Invariant subspaces of compact-friendly-like operators: the state-of-the-art and some open problems

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(joint work with Tunç Mısırlıoğlu)

Outline

Fix a Banach space X and $T \in \mathcal{L}(X)$. A subspace V of X is called **non-trivial** if $\{0\} \neq V \neq X$. If $TV \subseteq V$, then V is called a **T-invariant subspace.** If V is S-invariant for every S in the commutant

$$\{T\}' := \{S \in \mathcal{L}(X) \mid ST = TS\}$$

of T, then V is called a **T-hyperinvariant subspace.**

The Invariant Subspace Problem

When does a bounded operator on a Banach space have a non-trivial closed invariant subspace?

Fix a Banach space X and $T \in \mathcal{L}(X)$. A subspace V of X is called **non-trivial** if $\{0\} \neq V \neq X$. If $TV \subseteq V$, then V is called a ***T*-invariant subspace.** If V is S-invariant for every S in the commutant

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The Invariant Subspace Problem

When does a bounded operator on a Banach space have a non-trivial closed invariant subspace?

An operator $T : X \to Y$ between normed spaces is called **compact** if $\overline{TX_1}$ is a compact set in Y, where X_1 is the closed unit ball of X. The family of all compact operators from X to Y is denoted by $\mathcal{K}(X, Y)$, and one defines $\mathcal{K}(X) := \mathcal{K}(X, X)$.

An operator $T \in \mathcal{L}(X)$ on a Banach space X is said to be:

- **quasi-nilpotent** if $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 0$,
- **locally quasi-nilpotent** at a vector x in X if $r_T(x) := \lim_{n\to\infty} \|T^n x\|^{1/n} = 0$,
- **essentially quasi-nilpotent** if $r_{ess}(T) := r(\pi(T)) = 0$, where $\pi : \mathcal{L}(X) \to \mathfrak{C}(X)$ is the quotient map of $\mathcal{L}(X)$ onto the Calkin algebra $\mathfrak{C}(X) := \mathcal{L}(X)/\mathcal{K}(X)$.

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A **non-scalar** operator is one which is not a multiple of the identity.

Theorem (V. Lomonosov–1973)

If an operator $T : X \to X$ on a complex Banach space commutes with a non-scalar operator $S \in \mathcal{L}(X)$ which in turn commutes with a non-zero compact operator, then T has a non-trivial closed invariant subspace.

Corollary

If a non-scalar operator T on a complex Banach space commutes with a non-zero compact operator, then T has a non-trivial closed hyperinvariant subspace.

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An operator $T \in \mathcal{L}(X)$ is a **Lomonosov operator** if there is an operator $S \in \mathcal{L}(X)$ such that:

- S is a non-scalar operator.
- S commutes with T.
- S commutes with a non-zero compact operator.

Lomonosov's Theorem redux

Every Lomonosov operator on a complex Banach space has a non-trivial closed invariant subspace.

But not vice versa: not every operator with a non-trivial closed invariant subspace is a Lomonosov operator (D.W. Hadwin, E.A. Nordgren, H. Radjavi & P. Rosenthal–1980).

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- A **Banach lattice** is a real Banach space E equipped with a partial order $\leqslant$, which makes E into a lattice in the algebraic sense and which is compatible with the linear and the norm structures.
- Complex Banach lattices are obtained via complexification of real ones.
- All classical Banach spaces are Banach lattices under their natural orderings and norms.
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- The infimum and the supremum operations $\wedge$ and $\vee$ in a Banach lattice E generate the positive elements

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad |x| := x \vee (-x),$$

for which the identities

$$x = x^+ - x^-, \quad |x| = x^+ + x^-, \quad x^+ \wedge x^- = 0$$

hold for every x in E.

- If V is a subspace of a Banach lattice and if $v \in V$ and $|u| \leqslant |v|$ imply $u \in V$, then V is called an **ideal.**

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- A linear operator $T : E \to F$ between Banach lattices is called **positive** and is denoted by $T \geqslant 0$ if $TE^+ \subseteq F^+$. The family of positive operators from E to F is denoted by $\mathcal{L}(E, F)_+$.
- The family $\mathcal{L}(E, F)$ becomes an ordered vector space with $\mathcal{L}(E, F)_+$ being its positive cone by declaring

$$T \geqslant S \quad \text{whenever} \quad T - S \geqslant 0.$$

- An operator T on E is said to be **dominated** by a positive operator B on E, denoted by $T \prec B$, provided

$$|Tx| \leqslant B|x|$$

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- An operator on E which is dominated by a multiple of the identity operator is called a **central operator.** The collection of all central operators on E is denoted by $\mathcal{Z}(E)$ and is referred to as the **center** of the Banach lattice E.
- An operator $T : E \to F$ is said to be *AM*-**compact**, provided that T maps order bounded sets to norm-precompact sets. Each compact operator is necessarily *AM*-compact.

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The ISP for positive operators on Banach lattices

Does every positive operator on an infinite-dimensional, separable Banach lattice have a non-trivial closed invariant subspace?

Notation: Throughout, the letters X and Y will denote infinite-dimensional Banach spaces while E and F will be fixed infinite-dimensional Banach lattices.

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Compact-friendliness (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw−1994)

A positive operator B on E is said to be **compact-friendly** if there is a positive operator that commutes with B and dominates a non-zero operator which is dominated by a compact positive operator.

In other words, a positive operator B on E is compact-friendly if there exist three non-zero operators $R, K, C : E \to E$ such that R, K are positive, K is compact, $RB = BR$, and for each $x \in E$ one has

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Some examples of compact-friendly operators are:

- Compact positive operators: if $B \geqslant 0$ is compact, then take $R = K = C := B$ in the definition.
- The identity operator: having fixed an arbitrary non-zero compact positive operator K, set $R = C := K$ (which also shows that a compact-friendly operator need not be compact).
- Every power (even every polynomial with non-negative coefficients) of a compact-friendly operator.
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A continuous function $\varphi : \Omega \to \mathbb{R}$, where Ω is a topological space, has a **flat** if there exists a non-empty open set Ω_0 in Ω such that φ is constant on Ω_0.

If Ω is a compact Hausdorff space, then each $\varphi \in \mathcal{C}(\Omega)$ generates the **multiplication operator** $M_\varphi : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ defined for each $f \in \mathcal{C}(\Omega)$ by

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The function φ is called the **multiplier** of M_φ.

A multiplication operator M_φ is positive if and only if the multiplier φ is positive.

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Apart from its counterparts, this is the only known characterization of compact-friendliness on a concrete Banach lattice. A similar characterization of compact-friendly multiplication operators on L_p-spaces was obtained by Y.A. Abramovich, C.D. Aliprantis, O. Burkinshaw and A.W. Wickstead in 1998. G. Sirotkin has managed to extend the latter to arbitrary Banach function spaces in 2002.

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If a non-zero compact-friendly operator $B : E \to E$ on a Banach lattice E is quasi-nilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal.

The motivation

Let B and R be two commuting positive operators on E such that B is compact-friendly and R is locally quasi-nilpotent at some non-zero positive vector in E. Does there exist a non-trivial closed B-invariant subspace, or an R-invariant subspace, or a common invariant subspace for B and R?

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Quasi-similarity (B. Sz.-Nagy & C. Foiaş – 1970)

- An operator $Q \in \mathcal{L}(X, Y)$ is a **quasi-affinity** if Q is one-to-one and has dense range.
- An operator $T \in \mathcal{L}(X)$ is said to be a **quasi-affine transform** of an operator $S \in \mathcal{L}(Y)$ if there exists a quasi-affinity $Q \in \mathcal{L}(X, Y)$ such that $QT = SQ$.
- If both T and S are quasi-affine transforms of each other, the operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are called **quasi-similar** and this is denoted by $T \overset{qs}{\sim} S$.

- Similarity implies quasi-similarity; the implication is generally not reversible.
- Quasi-similarity is an equivalence relation on the class of all operators.
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Positive quasi-similarity

Two positive operators $S \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F)$ are **positively quasi-similar**, denoted by $S \overset{pqs}{\sim} T$, if there exist positive quasi-affinities $P \in \mathcal{L}(E, F)$ and $Q \in \mathcal{L}(F, E)$ such that $TP = PS$ and $QT = SQ$.

- Since positive operators between Banach lattices are continuous, each operator dominated by a positive operator is automatically continuous. This guarantees the non-triviality of an operator of the form QTP if P and Q are positive quasi-affinities on E, whenever T is a non-trivial operator dominated by a positive operator on E.

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Lemma

Let B and T be two positive operators on E. If B is compact-friendly and T is positively quasi-similar to B, then T is also compact-friendly.

Sketch of proof.

- Since $T \overset{pqs}{\sim} B$, there exist quasi-affinities P and Q such that $BP = PT$ and $QB = TQ$.
- As B is compact-friendly, there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $RB = BR$, $C \prec R$, and $C \prec K$.
- Take $R_1 := QRP$, $K_1 := QKP$, and $C_1 := QCP$ as the required three operators for the compact-friendliness of T.

□

Lemma

Let B and T be two positive operators on E. If B is compact-friendly and T is positively quasi-similar to B, then T is also compact-friendly.

Sketch of proof.

- Since $T \overset{pqs}{\sim} B$, there exist quasi-affinities P and Q such that $BP = PT$ and $QB = TQ$.
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- Although quasi-similarity need not preserve compactness (T.B. Hoover$-$1972), positive quasi-similarity does preserve compact-friendliness.
- There exists a non-zero quasi-nilpotent operator on ℓ_2 that does not commute with any non-zero compact operator, and hence is not quasi-similar to any compact operator (C. Foiaş & C. Pearcy$-$1974). Combined with a result of H.H. Schaefer which dates back to 1970, an example in the same spirit for Banach lattices is obtained: *there exists a positive quasi-nilpotent operator on the Banach lattice $L_p(\mu)$, where $1 \leqslant p < \infty$ and μ is the Lebesgue measure on the unit circle $\mathbb{T}$, which is not positively quasi-similar to any non-zero compact-friendly operator.*

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Theorem

Let B and T be two positive operators on E such that B is compact-friendly and T is locally quasi-nilpotent at a non-zero positive element of E. If $B \overset{pqs}{\sim} T$, then T has a non-trivial closed invariant ideal.

Strongly compact-friendly operators

A positive operator B on a Banach lattice E is called **strongly compact-friendly** if there exist three non-zero operators R, K, and C on E with R, K positive, K compact such that $B \overset{pqs}{\sim} R$, and C is dominated by both R and K.

Denote the families of positive compact operators, strongly compact-friendly operators and compact-friendly operators on E by $\mathcal{K}(E)_+$, $\mathcal{SKF}(E)$ and $\mathcal{KF}(E)$, respectively.

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Denote the families of positive compact operators, strongly compact-friendly operators and compact-friendly operators on E by $\mathcal{K}(E)_+$, $\mathcal{SKF}(E)$ and $\mathcal{KF}(E)$, respectively.

Lemma

(i) *If a positive operator B on E is positively quasi-similar to an operator on E which is dominated by a positive compact operator or which dominates a positive compact operator, then B is strongly compact-friendly, and the commutant $\{B\}'$ of B contains an operator which is dominated by a positive compact operator or which dominates a positive compact operator, respectively. In particular, every positive operator which is positively quasi-similar to a positive compact operator is strongly compact-friendly and commutes with a positive compact operator.*

(ii) *A non-zero positive operator B on E is strongly compact-friendly if and only if λB is strongly compact-friendly for some scalar $\lambda > 0$. However, B need not be quasi-similar to λB for $\lambda \neq 1$.*

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(iii) *A positive compact perturbation of a positive operator on E is strongly compact-friendly.*

(iv) *For every positive operator B on E, there exists a strongly compact-friendly operator T on E which dominates B.*

(v) *If $B \geqslant I$ on E and $\{B\}'$ does not contain a non-zero compact operator, then there exists a non-zero strongly compact-friendly, non-compact operator on E which is not positively quasi-similar to B.*

(vi) *Positive kernel operators on order-complete Banach lattices are strongly compact-friendly.*

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Theorem

For an infinite-dimensional Banach lattice E, one has

$$\mathcal{K}(E)_+ \subset \mathcal{SKF}(E) \subset \mathcal{KF}(E)$$

and the inclusions are generally proper.

Sketch of proof.

- That both inclusions hold and that the former is proper follow from (i) and (iii) of the previous Lemma.
- For a compact Hausdorff space Ω without isolated points, the space $E := \mathcal{C}(\Omega)$ and the identity operator on E provide together an example which reveals that the second inclusion may well be proper.

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$\square$

Example. ***A strongly compact-friendly operator which is not polynomially compact***

Let $T : \ell_2 \to \ell_2$ be the backward weighted shift defined by

$$Te_0 = 0 \quad \text{and} \quad Te_{n+1} = \tau_n e_n, \quad n \geqslant 0,$$

where $(e_n)_{n=0}^{\infty}$ is the canonical basis of ℓ_2 and $(\tau_n)_{n=0}^{\infty}$ is the sequence

$$\left(\frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{16}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{64}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{256}}, \dots\right).$$

(C. Foiaş & C. Pearcy$-$1974)

One can observe that:

- T is a positive, non-compact operator.
- T is strongly compact-friendly.
- $\|T^n\|^{1/n} \to 0$, that is, T is quasi-nilpotent, and hence is essentially quasi-nilpotent.
- No power of T is compact.

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On a subclass of $\mathcal{SKF}(E)$, the local quasi-nilpotence assumption in (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw$-$1998) can be removed.

Theorem

Let B be a positive operator on E. If B is positively quasi-similar to a positive operator R on E which is dominated by a positive compact operator K on E, then B has a non-trivial closed invariant subspace. Moreover, for each sequence $(T_n)_{n\in\mathbb{N}}$ in $\{B\}'$, there exists a non-trivial closed subspace that is invariant under B and under each T_n.

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For a positive operator B on a Banach lattice E, the **super right-commutant** $[B\rangle$ of B is defined by

$$[B\rangle := \{A \in \mathcal{L}(E)_+ \mid AB - BA \geqslant 0\}.$$

A subspace of E which is A-invariant for every operator A in $[B\rangle$ is called a $[B\rangle$**-invariant subspace.**

Weakly compact-friendly operators

A positive operator $B \in \mathcal{L}(E)$ is called **weakly compact-friendly** if there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that $R \in [B\rangle$, and C is dominated by both R and K.

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Weak positive quasi-similarity

Two positive operators $B \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F)$ are **weakly positively quasi-similar**, denoted by $B \overset{w}{\sim} T$, if there exist positive quasi-affinities $P \in \mathcal{L}(F, E)$ and $Q \in \mathcal{L}(E, F)$ such that $BP \leqslant PT$ and $TQ \leqslant QB$.

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The binary relation $\overset{w}{\sim}$ is an equivalence relation on the class of all positive operators, under which weak compact-friendliness is preserved.

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Example. ***A weakly compact-friendly operator which is not compact-friendly***

Let $E := \mathcal{C}[0,1/2]$, equipped with the uniform norm. Define $\varphi : [0,1/2] \to \mathbb{R}$ by $\varphi(\omega) := 1 - 2\omega$ for all $\omega \in [0,1/2]$. The multiplication operator $M_\varphi : E \to E$ is not compact-friendly since φ has no flats. But M_φ is weakly compact-friendly: take the linear functional $\psi \in E^*$ given by $\psi(f) := f(0)$ for all $f \in E$ and define the rank-one (and hence, compact) positive operator $K : E \to E$ by

$$Kf := (\psi \otimes \varphi)(f), \quad f \in E.$$

Set $R = C := K$.

The arguments, with slight modifications, used in the proof of the corresponding theorem of Abramovich, Aliprantis and Burkinshaw concerning the commutant of the operator B can be shown to work for weakly compact-friendly operators as well.

Theorem

If a non-zero weakly compact-friendly operator $B : E \to E$ on a Banach lattice is quasi-nilpotent at some $x_0 > 0$, then B has a non-trivial closed invariant ideal. Moreover, for each sequence $(T_n)_{n\in\mathbb{N}}$ in $[B\rangle$ there exists a non-trivial closed ideal that is invariant under B and under each T_n.

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Theorem

Let T be a locally quasi-nilpotent positive operator which is weakly positively quasi-similar to a compact operator. Then T has a non-trivial closed invariant subspace.

Sketch of proof.

- If $T \overset{w}{\sim} K$ with K compact, then there exist positive quasi-affinities P and Q such that $TP \leqslant PK$ and $KQ \leqslant QT$. Thus, $TPKQ \leqslant PK^2Q \leqslant PKQT$, i.e., the compact operator $K_0 := PKQ$ belongs to $[T\rangle$.
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Topological fullness of the center (A.W. Wickstead–1981)

The center $\mathcal{Z}(E)$ of a Banach lattice E is called **topologically full** if whenever $x, y \in E$ with $0 \leqslant x \leqslant y$, one can find a sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}(E)$ such that $\|T_n y - x\| \to 0$.

Some examples are:

- Banach lattices with quasi-interior points–such as separable Banach lattices.
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The result in (J. Flores, P. Tradacete & V.G. Troitsky–2008), which uses the existence of a quasi-interior point, can further be improved.

Theorem

Suppose that B is a positive operator on a Banach lattice E with topologically full center such that

(i) *B is locally quasi-nilpotent at some $x_0 > 0$, and*

(ii) *there is an $S \in [B\rangle$ such that S dominates a non-zero AM-compact operator K.*

Then $[B\rangle$ has an invariant closed ideal.

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- Since the null ideal N_B of B is $[B\rangle$-invariant, assume that $N_B = \{0\}$.
- Use the topological fullness of $\mathcal{Z}(E)$ to show that there exists an operator M in
$$\mathcal{Z}(E)_{1+} := \{T \in Z(E) \mid 0 \leqslant T \leqslant I\}$$
with $M|Kz| \neq 0$, where $z \in E$ is such that $Kz \neq 0$.
- Put $K_1 := MK$ and observe that $BK_1 \neq 0$, that BK_1 is AM-compact, and that BK_1 is dominated by BS.
- Observe that the semigroup ideal $\mathcal{J}$ in $[B\rangle$ generated by BS is finitely quasi-nilpotent at x_0, whence $\mathcal{J}$ has an invariant closed ideal.

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Dedekind completeness and compact-friendliness can be relaxed, respectively, to topological fullness of the center and weak compact-friendliness in (Y.A. Abramovich, C.D. Aliprantis & O. Burkinshaw−1998).

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- For each $x > 0$ denote by J_x the ideal generated by the orbit $[B\rangle x$, and suppose that $\overline{J_x} = E$ for each $x > 0$.
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- Put $\pi_1 := M_1 C$ and observe that π_1 is dominated by R and K.
- Repeat the preceding argument twice more to get a non-zero positive operator S in $[B\rangle$ which dominates a compact operator.
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- Put $\pi_1 := M_1 C$ and observe that π_1 is dominated by R and K.
- Repeat the preceding argument twice more to get a non-zero positive operator S in $[B\rangle$ which dominates a compact operator.
- Invoke the previous Theorem to get the assertion.

Sketch of proof.

- For each $x > 0$ denote by J_x the ideal generated by the orbit $[B\rangle x$, and suppose that $\overline{J_x} = E$ for each $x > 0$.
- Use the topological fullness of $\mathcal{Z}(E)$ to show that there exists an operator M_1 in

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SOME OPEN PROBLEMS

- Does every positive operator on ℓ_1 have a non-trivial closed invariant subspace?
- Fix a positive operator B on E, and suppose that there exists a non-zero compact operator dominated by B. Does it follow that there exists a non-zero compact *positive* operator dominated by B?
- It is not known whether the set $\mathcal{KF}(E)$ is order-dense in $\mathcal{L}(E)$ for an arbitrary Banach lattice E. Is the set $\mathcal{SKF}(E)$ order-dense in $\mathcal{L}(E)$? In other words, does every strictly positive operator dominate some strongly compact-friendly operator?

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