On the structure theory of Fréchet-Hilbert power series spaces

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Die Welt ist aus den Fugen, die jungen Leute werden 60

G. Köthe

The world is in disorder, the youngsters become 60

Power series spaces and their invariants

All spaces E will be Fréchet-Hilbert spaces with Hilbert semi-norms $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \ldots$, the local Banach spaces E_k will be Hilbert spaces.

Exponent sequence $\alpha : 0 \leq \alpha_1 < \alpha_2 < \ldots \uparrow \infty$.

$$\Lambda_\infty(\alpha) = \{x = (x_0, x_1, \ldots) \,:\, |x|_t^2 = \sum_{j=0}^{\infty} |x_j|^2 e^{2t\alpha_j} < \infty \text{ for all } t\}$$

Invariants:

1. $(\mathrm{DN}) \quad \exists\, p \;\forall\, k, 0 < \tau < 1 \;\exists\, K, C \quad \|\;\|_k \leq C \|\;\|_p^\tau \|\;\|_K^{1-\tau}$
2. $(\Omega) \quad \forall\, n \;\exists\, m \;\forall\, N \;\exists\, 0 < \vartheta < 1, C \quad \|\;\|_m \leq C \|\;\|_n^{*\,\vartheta} \|\;\|_N^{*\,1-\vartheta}$

Theorem 1 (V.-Wagner). *Let E be nuclear. Then E is isomorphic to a complemented subspace of s if and only if E satisfies* (DN) *and* (Ω).

Problem of Mityagin. Does every complemented subspace of s have a basis?

EQUIVALENT: Is every nuclear space with (DN) and (Ω) isomorphic to a power series space?

The Aytuna-Krone-Terzioğlu Theorem

Let E be a Fréchet-Hilbert-Schwartz space with (DN) and (Ω). We may assume that "p" in (DN) is 0 and the "m" for p in (Ω) is 1, moreover the canonical map $j_1^0 : E_1 \longrightarrow E_0$ is compact. Let s_n be the singular numbers for j_1^0. We set

$$\alpha_n := -\log s_n.$$

The sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}_0}$ is called *associated exponent sequence* for E.
Lemma 2 (Terzioğlu). *If E is nuclear and has a basis then $E \cong \Lambda_\infty(\alpha)$.*

An exponent sequence is called stable if $\sup_n \frac{\alpha_{2n}}{\alpha_n} < +\infty$.
Theorem 3 (Aytuna-Krone-Terzioğlu). *If E is nuclear, satisfies* (DN) *and* (Ω) *and its associated exponent sequence α is stable then $E \cong \Lambda_\infty(\alpha)$.*

Sketch of proof:
Reduction part: E is α-nuclear, satisfies (DN) and (Ω). By structure theory: E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$. Use Lemma:
Lemma 4 (V.). *Let α be stable. If E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of E then $E \cong \Lambda_\infty(\alpha)$.*

Main part: Show that $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of E.

1. Step: Show that there is a local imbedding $\Lambda_\infty(\alpha) \hookrightarrow E$, that is $T \in L(\Lambda_\infty(\alpha), E)$ such that $|x|_0 \le C\|Tx\|_q$ for suitable C and q.

2. Step: Prove the following

Lemma 5. *If α is stable, $\Lambda_\infty(\alpha)$ nuclear and $T : \Lambda_\infty(\alpha) \hookrightarrow \Lambda_\infty(\alpha)$ is a local imbedding. Then $\overline{R(\varphi)}$ contains a complemented subspace isomorphic to $\Lambda_\infty(\alpha)$.*

The essential step in the **Proof** is the following (assuming for the sake of simplicity $|Tx|_0 = |x|_0$):

Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \Lambda_\infty(\alpha)$ and $f_j = Te_j$. We choose inductively vectors $g_n \in \Lambda_\infty(\alpha)$ with following properties:

$g_n \in \operatorname{span}\{f_0, \ldots, f_{2n}\}$
$g_n \perp g_0, \ldots, g_{n-1}$ in ℓ_2
$g_n \perp e_0, \ldots, e_{n-1}$ in ℓ_2
$|g_n|_0 = 1.$

Notice that $\dim \operatorname{span}\{f_0, \ldots, f_{2n}\} = 2n + 1$. The map $Px = \sum_{n=0}^\infty \langle x, g_n\rangle_0 g_n$ is what we are looking for.

Adjustment of norms

Theorem 6. *If* $\| \ \|$ *is a Hilbert seminorm on* $\Lambda_\infty(\alpha)$ *and* $\| \ \| \leq C| \ |_\tau$, $\tau \geq 0$, $C \geq 1$. *Then there is* $U \in L(\Lambda_\infty(\alpha))$ *so that*

$$|Ux|_0 = \|x\| \text{ and } |Ux|_t \leq C|x|_{t+\tau}$$

for all $x \in \Lambda_\infty(\alpha)$, $t \geq 0$.

If, moreover, we have $| \ |_0 \leq \| \ \|$ *then* U *can be chosen as an automorphism of* $\Lambda_\infty(\alpha)$ *with* $|Ux|_t \geq |x|_t$ *for all* x *and* t.

Consequences:
1. If $T : \Lambda_\infty(\alpha) \hookrightarrow \Lambda_\infty(\alpha)$ is a local imbedding we may assume that $|Tx|_0 = |x|_0$ for every x in $\Lambda_\infty(\alpha)$.
2. If P is a continuous projection in $\Lambda_\infty(\alpha)$ we may assume that P is an orthogonal projection in ℓ_2.

Proof. for 2.: Apply Theorem to $\| \ \|^2 = |Px|_0^2 + |Qx|_0^2$ where $Q = I - P$. Notice that $(\Lambda_\infty(\alpha))_0 = \ell_2$.

From now on projections in $\Lambda_\infty(\alpha)$ will be assumed to be orthogonal in ℓ_2.

Triangular matrices

Definition: An endomorphism U of $\Lambda_\infty(\alpha)$ for which there are $C > 0$ and $\tau \geq 0$ such that

$$|Ux|_t \leq C\,|x|_{t+\tau}, \quad x \in \Lambda_\infty(\alpha)$$

for all $t > 0$ is called uniformly tame.

For any endomorphism $U \in L(\Lambda_\infty(\alpha))$ there is a matrix $(u_{k,j})_{k,j\in\mathbb{N}_0}$ so that

$$Ux = \left(\sum_{j=0}^{\infty} u_{k,j} x_j\right)_{k\in\mathbb{N}_0}, \quad x \in \Lambda_\infty(\alpha).$$

We have $u_{k,j} = \langle Ue_j, e_k\rangle$ if e_k and e_j denote the canonical basis vectors and $\langle\ ,\ \rangle$ the ℓ_2-scalar product.

Lemma 7. *Let $U \in L(\Lambda_\infty(\alpha))$ be uniformly tame, then its matrix is upper triangular.*

Proof From the continuity estimates we get $|u_{k,j}| \leq C\, e^{t(\alpha_j-\alpha_k)+\tau\alpha_j}$ for all $t > 0$.

Return to Theorem 6: Assume $|\ |_0 \leq \|\ \| \leq |\ |_\tau$
$\Rightarrow$ exists uniformly tame automorphism U with $|Ux|_0 = \|x\|$
$\Rightarrow$ exists automorphism U with upper triangular matrix and $|Ux|_0 = \|x\|$

Consequence: The columns in U must be the Gram-Schmidt orthogonalization of the unit vectors e_j with respect to $\|\ \|$.

Proposition 8. *Assume $|\ |_0 \leq \|\ \| \leq |\ |_\tau$ for the Hilbert norm $\|\ \|$ on $\Lambda_\infty(\alpha)$. Then the Gram-Schmidt orthogonalization of the unit vectors e_j with respect to $\|\ \|$ gives a tamely equivalent basis of $\Lambda_\infty(\alpha)$.*

Lemma 9. *Let $S \in L(\ell_2)$ have an upper triangular matrix, then $S \in L(\Lambda_\infty(\alpha))$ and S is uniformly tame with $\tau = 0$.*

If α is not necessarily *strictly* increasing, we can apply Lemma 9 to a slightly changed α and obtain that $S \in L(\Lambda_\infty(\alpha))$.

Lemma 10. *Let α be stable $S \in L(\ell_2)$ and $S_{k,j} = 0$ for $k > 2j$, then $S \in L(\Lambda_\infty(\alpha))$.*

Proof. Apply Lemma 9 to $\tilde{S} = S \circ A$ where $Ax = (x_{2n})_{n \in \mathbb{N}_0}$ and observe that $S = \tilde{S} \circ B$ where $(Bx)_j = x_\nu$ for $j = 2\nu$, $(Bx)_j = 0$ otherwise. □

Lemma 11. *Let α be stable, $T \in L(\Lambda_\infty(\alpha))$ so that T induces a unitary map in $L(\ell_2)$. Then there is $S \in L(\Lambda_\infty(\alpha))$, so that $P = T \circ S$ is a projection in $\Lambda_\infty(\alpha)$, orthogonal in ℓ_2, and $R(P) \cong \Lambda_\infty(\alpha)$.*

Proof. Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \Lambda_\infty(\alpha)$ and $f_j = Te_j$. We choose inductively vectors $g_n \in \Lambda_\infty(\alpha)$ with following properties:

(1) $g_n \in \operatorname{span}\{f_0, \ldots, f_{2n}\}$
(2) $g_n \perp g_0, \ldots, g_{n-1}$ in ℓ_2
(3) $g_n \perp e_0, \ldots, e_{n-1}$ in ℓ_2
(4) $|g_n|_0 = 1$.

This is possible since $\dim \operatorname{span}\{f_0, \ldots, f_{2n}\} = 2n + 1$. Due to (1) we have

$$g_n := \sum_{k=0}^{2n} \mu_{k,n} f_k = T(\sum_{k=0}^{2n} \mu_{k,n} e_k).$$

We set $h_n = \sum_{k=0}^{2n} \mu_{k,n} e_k$ and obtain an orthonormal system $(h_n)_{n \in \mathbb{N}_0}$. We set $\mu_{k,n} = 0$ for $k > 2n$.

We define

$$Sx := \sum_{n=0}^{\infty} \langle x, g_n \rangle h_n.$$

This means $S = T^{-1} \circ P$ where P is the orthogonal projection onto $\overline{\mathrm{span}\{g_0, g_1, \ldots\}}$. We have to show that S defines a map in $L(\Lambda_\infty(\alpha))$.

We do that in two steps. First we define a map $\varphi \in L(\ell_2)$ by

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x, g_n \rangle e_n.$$

For the matrix elements $\varphi_{k,j} = \langle \varphi e_j, e_k \rangle = \langle e_j, g_k \rangle$ we have $\varphi_{k,j} = 0$ for $k > j$. Therefore, by Lemma 9, $\varphi \in L(\Lambda_\infty(\alpha))$.

Next we define a map $\psi \in L(\ell_2)$ by

$$\psi(x) = \sum_{n=0}^{\infty} \langle x, e_n \rangle h_n.$$

For the matrix elements $\psi_{k,j} = \langle \psi e_j, e_k \rangle = \langle h_j, e_k \rangle$ we obtain that $\psi_{k,j} = 0$ for $k > 2j$. Therefore, by Lemma 10, $\varphi \in L(\Lambda_\infty(\alpha))$.

Since obviously $S = \psi \circ \varphi$ we have shown that $S \in L(\Lambda_\infty(\alpha))$. It remains to show that $R(P) \cong \Lambda_\infty(\alpha)$.

The map $T \circ \psi \in L(\Lambda_\infty(\alpha), R(P))$ is injective and, because of $(T \circ \psi) \circ \varphi = T \circ S = P$, also surjective. Therefore it is an isomorphism. $\square$

This yields a non-nuclear version of the Aytuna-Krone-Terzioğlu Theorem:

Theorem 12. *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated exponent sequence α is stable, then $E \cong \Lambda_\infty(\alpha)$.*

The existence of the local imbedding is also nontrivial. It is based on the following Lemma:

Lemma 13. *Let E be a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω). *Let α be the associated exponent sequence. Then there exist maps $\psi \in L(\Lambda_\infty(\alpha), E)$, $\varphi \in L(E, \Lambda_\infty(\alpha))$ so that ψ extends to an isomorphism $\psi_0 \colon \ell_2 \longrightarrow E_0$, φ extends to an isomorphism $\varphi_0 \colon E_0 \longrightarrow \ell_2$ and we have $\sup_{|\xi|_0 \leq 1} |\xi - \varphi_0 \circ \psi_0(\xi)|_0 < \frac{1}{2}$.*

This Lemma does not use stability. The local imbedding exists for any associated exponent sequence.

Assume $E \subset \Lambda_\infty(\beta)$ and let α be its associated exponent sequence. Then $\beta_n \leq C\alpha_n$ for some C and all n.

We use the same line of arguments as before with g_n chosen to satisfy:

(1) $g_n \in \mathrm{span}\{f_0, \dots, f_{n+m(n)}\}$
(2) $g_n \perp g_0, \dots, g_{n-1}$ in ℓ_2
(3) $g_n \perp e_0, \dots, e_{m(n)-1}$ in ℓ_2
(4) $|g_n|_0 = 1$.

We obtain:

Theorem 14. *Let* $(m(n))_{n\in\mathbb{N}_0}$ *be a nondecreasing unbounded sequence of integers, and*

$$\limsup_{n\to\infty} \frac{\alpha_{n+m(n)}}{\beta_{m(n)}} < \infty. \tag{1}$$

Then E *contains a complemented subspace isomorphic to* $\Lambda_\infty(\gamma) = \Lambda_\infty(\delta)$ *where* $\gamma_n = \alpha_{m(n)}$ *and* $\delta_n = \beta_{m(n)}$.

Remark: $\alpha_{m(n)} \leq \alpha_{n+m(n)} \leq C_1\beta_{m(n)} \leq C_2\alpha_{m(n)}$ for large n.

To consider a concrete non stable case we assume that $\alpha_n = \beta_n = e^{f(n)}$ where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuously differentiable, increasing and strictly concave for large t. We assume moreover that $\lim_{t\to\infty} f'(t) = 0$ and we put $h(t) = 1/f'(t)$.

Lemma 15. *In this case $m(n)$ may be chosen as $h^{-1}(cn)$ where $c > 0$. This means E contains a complemented subspace isomorphic to $\Lambda_\infty(\gamma)$ with $\gamma_n = e^{f(h^{-1}(cn))}$.*

Proof. We have to choose $m(n)$ so that $f(n + m(n)) - f(m(n)) \leq C$ for large n. With the choices we have made this follows from the mean value theorem. □

Examples:

1. If $\alpha_n = e^{n^{\frac{1}{s}}}$ with $s > 1$ then we may choose $\gamma_n = e^{n^{\frac{1}{s-1}}}$.

2. If $\alpha_n = e^{(\log(n+1))^s}$ with $s > 1$ then we may choose

$$\gamma_n = e^{(\log(n+1)+(s-1)\log\log(n+1))^s}.$$

Rotwein ist für
alte Knaben
eine von den
besten Gaben

W. Busch

Red wine is for old boys one of the best gifts