{\sim} T$, then T has a non-trivial closed invariant ideal.*

Proof. Being also compact-friendly by Lemma 2.2, the locally quasi-nilpotent operator T has a non-trivial closed invariant ideal by [1, Theorem 10.55]. □

Remark 2.5. It is easy to observe (cf. Lemma 3.2 (i) below) that an operator which is quasi-similar to a compact operator necessarily commutes with a compact operator. As is shown by C. Foiaş and C. Pearcy [6, Theorem 5], there exists a non-zero quasi-nilpotent operator on ℓ_2 that does not commute with any non-zero compact operator, and hence is not quasi-similar to any compact operator. An example in the same spirit for Banach lattices using positive quasi-similarity can also be provided: recall that H.H. Schaefer constructed in [10, Example 2, pp. 262-264] an example of a positive quasi-nilpotent operator T on a Banach lattice such that T has no non-trivial closed invariant ideals. It follows from Lemma 2.2 and [1, Theorem 10.55] that T cannot be positively quasi-similar to a non-zero compact-friendly operator.

3. Strongly compact-friendly operators

Considering the existence of the non-commuting quasi-similar operators constructed by T.B. Hoover mentioned in Remark 2.3 naturally suggests a modification of the notion of compact-friendliness by replacing the commutativity condition with positive quasi-similarity, which we introduce in the following definition. As Theorem 2.4 reveals, such an approach could be useful in connection with the invariant subspace problem for positive operators on Banach lattices.

Definition 3.1. *A positive operator B on a Banach lattice E is called* ***strongly compact-friendly*** *if there exist three non-zero operators R, K, and C on E with R, K positive, K compact such that $B \overset{pqs}{\sim} R$, and C is dominated by both R and K.*

One should be stressed that Lemma 2.2 still holds true if compact-friendliness of operators is replaced by strong compact-friendliness since the positive quasi-similarity relation is transitive.

Some examples and properties of strongly compact-friendly operators are presented next.

Lemma 3.2. *(i) If a positive operator B on E is positively quasi-similar to an operator on E which is dominated by a positive compact operator or which dominates a positive compact operator, then B is strongly compact-friendly, and the commutant $\{B\}'$ of B contains an operator which is dominated by a positive compact operator or which dominates a positive compact operator, respectively. In particular, every positive operator which is positively quasi-similar to a positive compact operator is strongly compact-friendly and commutes with a positive compact operator.*

(ii) A non-zero positive operator B on E is strongly compact-friendly if and only if λB is strongly compact-friendly for some scalar $\lambda > 0$. However, B need not be quasi-similar to λB for $\lambda \neq 1$.

(iii) A positive compact perturbation of a positive operator on E is strongly compact-friendly.

(iv) For every positive operator B on E, there exists a strongly compact-friendly operator T on E which dominates B.

(v) If $B \geq I$ on E and $\{B\}'$ does not contain a non-zero compact operator, then there exists a non-zero strongly compact-friendly, non-compact operator on E which is not positively quasi-similar to B.

(vi) Positive kernel operators on order-complete Banach lattices are strongly compact-friendly.

(vii) Every non-zero positive operator on ℓ_p $(1 \leq p < \infty)$ is strongly compact-friendly.

Proof. (i) Let $B \geq 0$ and $B \overset{pqs}{\sim} T$. Then there exist positive quasi-affinities P and Q such that $BP = PT$ and $QB = TQ$ satisfying $PTQ \in \{B\}'$. If $0 \leq K \prec T$, where K is compact, then take $R := T$ and $C := K$ as the required three operators for the strong compact-friendliness of T, and observe that $0 \leq PKQ \prec PTQ$. If $0 \leq T \prec K$, where K is compact, then taking $C = R := T$ suffices to establish that T is strongly compact-friendly with $0 \leq PTQ \prec PKQ$.

(ii) The first statement is trivial. Moreover, it is known [5, Theorem 4] that the Volterra operator V on $L_2(0, 1)$ is not quasi-similar to λV for every $\lambda \neq 1$.

(iii) If $K \geq 0$ compact and $B \geq 0$, then $K \prec B + K$, whence $B + K$ is strongly compact-friendly by (i).

(iv) If $B \geq 0$, then take a rank-one positive operator K and use (iii) with $T := B+K$.

(v) If K is a non-zero positive compact operator, then the positive operator $I + K$ is not compact and is strongly compact-friendly by (iii). Observe that $B \overset{pqs}{\sim} I+K$ implies $B-I \overset{pqs}{\sim} K$, from which it follows from the second part of (i) that the positive operator $B - I$ commutes with a compact operator, which in turn implies that B commutes with a compact operator, contradicting that $\{B\}'$ does not contain a compact operator. Thus, B is not positively quasi-similar to $I + K$.

(vi) The remark preceding Lemma 10.58 in [1] coupled with (i) justifies the assertion.

(vii) Let T be a non-zero positive operator on ℓ_p $(1 \leq p < \infty)$. Since $T \neq 0$, the matrix of T must have at least one non-zero entry; replacing all the other entries by zeros one gets a rank-one (and hence, compact) operator dominated by T, whence T is strongly compact-friendly by (i). □

We are now ready to determine the position of strongly compact-friendly operators with respect to positive compact and compact-friendly operators on a Banach lattice. Let us denote the sets of positive compact operators, strongly compact-friendly operators and compact-friendly operators on a Banach lattice E by $\mathcal{K}(E)_+$, $\mathcal{SKF}(E)$ and $\mathcal{KF}(E)$, respectively.

Theorem 3.3. *For an infinite-dimensional Banach lattice E, one has*

$$\mathcal{K}(E)_+ \subset \mathcal{SKF}(E) \subset \mathcal{KF}(E)$$

and the inclusions are proper.

Proof. Since the positive quasi-similarity relation is reflexive, Lemma 3.2 (i) implies that a compact operator on E is strongly compact-friendly; moreover, the positive operator $I + K$, where K is a positive compact operator on E, is not compact but is strongly compact-friendly by Lemma 3.2 (iii). Hence, the first proper inclusion is proved. For the second inclusion, let B be strongly compact-friendly. Then, there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $B \overset{pqs}{\sim} R$, $C \prec R$, and $C \prec K$. Since $B \overset{pqs}{\sim} R$, there exist quasi-affinities P and Q such that $BP = PR$ and $QB = RQ$. Now taking $R_1 := PRQ$, $K_1 := PKQ$ and $C_1 := PCQ$ and observing that $BR_1 = B(PRQ) = BPQB = (PRQ)B = R_1B$ and that $C_1 \prec R_1$ and $C_1 \prec K_1$, one gets that the operators R_1, K_1 and C_1 fulfill the requirements needed for the compact-friendliness of B.

To finish the proof of the theorem, it remains only to show that the last inclusion is proper. To see this, let Ω be a compact Hausdorff space without isolated points and $E := C(\Omega)$. It is known that the identity operator I on E is compact-friendly. Assume now that I is strongly compact-friendly with the corresponding three operators R, K, and C, respectively. Now, a straightforward observation yields that the positive quasi-similarity of I to R implies $R = I$, which brings the fact that the operator C is central. Since E is an AM-space with unit, [1, Theorem 3.33] implies that C is a non-zero multiplication operator. On the other hand, the domination of C by the compact operator K reveals by [3, Theorem 5.13] that the non-zero multiplication operator C^3 is compact, which we already know to be impossible by [1, Lemma 4.18] since Ω has no isolated points. Thus, I is not strongly compact-friendly. □

The proof of the last part of Theorem 3.3, combined with Lemma 3.2 (i), shows that a scalar operator on a $C(\Omega)$-space, where Ω is a compact Hausdorff space without isolated points, is an example of an operator which commutes with a (in fact, every) compact operator yet not positively quasi-similar to any non-zero compact operator, not being strongly compact-friendly. This proves that the implication in the second part of Lemma 3.2 (i) is generally not reversible.

Example 3.4. *A strongly compact-friendly operator which is not polynomially compact.* We will use an example of an operator due to C. Foiaş and C. Pearcy (cf. [6]). Let $T : \ell_2 \to \ell_2$ be the backward weighted shift defined by $Te_0 = 0$ and $Te_{n+1} = \tau_n e_n, n \geq 0$, where $(e_n)_{n=0}^{\infty}$ is the canonical basis of ℓ_2 and $(\tau_n)_{n=0}^{\infty}$ is the sequence

$$\left(\frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{16}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{64}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{256}}, \dots\right).$$

Clearly, T is a positive non-compact operator; on the other hand, by Lemma 3.2 (vii), T is strongly compact-friendly. An easy computation shows that $\|T^n\|^{1/n} \to 0$, that is, T is quasi-nilpotent and hence is essentially quasi-nilpotent; moreover, by its definition, it is readily seen that no power of T is compact. This implies by [7, Lemma 2] that T is not polynomially compact. Note, for the sake of completeness, that the assertion of the example can be improved: in the proof of Theorem 5 in [6] it was observed that not only T is not polynomially compact, but T does not commute with a non-zero compact operator.

We do not know whether every non-zero strongly compact-friendly operator on a Banach lattice has a non-trivial closed invariant subspace; but on a subclass of $\mathcal{SKF}(E)$, we will show now that the assertion of [1, Theorem 10.55] still holds true without any assumption of local quasi-nilpotence. Let us note that the phrase "Lomonosov theorem," below will be employed to operators on an arbitrary Banach lattice, being valid due to [11, Corollary 2.4] for non-scalar bounded operators on real Banach lattices as well as complex ones [1, Theorem 10.19].

Theorem 3.5. *Let B be a positive operator on E. If B is positively quasi-similar to a positive operator R on E which is dominated by a positive compact operator K on E, then B has a non-trivial closed invariant subspace. Moreover, for each sequence $(T_n)_{n\in\mathbb{N}}$ in $\{B\}'$, there exists a non-trivial closed subspace that is invariant under B and under each T_n.*

Proof. Our argument is similar to the one in [1] and uses the properties of quasi-affinities. Without loss of generality, we may assume that the operator B is not scalar, in which case the assertion holds trivially. Clearly, B is strongly compact-friendly by

Lemma 3.2 (i) with the corresponding quasi-affinities and the three operators, say, P, Q, R, K and C, respectively. Also, it follows from the proof of the second inclusion in Theorem 3.3 that using the operators $R_1 := PRQ \geq 0$, $K_1 := PKQ \geq 0$ and $C_1 := PCQ$, the operator B becomes compact-friendly since we have $BR_1 = R_1 B$, $C_1 \prec R_1$, and $C_1 \prec K_1$.

Now, if Ker R_1 is non-trivial, then Ker R_1 is a non-trivial closed R_1-hyperinvariant subspace of E whence it is a non-trivial closed B-invariant subspace since B commutes with R_1, and we are done.

Assume now that Ker R_1 is trivial, from which it follows that the operator R_1 is non-trivial, yielding that the operator R_1^3 is non-trivial, too. As $0 \leq R_1 \prec K_1$, [3, Theorem 5.13] implies that R_1^3 is compact. That B has a non-trivial closed hyperinvariant subspace now follows, by virtue of Lomonosov's theorem, from the fact that B commutes with the compact operator R_1^3, since B commutes with R_1.

The only assertion left, then, is to show that if Ker R_1 is non-trivial, then there exists not only an R_1-hyperinvariant subspace but a non-trivial closed subspace of E that is B-invariant and T_n-invariant for each n. Without loss of generality, we can suppose that $\|B\| < 1$. Pick arbitrary scalars $\alpha_n > 0$ that are small enough to guarantee that the positive operator $T := \sum_{n=1}^{\infty} \alpha_n T_n$ exists and $\|B + T\| < 1$. Clearly, $BT = TB$. Define the operator $A := \sum_{n=1}^{\infty} (B+T)^n$, and notice that $A \geq 0$, $AB = BA$, $AT = TA$, and $Ax \geq x$ for all $x \geq 0$.

For each $x > 0$, denote by J_x the principal ideal generated by Ax; that is

$$J_x := \{y \in E \mid |y| \leq \lambda Ax \text{ for some } \lambda > 0\}.$$

Since $x \leq Ax$, we have that $x \in J_x$, so this is a non-zero ideal.

Note that J_x is $(B+T)$-invariant: because, if $y \in J_x$, then $|y| \leq \lambda Ax$ for some $\lambda > 0$ and hence

$$|(B+T)y| \leq (B+T)|y| \leq \lambda (B+T) \sum_{n=0}^{\infty} (B+T)^n x = \lambda \sum_{n=1}^{\infty} (B+T)^n x \leq \lambda Ax.$$

It is also clear that J_x is also invariant under B and T, since B is positive and is dominated by $B+T$. In case where there exists a positive $x \in E$ such that the ideal J_x is not norm-dense in E, the proof is complete. So, suppose that Ax is a quasi-interior point in E for each $x > 0$.

Since $C \neq 0$ we have $C_1 \neq 0$, so there exists some $x_1 > 0$ such that $C_1 x_1 \neq 0$. Then $A|C_1 x_1|$ is a quasi-interior point satisfying $A|C_1 x_1| \geq |C_1 x_1|$. By [1, Lemma 4.16 (i)] there exists an operator $M_1 \in \mathcal{L}(E)$ dominated by the identity operator such that

$x_2 := M_1C_1x_1 > 0$. Let $\pi_1 := M_1C_1$. Note that π_1 is dominated by R_1 and hence by K_1.

Now we have $\overline{J_{x_2}} = E$, and since $C_1 \neq 0$ there exists $0 < y \leq Ax_2$ such that $C_1y \neq 0$. Since Ax_2 is a quasi-interior point, it follows from [1, Lemma 4.16 (ii)] that there is an operator $M \in \mathcal{L}(E)$ dominated by the identity operator such that $MAx_2 = y$. Notice that $A|C_1y|$ is a quasi-interior point. As $|C_1y| \leq A|C_1y|$, it follows from [1, Lemma 4.16 (i)] that there exists $M_2 \in \mathcal{L}(E)$ dominated by the identity operator such that $x_3 := M_2C_1y = M_2C_1MAx_2 > 0$. Put $\pi_2 := M_2C_1MA$ and observe that π_2 is dominated by R_1A and hence by K_1A. Repeating once more the preceding argument with x_2 replaced by x_3, we then obtain $\pi_3 \in \mathcal{L}(E)$ such that $\pi_3x_3 > 0$ and π_3 is dominated by R_1A whence by K_1A. From $\pi_3\pi_2\pi_1x_1 = \pi_3x_3 > 0$, we see that $\pi_3\pi_2\pi_1 \neq 0$.

Set $S := R_1AR_1AR_1 \geq 0$. Since $|\pi_3\pi_2\pi_1x| \leq S|x|$ for each $x \in E$, it follows that $S \neq 0$. Moreover, since K_1 is compact and dominates R_1, that A is positive, and that $S = (R_1A)(R_1A)R_1 \prec (K_1A)(K_1A)K_1$, we have by [3, Theorem 5.14] that S is compact. Lastly, because A and R_1 commute with B, so does S. Thus, B commutes with a non-zero compact operator, and hence has a non-trivial closed hyperinvariant subspace by Lomonosov's theorem. The proof of the theorem is now complete. □

Finally, we want to conclude with a problem related to the main subject matter of this article. It has become clear that the quasi-similarity relation is too intricate to establish a link between strong compact-friendliness and commutativity. Considering Theorem 3.3, the following question sounds then meaningful.

QUESTION: Let B and R be two commuting positive operators on E such that B is strongly compact-friendly and R is locally quasi-nilpotent at some non-zero positive vector in E. Does there exist a non-trivial closed B-invariant subspace, or an R-invariant subspace, or a common invariant subspace for B and R?

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