On the structure theory of Fréchet-Hilbert power series spaces

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Die Welt ist aus den Fugen, die jungen Leute werden 60
G. Köthe

The world is in disorder, the youngsters become 60
Power series spaces and their invariants

All spaces $E$ will be Fréchet-Hilbert spaces with Hilbert semi-norms $\| \cdot \|_0 \leq \| \cdot \|_1 \leq \ldots$, the local Banach spaces $E_k$ will be Hilbert spaces.

Exponent sequence $\alpha : 0 \leq \alpha_1 < \alpha_2 < \ldots \uparrow \infty$.

$$\Lambda_\infty(\alpha) = \{ x = (x_0, x_1, \ldots) : |x|_t^2 = \sum_{j=0}^{\infty} |x_j|^2 e^{2t\alpha_j} < \infty \text{ for all } t \}$$

**Invariants:**

1. (DN) $\exists \ p \ \forall \ k, 0 < \tau < 1 \ \exists \ K, C \ \| k \| \leq C \| \|_{K}^{1-\tau}$
2. (Ω) $\forall \ n \ \exists \ m \ \forall \ N \ \exists \ 0 < \vartheta < 1, C \ \| m \| \leq C \| *_{n}^{\vartheta} \| *_{N}^{1-\vartheta}$

**Theorem 1 (V.-Wagner).** Let $E$ be nuclear. Then $E$ is isomorphic to a complemented subspace of $s$ if and only if $E$ satisfies (DN) and (Ω).

**Problem of Mityagin.** Does every complemented subspace of $s$ have a basis?

**Equivalent:** Is every nuclear space with (DN) and (Ω) isomorphic to a power series space?
The Aytuna-Krone-Terzioğlu Theorem

Let $E$ be a Fréchet-Hilbert-Schwartz space with (DN) and ($\Omega$). We may assume that “p” in (DN) is 0 and the “m” for p in ($\Omega$) is 1, moreover the canonical map $j_1^0 : E_1 \rightarrow E_0$ is compact. Let $s_n$ be the singular numbers for $j_1^0$. We set

$$\alpha_n := -\log s_n.$$ 

The sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}_0}$ is called associated exponent sequence for $E$.

**Lemma 2 (Terzioğlu).** If $E$ is nuclear and has a basis then $E \cong \Lambda_\infty(\alpha)$.

An exponent sequence is called stable if $\sup_n \frac{\alpha_{2n}}{\alpha_n} < +\infty$.

**Theorem 3 (Aytuna-Krone-Terzioğlu).** If $E$ is nuclear, satisfies (DN) and ($\Omega$) and its associated exponent sequence $\alpha$ is stable then $E \cong \Lambda_\infty(\alpha)$.

**Sketch of proof:**
Reduction part: $E$ is $\alpha$-nuclear, satisfies (DN) and ($\Omega$). By structure theory: $E$ is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$. Use Lemma:

**Lemma 4 (V.).** Let $\alpha$ be stable. If $E$ is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of $E$ then $E \cong \Lambda_\infty(\alpha)$. 

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Main part: Show that $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of $E$.

1. Step: Show that there is a local imbedding $\Lambda_\infty(\alpha) \hookrightarrow E$, that is $T \in L(\Lambda_\infty(\alpha), E)$ such that $|x|_0 \leq C\|Tx\|_q$ for suitable $C$ and $q$.

2. Step: Prove the following

**Lemma 5.** If $\alpha$ is stable, $\Lambda_\infty(\alpha)$ nuclear and $T : \Lambda_\infty(\alpha) \hookrightarrow \Lambda_\infty(\alpha)$ is a local imbedding. Then $R(\varphi)$ contains a complemented subspace isomorphic to $\Lambda_\infty(\alpha)$.

The essential step in the **Proof** is the following (assuming for the sake of simplicity $|Tx|_0 = |x|_0$):

Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \Lambda_\infty(\alpha)$ and $f_j = Te_j$. We choose inductively vectors $g_n \in \Lambda_\infty(\alpha)$ with following properties:

\begin{align*}
  g_n &\in \text{span}\{f_0, \ldots, f_{2n}\} \\
  g_n &\perp g_0, \ldots, g_{n-1} \text{ in } \ell_2 \\
  g_n &\perp e_0, \ldots, e_{n-1} \text{ in } \ell_2 \\
  |g_n|_0 &= 1.
\end{align*}

Notice that \(\text{dim span}\{f_0, \ldots, f_{2n}\} = 2n + 1\). The map $Px = \sum_{n=0}^{\infty} \langle x, g_n \rangle_0 g_n$ is what we are looking for.
Adjustment of norms

**Theorem 6.** If $\| \|$ is a Hilbert seminorm on $\Lambda_{\infty}(\alpha)$ and $\| \| \leq C|\tau|$, $\tau \geq 0$, $C \geq 1$. Then there is $U \in L(\Lambda_{\infty}(\alpha))$ so that

$$|Ux|_0 = \|x\| \text{ and } |Ux|_t \leq C|x|_{t+\tau}$$

for all $x \in \Lambda_{\infty}(\alpha)$, $t \geq 0$.

If, moreover, we have $|\|_0 \leq \| \|$ then $U$ can be chosen as an automorphism of $\Lambda_{\infty}(\alpha)$ with $|Ux|_t \geq |x|_t$ for all $x$ and $t$.

**Consequences:**

1. If $T : \Lambda_{\infty}(\alpha) \hookrightarrow \Lambda_{\infty}(\alpha)$ is a local imbedding we may assume that $|Tx|_0 = |x|_0$ for every $x$ in $\Lambda_{\infty}(\alpha)$.

2. If $P$ is a continuous projection in $\Lambda_{\infty}(\alpha)$ we may assume that $P$ is an orthogonal projection in $\ell_2$.

**Proof.** for 2.: Apply Theorem to $\| \|^2 = |Px|_0^2 + |Qx|_0^2$ where $Q = I - P$. Notice that $(\Lambda_{\infty}(\alpha))_0 = \ell_2$.

From now on projections in $\Lambda_{\infty}(\alpha)$ will be assumed to be orthogonal in $\ell_2$. 
Triangular matrices

Definition: An endomorphism $U$ of $\Lambda_\infty(\alpha)$ for which there are $C > 0$ and $\tau \geq 0$ such that

$$|Ux|_t \leq C|x|_{t+\tau}, \quad x \in \Lambda_\infty(\alpha)$$

for all $t > 0$ is called uniformly tame.

For any endomorphism $U \in L(\Lambda_\infty(\alpha))$ there is a matrix $(u_{k,j})_{k,j \in \mathbb{N}_0}$ so that

$$Ux = \left( \sum_{j=0}^{\infty} u_{k,j} x_j \right)_{k \in \mathbb{N}_0}, \quad x \in \Lambda_\infty(\alpha).$$

We have $u_{k,j} = \langle U e_j, e_k \rangle$ if $e_k$ and $e_j$ denote the canonical basis vectors and $\langle \cdot, \cdot \rangle$ the $\ell_2$-scalar product.

Lemma 7. Let $U \in L(\Lambda_\infty(\alpha))$ be uniformly tame, then its matrix is upper triangular.

Proof From the continuity estimates we get $|u_{k,j}| \leq C e^{t(\alpha_j-\alpha_k)+\tau\alpha_j}$ for all $t > 0$. 
Return to Theorem 6: Assume $|0 \leq \| \| \leq |_{\tau}$
\[ \Rightarrow \text{exists uniformly tame automorphism } U \text{ with } |Ux|_0 = \|x\| \]
\[ \Rightarrow \text{exists automorphism } U \text{ with upper triangular matrix and } |Ux|_0 = \|x\| \]

**Consequence:** The columns in $U$ must be the Gram-Schmidt orthogonalization of the unit vectors $e_j$ with respect to $\| \|$. 

**Proposition 8.** Assume $|0 \leq \| \| \leq |_{\tau}$ for the Hilbert norm $\| \|$ on $\Lambda_\infty(\alpha)$. Then the Gram-Schmidt orthogonalization of the unit vectors $e_j$ with respect to $\| \|$ gives a tamely equivalent basis of $\Lambda_\infty(\alpha)$.

**Lemma 9.** Let $S \in L(\ell_2)$ have an upper triangular matrix, then $S \in L(\Lambda_\infty(\alpha))$ and $S$ is uniformly tame with $\tau = 0$.

If $\alpha$ is not necessarily strictly increasing, we can apply Lemma 9 to a slightly changed $\alpha$ and obtain that $S \in L(\Lambda_\infty(\alpha))$.

**Lemma 10.** Let $\alpha$ be stable $S \in L(\ell_2)$ and $S_{k,j} = 0$ for $k > 2j$, then $S \in L(\Lambda_\infty(\alpha))$.

**Proof.** Apply Lemma 9 to $\tilde{S} = S \circ A$ where $Ax = (x_{2n})_{n \in \mathbb{N}_0}$ and observe that $S = \tilde{S} \circ B$ where $(Bx)_j = x_{\nu}$ for $j = 2\nu$, $(Bx)_j = 0$ otherwise. \[ \square \]
Lemma 11. Let $\alpha$ be stable, $T \in L(\Lambda_\infty(\alpha))$ so that $T$ induces a unitary map in $L(\ell_2)$. Then there is $S \in L(\Lambda_\infty(\alpha))$, so that $P = T \circ S$ is a projection in $\Lambda_\infty(\alpha)$, orthogonal in $\ell_2$, and $R(P) \cong \Lambda_\infty(\alpha)$.

Proof. Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \Lambda_\infty(\alpha)$ and $f_j = Te_j$. We choose inductively vectors $g_n \in \Lambda_\infty(\alpha)$ with following properties:

(1) $g_n \in \text{span}\{f_0, \ldots, f_{2n}\}$
(2) $g_n \perp g_0, \ldots, g_{n-1}$ in $\ell_2$
(3) $g_n \perp e_0, \ldots, e_{n-1}$ in $\ell_2$
(4) $|g_n|_0 = 1$.

This is possible since $\dim \text{span}\{f_0, \ldots, f_{2n}\} = 2n + 1$. Due to (1) we have

$$g_n := \sum_{k=0}^{2n} \mu_{k,n} f_k = T \left( \sum_{k=0}^{2n} \mu_{k,n} e_k \right).$$

We set $h_n = \sum_{k=0}^{2n} \mu_{k,n} e_k$ and obtain an orthonormal system $(h_n)_{n \in \mathbb{N}_0}$. We set $\mu_{k,n} = 0$ for $k > 2n$.  

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We define
\[ Sx := \sum_{n=0}^{\infty} \langle x, g_n \rangle h_n. \]
This means \( S = T^{-1} \circ P \) where \( P \) is the orthogonal projection onto \( \text{span}\{g_0, g_1, \ldots\} \).
We have to show that \( S \) defines a map in \( L(\Lambda_\infty(\alpha)) \).

We do that in two steps. First we define a map \( \varphi \in L(\ell_2) \) by
\[ \varphi(x) = \sum_{n=0}^{\infty} \langle x, g_n \rangle e_n. \]
For the matrix elements \( \varphi_{k,j} = \langle \varphi e_j, e_k \rangle = \langle e_j, g_k \rangle \) we have \( \varphi_{k,j} = 0 \) for \( k > j \). Therefore, by Lemma 9, \( \varphi \in L(\Lambda_\infty(\alpha)) \).

Next we define a map \( \psi \in L(\ell_2) \) by
\[ \psi(x) = \sum_{n=0}^{\infty} \langle x, e_n \rangle h_n. \]
For the matrix elements $\psi_{k,j} = \langle \psi e_j, e_k \rangle = \langle h_j, e_k \rangle$ we obtain that $\psi_{k,j} = 0$ for $k > 2j$. Therefore, by Lemma 10, $\varphi \in L(\Lambda_\infty(\alpha))$.

Since obviously $S = \psi \circ \varphi$ we have shown that $S \in L(\Lambda_\infty(\alpha))$. It remains to show that $R(P) \cong \Lambda_\infty(\alpha)$.

The map $T \circ \psi \in L(\Lambda_\infty(\alpha), R(P))$ is injective and, because of $(T \circ \psi) \circ \varphi = T \circ S = P$, also surjective. Therefore it is an isomorphism.  

This yields a non-nuclear version of the Aytuna-Krone-Terzioğlu Theorem:

**Theorem 12.** If $E$ is a Fréchet-Hilbert-Schwartz space with properties (DN) and (Ω) and its associated exponent sequence $\alpha$ is stable, then $E \cong \Lambda_\infty(\alpha)$.

The existence of the local imbedding is also nontrivial. It is based on the following Lemma:

**Lemma 13.** Let $E$ be a Fréchet-Hilbert-Schwartz space with properties (DN) and (Ω). Let $\alpha$ be the associated exponent sequence. Then there exist maps $\psi \in L(\Lambda_\infty(\alpha), E)$, $\varphi \in L(E, \Lambda_\infty(\alpha))$ so that $\psi$ extends to an isomorphism $\psi_0 : \ell_2 \to E_0$, $\varphi$ extends to an isomorphism $\varphi_0 : E_0 \to \ell_2$ and we have $\sup_{|\xi|_0 \leq 1} |\xi - \varphi_0 \circ \psi_0(\xi)|_0 < \frac{1}{2}$.
This Lemma does not use stability. The local imbedding exists for any associated exponent sequence.

Assume $E \subset \Lambda_\infty(\beta)$ and let $\alpha$ be its associated exponent sequence. Then $\beta_n \leq C\alpha_n$ for some $C$ and all $n$.

We use the same line of arguments as before with $g_n$ chosen to satisfy:

1. $g_n \in \text{span}\{f_0, \ldots, f_{n+m(n)}\}$
2. $g_n \perp g_0, \ldots, g_{n-1}$ in $\ell_2$
3. $g_n \perp e_0, \ldots, e_{m(n)-1}$ in $\ell_2$
4. $|g_n|_0 = 1$.

We obtain:

**Theorem 14.** Let $(m(n))_{n \in \mathbb{N}_0}$ be a nondecreasing unbounded sequence of integers, and

$$\limsup_{n \to \infty} \frac{\alpha_{n+m(n)}}{\beta_{m(n)}} < \infty.$$ \hfill (1)

Then $E$ contains a complemented subspace isomorphic to $\Lambda_\infty(\gamma) = \Lambda_\infty(\delta)$ where $\gamma_n = \alpha_{m(n)}$ and $\delta_n = \beta_{m(n)}$.

**Remark:** $\alpha_{m(n)} \leq \alpha_{n+m(n)} \leq C_1\beta_{m(n)} \leq C_2\alpha_{m(n)}$ for large $n$. 
To consider a concrete non stable case we assume that $\alpha_n = \beta_n = e^{f(n)}$ where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuously differentiable, increasing and strictly concave for large $t$. We assume moreover that $\lim_{t \rightarrow \infty} f'(t) = 0$ and we put $h(t) = 1/f'(t)$.

**Lemma 15.** In this case $m(n)$ may be chosen as $h^{-1}(cn)$ where $c > 0$. This means $E$ contains a complemented subspace isomorphic to $\Lambda_\infty(\gamma)$ with $\gamma_n = e^{f(h^{-1}(cn))}$.

**Proof.** We have to choose $m(n)$ so that $f(n + m(n)) - f(m(n)) \leq C$ for large $n$. With the choices we have made this follows from the mean value theorem. \qed

**Examples:**

1. If $\alpha_n = e^{n^{\frac{s}{2}}}$ with $s > 1$ then we may choose $\gamma_n = e^{n^{\frac{1}{2s-1}}}$.

2. If $\alpha_n = e^{(\log(n+1))^{s}}$ with $s > 1$ then we may choose

$$\gamma_n = e^{(\log(n+1) + (s-1) \log \log(n+1))^{s}}.$$
Rotwein ist für alte Knaben eine von den besten Gaben
W. Busch

Red wine is for old boys one of the best gifts