The Schur-Horn theorem in von Neumann algebras

Mohan Ravichandran, Sabanci University

November 4, 2011, Istanbul Analysis Seminar

Joint with Rajarama Bhat, ISI Bangalore

Pythagoras and Carpenter

The Pythagoras theorem: Let $0 < a \leq b < c$ be the sides of a right angled triangle: Then,

$$a^2 + b^2 = c^2$$

The Carpenter's rule: Suppose $a^2 + b^2 = 1$, then there is a right angled triangle with sides $a, b, 1$.

Carpenter's rule, restatement: Suppose $0 \leq a, b, \leq 1$ and $a + b = 1$ and e_1, e_2 is an orthonormal basis for $\mathbb{C}^2$. Then, there is a one dimensional subspace so that the squares of the norms of the projections of e_1, e_2 are a and b.

Note: Take the 1-dimensional subspace to be the span of $\sqrt{a_1}e_1 + \sqrt{a_2}e_2$.

Carpenter continued

Carpenter redux 1: Suppose $0 \leq a_i \leq 1$ and $a_1 + \cdots + a_n = 1$ and $e_1, \cdots, e_n$ is an orthonormal basis for $\mathbb{C}^n$. Then there is a one dimensional subspace $\mathcal{M}$ so that the squares of the norms of the projections of e_i onto $\mathcal{M}$ are $a_1, \cdots, a_n$.

Note: Take the one dimensional subspace to be the span of $\sqrt{a_1}e_1 + \cdots + \sqrt{a_n}e_n$.

Carpenter redux 2: Suppose $0 \leq a_i \leq 1$ and $a_1 + \cdots + a_n = m$ and $e_1, \cdots, e_n$ is an orthonormal basis for $\mathbb{C}^n$. Then there is an m dimensional subspace $\mathcal{M}$ so that the squares of the norms of the projections of e_i onto $\mathcal{M}$ are $a_1, \cdots, a_n$.

Carpenter redux 3: Suppose $0 \leq a_i \leq 1$ and $a_1 + a_2 + \cdots = m$ is a possibly infinite sequence and $e_1, e_2, \cdots$ is an orthonormal basis for $\mathcal{H}$, a separable Hilbert space. Then there is an m dimensional subspace $\mathcal{M}$ so that the squares of the norms of the projections of e_i onto $\mathcal{M}$ are a_i for $i = 1, 2, \cdots$.

Operator matrix formulation

Carpenter redux 4: Suppose $0 \leq a_i \leq 1$ and $a_1 + a_2 + \cdots = m$ is an infinite sequence. Then there is a projection P of rank m so that the diagonal of P is precisely $a_1, a_2, \cdots$.
Question: What happens if $a_1 + a_2 + \cdots = \infty$?

First guess: There is always a projection so that the diagonal is $a_1, a_2, \cdots$.

But this is wrong: If P is a projection, so is $I - P$. Consider the sequence $a = \frac{1}{2}, 1, 1, \cdots$. Then, $1 - a = \frac{1}{2}, 0, 0, \cdots$. This is not the diagonal of a projection as the trace is not an integer.

Note: Suppose $a_1 + a_2 + \cdots = \infty$ and $(1 - a_1) + (1 - a_2) + \cdots$ is a finite integer. Then, there is a projection with diagonal $a_1, a_2, \cdots$.

Second guess: Suppose $a_1 + a_2 + \cdots = \infty$ and $(1 - a_1) + (1 - a_2) + \cdots = \infty$. Then, there is always a projection with diagonal $a_1, a_2, \cdots$.

Operator matrices, continued

But this is also wrong: Consider $(0, 0, \cdots) \cup (1, 1, \cdots) \cup \frac{1}{2}$ with infinitely may zeros and ones. This cannot be the diagonal of a projection.

Answer due to Kadison: Let $\{a_1, a_2, \cdots, b_1, b_2, \cdots\}$ be a sequence with $a_i \in [0, \frac{1}{2})$ and $b_i \in [\frac{1}{2}, 1]$. Let

$$A = \sum_i a_i \quad \text{and} \quad B = \sum_i (1 - b_i)$$

Then, the sequence is the diagonal of a projection iff one of the two possibilities occurs

1. Either A or B is infinite.
2. Both A and B are finite and $A - B \in \mathbb{Z}$.

Conclusion: Diagonals of projections completely characterized.

Majorization and the Schur-Horn theorem

Let $\overline{a} = (a_1, a_2, \cdots)$ and $\overline{b} = (b_1, b_2, \cdots)$ be positive non-increasing sequences. Say $\overline{b} \prec \overline{a}$($\overline{a}$ **majorizes** $\overline{b}$ if)

$$\sum_{i=1}^{N} a_i \geq \sum_{i=1}^{n} b_i \quad \text{and} \quad \sum_i a_i = \sum_i b_i$$

If the sequences are not ordered, order them in non-increasing order first.

Eigenvalues and diagonal entries of positive matrices(Schur): Suppose a positive matrix A has eigenvalues $\overline{\lambda}$ and diagonal entries $\overline{b}$. Then, $\overline{b} \prec \overline{\lambda}$.

Eigenvalues and diagonal entries of positive matrices(Horn): Suppose $\overline{b} \prec \overline{\lambda}$. Then, there is a positive matrix A with eigenvalues $\overline{\lambda}$ and diagonal entries $\overline{b}$.

Note: Suppose $a_1 + \cdots + a_n = m$. Then,

$$\left(\frac{m}{n}, \frac{m}{n}, \cdots, \frac{m}{n}\right) \preceq (a_1, \cdots, a_n) \preceq (1, 1, \cdots, 1, 0, \cdots, 0).$$

The Schur-Horn theorem, continued

Another form: The set of diagonals is the convex polytope generated by the permutations of the eigenvalue list.

Normal matrices: The above theorem also holds for normal matrices.

Generalized by Kostant to the setting of actions of compact Lie groups.

Generalized further by Guillemin, Sternberg, Atiyah and put in the setting of symplectic geometry and Hamiltonian manifolds.

Theorems concerning majorization

Birkhoff's theorem: If $a \preceq b$, then, there is a (finite)convex combination of permutation matrices A so that $Ab = a$.

Hardy, Littlewood and Polya's theorem: (Defn: A matrix is said to be doubly stochastic if the entries are positive and the rows and columns sum up to 1). If $a \preceq b$, then, there is a doubly stochastic matrix A so that $Ab = a$. Also, Birkhoff showed that unitary conjugates of permutation matrices are the extreme doubly stochastic matrices.

Horn's theorem: (Defn: A matrix is said to be orthostochastic if its entries are u_{ij}^2 for a unitary matrix U with real entries u_{ij}). If $a \preceq b$, then, there is an orthostochastic matrix A so that $Ab = a$.

Hardy, Littlewood and Polya: Say $A \prec B$ for two positive matrices A, B if the eigenvalue list of B majorizes the eigenvalue list of A. Then, $A \preceq B$ iff

$$\mathrm{Tr}(f(A)) \leq \mathrm{Tr}(f(B))$$

for every continuous convex function f defined on an interval containing the spectra of both A and B.

Diagonals of trace class and compact operators

Trace class operators: Let S be a positive trace class operator with eigenvalue list $(\mu_1, \mu_2, \cdots)$ arranged in non-increasing order let and $\mathcal{O}(S)$ be closure of the unitary orbit of S in the trace norm.

$$\mathcal{O}(S) = \overline{\{USU^* \mid U \in \mathcal{U}(\mathcal{H})\}}^{||_1}$$

Fact: A trace class operator T belongs to $\mathcal{O}(S)$ iff $T \oplus 0_{\mathcal{K}}$ is unitarily equivalent to $S \oplus 0_{\mathcal{K}}$ where $\mathcal{K}$ is an infinite dimensional Hilbert space.

Schur-Horn for trace class operators(Arveson and Kadison): Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for $\mathcal{H}$. Then, the diagonals of elements in $\mathcal{O}(S)$ are precisely positive sequences $(\lambda_1, \lambda_2 \cdots)$ such that

$$\sum_{n=1}^{N} \lambda_n \leq \sum_{n=1}^{N} \mu_n \quad \text{and} \quad \sum_{n} \lambda_n = \sum_{n} \mu_n$$

Schur-Horn for compact operators: Kaftal and Weiss.

Finite von Neumann algebras

$\mathcal{M} \subset \mathcal{B}(\mathcal{H})$.

1. Closed under $*$ operation.
2. Closed in the strong operator topology. Alternately, given the spectral decomposition of a self-adjoint element,

$$T = \int_{\mathbb{R}} \lambda dP_\lambda$$

   the projections P_λ in the projection valued measure all belong to $\mathbb{M}$.
3. Admits a finite trace τ, normalise so that $\tau(I) = 1$.

Note: Any finite von Neumann algebra is a direct integral of ones with trivial center, called factors.

Note: A finite factor is said to be of type II(type II_1) if the trace on projections takes all values on $[0, 1]$.

Weak* topology: von Neumann algebras are dual Banach spaces and admit a (unique)weak* topology. A weak* continuous map between von Neumann algebras is also called **normal**.

Masas: Given a maximal abelian subalgebra $\mathcal{A}$ in $\mathcal{M}$, it is also a von Neumann algebra.

Conditional expectations: $\mathcal{N} \subset \mathcal{M}$, inclusion of von Neumann algebras. A conditional expectation $E : \mathcal{M} \to \mathcal{N}$ is a projection of norm one that is onto.

There are plenty of conditional expectations(norm 1 projections) from a II_1 factor $\mathcal{M}$ onto a masa $\mathcal{A}$. However, there is a unique one that is trace preserving. This "canonical" conditional expectation $E : \mathcal{M} \to \mathcal{A}$ is weak*(and SOT) continuous. It is also the unique normal continuous conditional expectation.

Normal Conditional expectations onto masas: These are the analogues of the map that sends a matrix to its diagonal.

Some examples of II_1 factors

Group algebras: Let G be a discrete group. Consider the left regular representation of G on $\ell^2(G)$.

$$G \ni g \to U_g \quad [U_g(f)](h) = f(g^{-1}h)$$

The von Neumann algebra, denote $L(G)$ generated by $\{U(g) \mid g \in G\}$ is a finite von Neumann algebra.

Note: If every non-trivial conjugacy class of the group G is infinite(ICC), then $L(G)$ has trivial center, i.e it is a II_1 factor.

Masas: If $H \subset G$ is a maximal abelian subgroup, then $L(H) \subset L(G)$ is a masa.
Example: $L_u \subset L(F_2)$, where $< u, v >$ are generators for F_2

The group measure space construction:

Let α be a measure preserving action of the discrete group G on a probability space (X, μ) that is fixed point free and ergodic. Then, the subalgebra of $\ell^2(G) \otimes L^2(X, \mu)$ generated by

$$f \in L^\infty(X, \mu) \quad M_f : M_f(\delta_g \otimes f_1) = \delta_g \otimes f\alpha_{g^{-1}}(f_1)$$

and

$$h \in G \quad (U_h \otimes I)(\delta_g \otimes f) = \delta_{h^{-1}g} \otimes f$$

is called the crosesd product von Neumann algebra $L^\infty(X, \mu) \ltimes_\alpha G$(the analogue of the semi-direct product).

Note: When the groups are amenable, the II_1 factors are all isomorphic and they are denoted the hyperfinite II_1 factor R. Every II_1 factor contains R.

Normalisers: Masa $\mathcal{A}$ inside $\mathcal{M}$. A unitary U in $\mathcal{M}$ is said to normalise $\mathcal{A}$ if $U^*\mathcal{A}U = \mathcal{A}$. The set of all normalisers is denoted $\mathcal{U}_\mathcal{M}(\mathcal{A})$.

Singular masas: No non-trivial normalisers.
Cartan masas: The span of $\mathcal{U}_\mathcal{M}(\mathcal{A})$ is weak* dense in $\mathcal{M}$.
Note: In the group measure space construction, the masas $L^\infty(X, \mu)$ are Cartan and by results of Feldman and Moore, every Cartan masa(essentially) arises this way.

Majorization in finite von Neumann algebras

One definition: Say that $A \prec S$ for self-adjoint elements in a II_1 factor $\mathcal{M}$ if $\tau(f(A)) \leq \tau(f(S))$ for every continuous convex function f defined on an interval containing the spectra of both A and S.

Alternate formulation: For every self-adjoint element A, there is an essentially unique function f_A from $[0,1] \to [0,1]$ that is right-continuous, non-increasing and so that

$$\tau(A^n) = \int_0^1 f(t)^n dm$$

This is often called the **non-increasing re-arrangement** of A. Say that $A \prec S$ if

$$\int_0^r f_A(t)dm \leq \int_0^r f_S(t)dm \quad \text{and} \quad \int_0^1 f_A(t)dm = \int_0^1 f_S(t)dm$$

Equimeasurable operators: Two self-adjoint operators are said to be equimeasurable if $\tau(A^n) = \tau(S^n)$ for every $n = 0, 1, 2, \cdots$ or alternately,

$$f_A = f_S \text{ a.e.} \quad \text{equivalently} \quad A \prec S \quad \text{and} \quad S \prec A$$

Approximate unitary equivalence: If two operators A, S are equimeasurable, then, there is a sequence of unitaries U_n so that

$$||A - U_n^* S U_n|| \to 0$$

Schur-Horn and Carpenter in finite von Neumann algebras

The Schur Inequalities(Hiai, Arveson, Kadison): $E(S) \prec S$, where E is a conditional expectation onto a masa.

Kadison's Carpenter problem: Given a masa $\mathcal{A}$ inside a II_1 factor $\mathcal{M}$, $E : \mathcal{M} \to \mathcal{A}$ the normal conditional expectation and $0 \leq A \leq I$ inside $\mathcal{A}$, is there a projection P in $\mathcal{M}$ so that

$$E(P) = A?$$

Arveson and Kadison's Schur-Horn problem: Given $A \in \mathcal{A}$, $S \in \mathcal{M}$ so that $A \prec S$, is there an element $T \in \mathcal{O}(S) = \overline{\{U^*SU \mid U \in \mathcal{U}(\mathcal{M})\}}^{||}$ so that

$$E(T) = A?$$

Remark: Two equimeasurable operators need not be unitarily equivalent. This is why we need to take the norm closure in the statement of the problem.

Approaching the Schur-Horn and Carpenter problem for finite von Neumann algebras

An existence method due to Dykema, Fang, Hadwin, Smith, 2011: Given $0 \leq A \leq 1$, let

$$\Phi(A) = \{S \in \mathcal{M} \mid \ ||S|| \leq 1 \text{ and } \Phi(S) = A\}$$

This is weak* compact and non-empty. Let T be an extreme point(Krein-Milman).

What can we say about the extreme points? Suppose T is not a projection, then, there is a non-zero spectral projection $Q = E_T([\epsilon, 1-\epsilon])$ for some ϵ. Then, if $E \mid QMQ : Q\mathcal{M}Q \to \mathcal{A}$ is non-injective, then we have a contradiction.

Theorem(DFHS): Suppose for every projection Q in $\mathcal{M}$, the map $E \mid Q\mathcal{M}Q$ is non-injective, then the Carpenter and Schur-Horn problems have a positive solution.

Single generation(DFHS): Suppose a von Neumann algebra is not singly generated, then the Carpenter and Schur-Horn problems have a positive solution.

A restatement: The map $E \mid Q\mathcal{M}Q$ is injective iff any weak* continuous functional on $Q\mathcal{M}Q$ is of the form $X \to \tau(XA)$ for some $A \in L^1(\mathcal{A})$. Alternately, $L^1(\mathcal{A})$ surjects onto $L^1(Q\mathcal{M}Q)$. This looks very unlikely, but is hard to prove.

Generator masas in free group factors and Cartan masas in R

Generator masas(DFHS): The Carpenter and Schur-Horn problems have a positive solution for $L_u \subset L(F_I)$ where I is uncountable. Consider Carpenter: Given A, there is a projection P so that $E(P) = A$. But P must be supported on countably many words!

Special cases(DFHS): The map is non-injective when $\mathcal{A}$ is a separable masa in a non-separable II_1 factor.

1. The problems have a positive solution for the generator and radial masas in the free group factors.
2. For a semi-regular masa and $0 \leq A \leq 1$, there is an automorphism θ of $\mathcal{A}$ and a projection P so that $E(P) = \theta(A)$.

Question

What kind of problem is the Schur-Horn problem? Is it just a problem in noncommutative measure theory? Or does it have to do with the geometry of von Neumann algebras? Could the problem depend on what algebra and masa we work with?

The Schur-Horn theorem for atomic elements

Atomic elements: Say an operator A is atomic if $A = \sum_n \lambda_n E_n$ where the λ_n are scalars and the E_n are orthogonal projections. Any self-adjoint element with purely atomic spectrum is atomic.

Theorem (Bhat, Ravichandran)

Let $\mathcal{A}$ be a masa in a II_1 factor $\mathcal{M}$ and let E be the normal conditional expectation from $\mathcal{M}$ to $\mathcal{A}$. Then for any $0 \le \lambda \le 1$, there is a projection P in $\mathcal{M}$ such that $E(P) = \lambda I$.

Note: When λ is rational, this follows from the matricial Schur-Horn theorem.

Theorem (BR)

Let A and S be positive atomic operators in $\mathcal{A}$ and $\mathcal{M}$ respectively and so that $A \prec S$. Then, there is a unitary U in $\mathcal{M}$ so that $E(USU^*) = A$.

Reminder - What does the Schur-Horn problem ask? Given $A \in \mathcal{A}$, $S \in \mathcal{M}$ so that $A \prec S$, is there an element $T \in \mathcal{O}(S)$ so that

$$E(T) = A?$$

Theorem (BR)

Let S be a positive operator in a II_1 factor $\mathcal{M}$ and let $\mathcal{A}$ be a masa in $\mathcal{M}$. Then, the norm closure of $E(\mathcal{U}(S))$ equals $\{A \in \mathcal{A}^+ \mid A \prec S\}$.

Note: SOT closure, proved by Argerami-Massey(2007). Same result for special case of Cartan masa in the hyperfinite II_1 factor, proved by DFHS, et al(2011).

What Frame theory is about

Definition: A sequence of vectors in a Hilbert space $\{f_i\}_{i\in I}$ is called a frame if there exist positive constants A, B so that for every vector f,

$$A||f||^2 \leq \sum_n | < f, f_i > |^2 \leq B||f||^2$$

An orthonormal basis is a frame with $A = B = 1$. A frame where $A = B$ is called a tight frame and if $A = B = 1$, then it is called a Parseval frame.

Note: Given an onb $\{e_i\}$, the sequence $\{\frac{e_1}{\sqrt{2}}, \frac{e_1}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \cdots\}$ is a Parseval frame that is not an onb.

Synthesis and analysis operator: The synthesis operator for the frame is the operator $T : \ell^2(I) \to \mathcal{H}$, $Te_i = f_i$ for the canonical basis $\{e_i\}$ of $\ell^2(I)$. T^* is called the analysis operator and $S = TT^*$ is called the **frame operator**. Easy to see that

$$f = \sum_i < f, f_i > S^{-1} f_i$$

Basic fact(Graduate students whould work this out): $\{f_i\}_{i\in I}$ is a Parseval frame in $\mathcal{H}$ iff there is a containing Hilbert space $\ell^2(I)$ and an onb $\{e_i\}$ so that

$$P(e_i) = f_i \quad \text{for} \quad i \in I.$$

Schur-Horn and Frame theory

Frames with prescribed norms and frame operator: Characterize pairs (S, c), where S is a frame operator and $c = (c_1, c_2, \cdots)$ where $c_i = ||f_i||^2$.

Alternate reformulation: This is equivalent to characterizing operators S so that

$$S \oplus O_{\mathcal{K}} \quad \text{has diagonal } (c_1, c_2, \cdots) \oplus (0, 0, \cdots).$$

Lyapunov theorems

Classical Theorem of Lyapunov: The range of any non-atomic vector valued measure taking values in $\mathbb{C}^n$ is compact and convex.

Lindenstrauss' reformulation: Let Φ be a weak* continuous linear map from a non-atomic abelian von Neumann algebra into $\mathbb{C}^n$. Then, for any positive contraction A, there is a projection P such that $\Phi(A) = \Phi(P)$.

Singular functionals on von Neumann algebras: The predual of a von Neumann algebra is complemented in the dual space and there is a norm 1 projection onto it.

$$\mathcal{M}^* = \mathcal{M}_* \oplus \mathcal{M}_s^*$$

The elements of $\mathcal{M}_s$ are called singular functionals.

Takesaki's lemma: A positive functional ϕ is in $\mathcal{M}_s^*$ iff for any projection P in $\mathcal{M}$, there is a subprojection Q so that $\phi(Q) = 0$.

Normal functionals: For normal functionals θ, there is a support projection P so that θ is faithful on $P\mathcal{M}P$.(faithful means that it is non-vanishing on any positive element).

Lyapunov theorems for singular maps

Singular maps: A map between a von Neumann algebra and a normed linear space is said to be singular if the pullback of any functional is singular.

Theorem (Akemann,Anderson, 1989)

Let $\mathcal{M}$ be a von Neumann algebra(satisfying some very weak conditions). Let $\Phi : \mathcal{M} \to X$ where X is a normed linear space whose dual is weak separable, be a singular map. Then, for every contraction $A \in \mathcal{M}$, there is a projection P so that $\Phi(P) = \Phi(A)$.*

Note: The extreme points of the positive unit ball of any II_1 factor consists of projections.

Note: Means that the Carpenter problem can be solved for singular conditional expectations onto masas. There are many singular conditional expectations, as pointed out by work of Akemann and Sherman(2008)

Question

What about the Schur-Horn theorem for singular conditional expectations? My guess: Let $A \in \mathcal{A}^+$ and $S \in \mathcal{M}^+$ so that $||A|| \leq ||S||$ and $\alpha(A) \geq \alpha(S)$ where $\alpha(A) = \inf(\{s \in \sigma(A)\})$. Then, there is an element $T \in \mathcal{O}(S)$ such that $E(T) = A$.