Joint Spectra of Toeplitz Operators, Generalized Schwarz Lemma and Optmal Recovery of Analytic Functions

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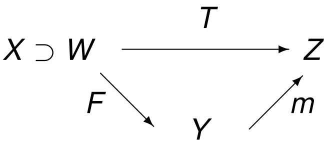
Acknowledgement. This talk is partially based on a joint work with Pablo Gonzalez-Vera. Since then Pablo past away due to an unexpected and severe illnes. I dedicate this talk to his memory.

General Optimal Recovery Problem

Let X be a linear space, Z be a normed linear space, and $T\colon X \to Z$ be a linear operator. We consider the problem of optimal recovery of the operator T on a set $W \subset X$ based on the information given by a multi-valued operator $F\colon W \to Y$ (for each $x \in W$, $F(x)$ is a set in Y). We assume that for every $x \in W$ we know an element $y \in F(x)$. Knowing y we have to approximate the value Tx. Every mapping $m\colon Y \to Z$ is admitted as a recovery method (or an algorithm).



Our goal to find the best recovery algorithm

Recovery in spaces of analytic functions

Let $D \subset \mathbb{C}^k$ be a bounded domain, ν be a probability measure on $\bar{D}$ and X be a closed subspace of $L^2_a(\nu)$. Consider $D_0, \ldots, D_n \subset D$ and probability measures $\mu_0, \ldots, \mu_n$ on $D_0, \ldots, D_n$ respectively.

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Suppose that $f \in X$ is approximately known on $D_1, \ldots, D_n$. It is required to find an optimal method of recovery of f on D_0. This means that we are given $y_1, \ldots, y_n$ defined on $D_1, \ldots, D_n$ such that

$$\|f_j - y_j\|_{L^2(\mu_j)} \leq \delta_j, \quad j = 1, \ldots, n,$$

where f_j is the restriction of f to D_j and $\delta_j \geq 0$, $j = 1, \ldots, n$ are accuracy levels. In particular, $\delta_j = 0$ means that f is known precisely on D_j.

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A recovery algorithm (method, procedure, etc.) is an operator

$$A\colon L^2(\mu_1) \times \ldots \times L^2(\mu_n) \to L^2(\mu_0) \cap X.$$

We consider $A(y)$, $y = (y_1, \ldots, y_n)$, to be the recovered value of f on D_0.

Recovery error

Given a recovery method A its accuracy is characterized by the maximal possible error

$$e(X, \mathbb{D}, \mu, \delta, A) = \sup\{\|f - A(y)\|_{L^2(\mu_0)} : f \in X,$$
$$y \in L^2(\mu_1) \times \ldots \times L^2(\mu_n),\ \|f - y_j\|_{L^2(\mu_j)} \le \delta_j,\ j = 1, \ldots, n\}.$$

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We futher introduce the optimal recovery error as

$$E(X, \mathbb{D}, \mu, \delta) = \inf_{A:\ L^2(\mu_1)\times\ldots\times L^2(\mu_n)\to L^2(\mu_0)} e(X, \mathbb{D}, \mu, \delta, A). \qquad (1)$$

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A method $\widehat{A}$ such that

$$E(X,\mathbb{D},\mu,\delta)=e(X,\mathbb{D},\mu,\delta,\widehat{A})$$

is called an *optimal recovery method*. The problem of finding an optimal recovery method (and sometimes an extremal function at which the optimal recovery error is attained) is usually referred to as *optimal recovery problem*.

Dual problem

It is known that this problem is closely related to the following extremal problem which is called the *dual problem*. Find

$$\sup\left\{ \|f\|^2_{L^2(\mu_0)} : f \in X,\ \|f\|^2_{L^2(\mu_j)} \leq \delta_j^2,\ j = 1, \dots, n \right\}. \qquad (2)$$

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Let $K(z, w)$ be the reproducing kernel of X.
Write

$$\widetilde{\mu} = -\mu_0 + \sum_{j=1}^{n} \lambda_j \mu_j,$$

where $\lambda_1, ..., \lambda_n$ are coefficients.
For $w \in D$ we introduce

$$d\widetilde{\mu}_w(z) = K(z, w) d\widetilde{\mu}(z).$$

The measures $\widetilde{\mu}$ and $\widetilde{\mu}_w$ depend on $\lambda = (\lambda_1, \dots, \lambda_n)$. We explicitly indicate this dependence for the regular part of $\widetilde{\mu}_w$ and write

$$\tau_w^{\lambda}(z) = \int_D K(z, \tau)\, d\widetilde{\mu}_w(\tau).$$

Necessary condition

The following result gives a necessary condition of extremum in the dual problem.

THEOREM 1 (Osipenko, Stessin) *If* $\widehat{f} \in X$ *is a solution of problem* (2)*, then there exists a non-negative vector* $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ *such that*

$$\widehat{f} \in (\mathrm{span}\{\tau_w^{\widehat{\lambda}},\ w \in D\})^{\perp},$$

and the complimentary slackness conditions

$$\widehat{\lambda}_j(\|\widehat{f}\|_{L_2(\mu_j)} - \delta_j) = 0, \quad j = 1, \ldots, n.$$

are satisfied.

Spectrum of the dual problem

We say that a non-negative vector $\lambda = (\lambda_1, \dots, \lambda_n)$ belongs to the *spectrum* of problem (2), if there exists an admissible for problem (2) function $f \in X$ such that

$$\lambda_j(\|f\|_{L_2(\mu_j)} - \delta_j) = 0, \tag{3}$$

$$f \in (\text{span}\{\tau_w^\lambda,\ w \in D\})^\perp. \tag{4}$$

The function f is called the corresponding spectral function.

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THEOREM 2(Osipenko, Stessin). *Let* Λ *be the spectrum of problem* (2). *Then*

$$\sup_{\substack{f \in X \\ \|f\|_{L^2(\mu_j)} \le \delta_j,\ j=1,\dots,n}} \|f\|^2_{L^2(\mu_0)} = \sup_{\lambda \in \Lambda} \sum_{j=1}^{n} \lambda_j \delta_j^2. \tag{5}$$

Regularity condition

The following important condition was first introduced by Melkman and Micchelli (1980)

We call an optimal recovery problem *regular*, if for every $\widehat{\lambda} = (\widehat{\lambda}_1, \cdots, \widehat{\lambda}_n) \in Extr(\mu_0, \cdots, \mu_{n-1})$ the value of the extremal problem

$$\|f\|^2_{L^2(\mu_0)} \to \max, \quad \sum_{j=1}^{n} \widehat{\lambda}_j \|f\|^2_{L^2(\mu_j)} \leq \sum_{j=1}^{n} \widehat{\lambda}_j \delta_j^2, \quad f \in X, \qquad (6)$$

is the same as in (2).

The connection between the optimal recovery and its dual problems is given by the following result.

THEOREM 3(Osipenko, Stessin) *If an optimal recovery problem is regular, then the linear projection*

$$y = (y_1, \dots, y_n) \mapsto \widehat{A}(y) = (f_y),$$

where f_y is the solution of the extremal problem

$$\sum_{j=1}^{n} \widehat{\lambda}_j \|f - \widetilde{y}_j\|^2_{L^2(\mu_j)} \to \min, \quad f \in X. \tag{7}$$

gives an optimal recovery method and its error is equal to the value of the dual problem.

Weighted spaces in the ball

We will be dealing with the weighted Bergman space case when $X = A^2_\alpha(\mathbb{B}_k)$, where $\mathbb{B}_k$ is the unit ball of $\mathbb{C}^k$, $d\nu(z) = (1 - |z|^2)^\alpha dV(z)$, ($dV$ being the normalized volume measure) and concentrate on a special case when $D_n = \mathbb{B}_k$, $y_n = 0$ and $\delta_n = 1$.

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Thus, the dual problem (2) in this case looks as follows:

$$\sup \Big\{ \|f_0\|^2_{L^2(\mu_0)} : f \in A^2_\alpha(\mathbb{B}_k),\ \|f_j\|^2_{L^2(\mu_j)} \le \delta_j^2, \\ j = 1, \ldots, n-1,\ \|f\|_{A^2_\alpha(\mathbb{B}_k)} \le 1, \Big\} \tag{8}$$

Dual problem solution

If D_0 lies compactly in $\mathbb{B}_k$, bounded point evaluations in $A^2_\alpha(\mathbb{B}_k)$, Montel's theorem and a standard compactness argument imply that the supremum in (8) is finite and the dual problem has an extremal function. If D_0 (which we identify with the support of μ_0) is not a compact in $\mathbb{B}_k$, the value of the problem (8) may be infinite. A natural condition which guarantees that the supremum in (8) is finite is the requirement that μ_0 is an α-Carleson measure. Still, this does not imply that the dual problem has an extremal function, and, therefore, Theorem C is not applicable. The following result shows that under some mild assumptions about supports of $\mu_j,\ j = 1, ..., n-1$ if restrictions of μ_0 to balls of radii $r_m \nearrow 1,\ m \to \infty$ satisfy Theorem C, then the optimal recovery method exists (even if an extremal function for (8) does not).

For $0 < r < 1$ let us denote by $\mu_{0,r}$ the restriction of μ_0 to the ball of radius r, $\mu_{0,r} = \chi_r \mu_0$, where χ_r is the characteristic function of the ball of radius r:

$$\chi_r(z) = \begin{cases} 1 & \text{if } |z| \leq r \\ 0 & \text{if } |z| > r\,. \end{cases}$$

We will call extremal problem (8) with $\|f\|^2_{L^2(\mu_0)}$ replaced with $\|f\|^2_{L^2(\mu_{0,r})}$ *problem* (3r). For each $0 < r < 1$ the measure $\mu_{0,r}$ is compactly supported, and, therefore, problem (3r) has an extremal function.

Method existence theorem

The following result is due to Gonzalez-Vera and Stessin (2012).

Theorem

Suppose that there is a sequence $r_m \nearrow 1$ as $m \to \infty$ such that problems (3r_m) satisfy the conditions of Theorem C. Suppose also that there exists $0 < \rho < 1$ such that $supp(\mu_0) \cap supp(\mu_j) \cap \{|z| > \rho\} = \emptyset,\ j = 1, \dots, n-1$, and that μ_0 is α-Carleson. Then there is an optimal recovery method for the $L^2(\mu_0)$-optimal recovery problem of reconstruction of $f \in A^2_\alpha(\mathbb{B}_k)$ which is the limit of optimal recovery methods for problems (3r_m), and the error of this method is equal to the value of problem (3).

Compactly supported measures

Previous Theorem allows us to concentrate on the case when μ_0 is compactly supported. The following result (Gonzalez-Vera, Stessin 2012) states that the same is true for the measures $\mu_1, ..., \mu_{n-1}$.

Let $0 < r < 1$. Similar to the notation above we will call extremal problem (8) with $\|f\|^2_{L^2(\mu_1)}, \cdots, \|f\|^2_{L^2(\mu_{n-1})}$ replaced with $\|f\|^2_{L^2(\mu_{1,r})}, \cdots, \|f\|^2_{L^2(\mu_{n-1,r})}$ *problem* (3(r)).

Theorem

Let μ_0 be compactly supported. Then there is a sequence $r_m \nearrow 1$ such that the solutions of the problem (3(r_m)) converge to a solution of (8)*, and extremal spectral points of* (3(r_m)) $\widehat{\lambda}(r_m) = (\widehat{\lambda}_1(r_m), \ldots, \widehat{\lambda}_n(r_m))$ *converge to an extremal spectral point* $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ *of problem* (8).

Toeplitz operators

Recall that given a function φ on D, the Toeplitz operator T_φ is defined by $T_\varphi(f) = P(\varphi f)$, or in integral form,

$$T_\varphi(f)(z) = \int_D K(z, w)\varphi(w)f(w)\, dv(w). \tag{9}$$

Obviously, if $\varphi \in L^\infty(D)$, the operator T_φ is bounded on L^2_a with $\|T_\varphi\| \leq \|\varphi\|_\infty$.

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More generally, motivated by the integral representation in (9), we define the Toeplitz operator T_μ with symbol μ being a measure on D as follows:

$$T_\mu(f)(z) = \int_D K(z,w)f(w)\,d\mu(w). \tag{10}$$

It is easy to see that if μ is a finite measure compactly supported in D, then T_μ is not only bounded on L^2_a, it is also compact there.

Joint Spectra

Recall that the spectrum of T_μ, denoted $\sigma(T_\mu)$, consists of all complex numbers λ such that $\lambda I - T_\mu$ is not invertible on $A^2_\alpha(\mathbb{B}_k)$.

More generally, for $N \geq 2$, we define the joint spectrum $\sigma(T_{\mu_1}, \cdots, T_{\mu_N})$ as the set of points $z = (z_1, \cdots, z_N) \in \mathbb{C}^N$ such that $z_1 T_{\mu_1} + \cdots + z_N T_{\mu_N}$ is not invertible on A^2.

The joint spectrum defined in the previous paragraph was recently introduced and studied by R. Yang for a general N-tuple $(T_1, \cdots, T_N)$. The case of an N-tuple of Toeplitz operators is slightly easier, because a linear combination of Toeplitz operators again is a Toeplitz operator. Still very little information is known about the structure of $\sigma(T_\mu)$, except in a few very special cases.

Necessary condition in terms of Toeplitz operators

Let us come back to optimal recovery problems. Now and on we will assume that $\mu_n = \nu$ is the measure which defines X, and $\delta_n = 1$.

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It is not difficult to derive that condition (3) implies

$$T_{\mu_0} f = \sum_{j=1}^{n-1} \lambda_j T_{\mu_j} f + \lambda_n f, \tag{11}$$

Thus, the spectrum of the dual problem lies inside the joint spectrum of the operators $T_{\mu_0}, ..., T_{\mu_{N-1}}, I$. Theorems 2 and 3 show that the structure of the joint spectrum is important.

Examples: Generalized Schwarz Lemma

Recall that the classical Schwarz lemma states that an analytic function f which takes the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into itself and vanishes at the origin, satisfies the inequality

$$|f(z)| \leq |z|$$

for all z in the unit disk.

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We consider the following two similar problems.
1. Let $0 < r_1 < \rho < r_2 < 1$, $f_r(z) = f(rz)$. Consider the following extremal problem. Find

$$\sup \left\{ \|f_\rho\|_{H^2} : f \in H^2, \ \|f_{r_j}\|_{H^2} \leq \delta_j, \ j = 1, 2 \right\}. \qquad (12)$$

It is easily seen that this problem is the dual to optimal recovery of an H^2 function f on the circle of radius ρ given that it differs in L^2-metric from given functions on circles of radii r_1 and r_2 by no more than δ_1 and δ_2 respectively, and that its H^2-norm does not exceed one.

Examples: Theorem 1

The following result (Osipenko and Stessin, 2010) gives the solution of this problem.

Theorem

Let $0 < r_1 < \rho < r_2 < 1$ *and* $\delta_1, \delta_2 > 0$*. Then*

1. *If* $s \in \mathbb{Z}_+$ *is such that*

$$\left(\frac{r_1}{r_2}\right)^{s+1} < \frac{\delta_1}{\delta_2} < \left(\frac{r_1}{r_2}\right)^{s},$$

   then the unique extremal spectral point of (12) *is*

$$\left(\frac{r_2^2 - \rho^2}{r_2^2 - r_1^2}\left(\frac{\rho}{r_1}\right)^{2s}, \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2}\left(\frac{\rho}{r_2}\right)^{2s}\right).$$

Schwarz Lemma, Theorem 1, contd

2. If $\delta_1 > \delta_2$, then the unique extremal spectral point of (12) is $(0, 1)$.
3. If there is $s \in \mathbb{Z}_+$ such that

$$\frac{\delta_1}{\delta_2} = \left(\frac{r_1}{r_2}\right)^s,$$

then the set of extremal spectral point of (12) is $(\widehat{\lambda}_1, \widehat{\lambda}_2)$, where $\widehat{\lambda}_1, \widehat{\lambda}_2 \geq 0$ and

$$\widehat{\lambda}_1 r_1^{2s} + \widehat{\lambda}_2 r_2^{2s} = \rho^{2s}. \tag{13}$$

Examples, continues

2. Our next problem is slightly different.
Let $a \in \mathbb{D}$ and Γ be a circle inside of the unit disk which passes through the origin and its center lies on the real axis, so that

$$\Gamma = \{\, z \in \mathbb{C} : |z - \rho| = \rho \,\},$$

μ_0 be the normalized Lebesgue measure on Γ, and $\delta > 0$.
The problem is to find

$$\sup \left\{ \int_\Gamma |f|^2 \, d\mu_0 : f \in H^2, \ \|f\|_{H^2} \le 1, \ |f(a)| \le \delta \right\}. \tag{14}$$

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$$T_{\mu_0} f(z) = \int_\Gamma \frac{f(w)}{1 - z\bar{w}} d\mu_0(w) = \frac{1}{1 - \rho w} f\left(\frac{\rho}{1 - \rho w}\right),$$

$$T_{\mu_1} f(z) = \frac{f(a)}{1 - \bar{a}z},$$

Examples, contd

The Lagrange equation n this case gives

$$\frac{1}{1-\rho w} f\left(\frac{\rho}{1-\rho w}\right) = \lambda_1 \frac{f(a)}{1-\bar{a}w} + \lambda_2 f(w). \qquad (15)$$

Our next step is to describe the spectrum and spectral functions. Let

$$b = \frac{1-\sqrt{1-4\rho^2}}{2\rho}, \ \alpha_j = \frac{b^{2j}}{1-\rho b}, \ j = 0, 1, ...$$

It is also easy to see that the disk bounded by the circle Γ is a hyperbolic neighborhood of b.

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It is also easy to see that the disk bounded by the circle Γ is a hyperbolic neighborhood of b. Consider the following functions

$$\varphi_j(z) = \frac{\sqrt{1-b^2}}{1-bz}\left(\frac{b-z}{1-bz}\right)^j,$$

$$j = 0, 1, \ldots$$

Example, continues

These functions form an orthonormal basis of H^2, and they are eigenfunctions of the operator

$$Tf(z) = \frac{1}{1-\rho z} f\left(\frac{\rho}{1-\rho z}\right).$$

and the corresponding eigenvalues are

$$\alpha_j = \frac{b^{2j}}{1-\rho b}. \tag{16}$$

The following result (Osipenko, Stessin, 2010) describes the spectrum of problem (14).

Theorem

1. *Let* $a \neq b$. *If* $\left| a - \frac{\rho}{1-\rho^2} \right| \geq \frac{\rho^2}{1-\rho^2}$, *or* $\delta > \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \bar{a}\rho - |a|^2}$, *then the spectrum of Schwarz Lemma extremal problem consists of two parts* $\Lambda = \Lambda_1 \cup \Lambda_2$, *where*

$$\Lambda_1 = \left\{ (0, \alpha_j) : |\varphi_j(a)| \leq \delta \right\}, \ \Lambda_2 = \left\{ (\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0, \ F(\lambda_2) = \delta^{-2}, \ \lambda_1 = h(\lambda_2) \right\},$$

where

$$F(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{(\alpha_j - \lambda)^2} h^2(\lambda),$$

$$h(\lambda) = \left(\sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} \right)^{-1}.$$

Theorem

2. *If $a \neq b$ and*

$$\left| a - \frac{\rho}{1-\rho^2} \right| < \frac{\rho^2}{1-\rho^2}, \tag{17}$$

and

$$\delta \leq \frac{\sqrt{|a|^2\rho^2 - |\rho - a|^2}}{a\rho + \bar{a}\rho - |a|^2}, \tag{18}$$

then the spectrum includes in addition the point

$$\Lambda_3 = \left\{ \left(\frac{a\rho + \bar{a}\rho - |a|^2}{\rho^2}, 0 \right) \right\}.$$

Theorem

3. *If $a = b$,*

$$\Lambda_1 = \left\{ (0, \alpha_j) : \ j = 1, 2, ... \right\},$$
$$\Lambda_2 = \Big\{ ((1 - b^2)(\alpha_0 - \alpha_j), \alpha_j),$$
$$j = 1, 2, \ldots \} .$$

Then the spectrum of problem is $\Lambda = \Lambda_1 \cup \Lambda_2$, if $\delta < \dfrac{1}{\sqrt{1 - b^2}}$,

and $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \{(0, \alpha_0)\}$, if $\delta \geq \dfrac{1}{\sqrt{1 - b^2}}$.

Schwarz Lemma optimal recovery method

Using the above description of the spectrum and Theorem C it is possible to find the corresponding optimal recovery method. It is given by the following theorem (Osipenko, Stessin, 2010).

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Theorem

Suppose that one of the following conditions is satisfied

1. $\delta \geq |\varphi_0(a)|$,
2. $\delta \leq |\varphi_1(a)|$,
3. $|\varphi_1(a)| < \delta < |\varphi_0(a)|$, $\gamma \geq b^{2/3}$,
4. $a = b$,

and $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ *is the corresponding extremal spectral point. Then the error of optimal recovery is equal to* $\sqrt{\widehat{\lambda}_1 \delta^2 + \widehat{\lambda}_2}$, *and the optimal recovery method is given by*

$$\widehat{A}(y)(z) = \frac{\widehat{\lambda}_1 y}{\widehat{\lambda}_1 + \widehat{\lambda}_2(1 - |a|^2)} \frac{1 - |a|^2}{1 - \bar{a}z}.$$

Linear Spectrum problem

Of the two examples we have just considered the second one is much harder then the first one. The reason is that the joint spectrum of Toeplitz operators included in Lagrange's equation for the first problem has a very simple structure. It consists of a locally finite countable union of lines, and that makes the description of the extremal part of the spectrum easy. Contrary to that the joint spectrum for the second example is quite complicated, and finding its extremal part is hard. It seems worth the investigation when the joint spectrum has a simple geometrical structure.

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Question 1 *Describe the setting when the joint spectrum of operators acting on a Hilbert space consists of a locally finite union of planes.*

Recently there was some progress in this problem. We will return to these results by the end of the talk.

Stability problem

Another remark to what we have seen so far is the following. We saw the regularity condition is important for both the existence of a linear optimal recovery method and the approximation by compact case.

Question. *Is the regularity condition stable, that is if the regularity condition is satisfied in the optimal recovery problem defined by measures* $\mu_0, \cdots, \mu_{n-1}$*, does it hold for perturbed measures?*

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Question. *Is the regularity condition stable, that is if the regularity condition is satisfied in the optimal recovery problem defined by measures* $\mu_0, \cdots, \mu_{n-1}$*, does it hold for perturbed measures?*

A reason for this question is as follows. In general, it is very difficult to describe the joint spectrum and find solutions of equation (11) for arbitrary measures. The only exceptions are the situations described in the previous slide and when all the involved measures are finite combinations of point masses. In the latter case every non-trivial solution of (11) is a rational function, namely, a linear combination of kernels, and the joint spectrum is an algebraic manifold given by zeros of the determinant of the corresponding matrix.

Stability problem 2

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It turned out that the answer is positive. The following result holds.

Stability theorem

Theorem

Suppose that for every $\lambda \in Extr(\mu_0, ..., \mu_{n-1})$ $\lambda_n \neq 0$, *and let problem (2) be regular. If measures* $\widetilde{\mu}_o, ...\widetilde{\mu}_{n-1}$ *are such that the norms* $\| T_{\mu_j} - T_{\widetilde{\mu}_j} \|,\ j = 0, ..., n-1$ *are sufficiently small, then problem* $\widetilde{(2)}$ *is regular too.*

Auxiliary results,1

Proposition

If T_1 *and* T_2 *are normal operators on a Hilbert space* X, *then the distance between* $\sigma(T_1)$ *and* $\sigma(T_2)$ *in Hausdorff metric does not exceed* $\|T_1 - T_2\|$.

Auxiliary results,1

Proposition

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Consider a normal operator T_1, and let $E(z)$ be the spectral measure on $\sigma(T_1)$. For $a \in \sigma(T_1)$ and $\rho > 0$ let us denote by $P(a;\rho)$ the following projection

$$P(a;\rho) = \int_{\sigma(T_1)\cap\{z\in\mathbb{C}:|z-a|\le\rho\}} dE(z).$$

We also denote by $L(a;\rho) \subset X$ the range of $P(a;\rho)$.

Proposition

Suppose that T_1 and T_2 are bounded operators on a Hilbert space X and that T_1 is normal. If $\|T_1 - T_2\| < \rho$, then for every $\lambda \in \sigma(T_1),\ \nu \in \sigma(T_2)$ such that $|\lambda - \nu| \leq \|T_1 - T_2\|$, and every unit vector $\xi \in X$ the distance from ξ to $L(\lambda, \rho)$ satisfies the estimate

$$d\,(\xi, L(\lambda;\rho)) \leq \frac{(\|(T_2 - \nu I)\xi\| + \|T_1 - T_2\|}{\rho - \|T_1 - T_2\|}. \tag{19}$$

Auxiliary results, Corollary

As a direct corollary to this Proposition we obtain the following result.
Suppose that T_1 is normal and compact, and let $\lambda \in \sigma(T_1),\ \lambda \neq 0$. Then λ is an isolated point of $\sigma(T_1)$. Denote by ρ the distance from λ to the rest of $\sigma(T_1)$, and by L_λ the corresponding eigen-subspace of T_1. If ν is an eigenvalue of T_2, and $|\lambda - \nu| \leq \|T_1 - T_2\| < \rho$, and ζ is an eigenvector of T_2 with eigenvalue ν, equation (19) implies that the following estimate holds.

Corollary

For every eigenvector ξ of T_2 with eigenvalue ν

$$d(\xi, L_\lambda) \leq \frac{\|T_1 - T_2\|}{\rho - \|T_1 - T_2\|}.$$

Thus, ξ approaches L_λ as $T_2 \to T_1$ in norm topology.

Auxiliary results, 3

Let $a = (a_0, ..., a_{n-1})$ be a vector with positive coordinates and $b > 0,\ c > 0$ be positive numbers. Denote by $E(\mu_0, ..., \mu_{n-1}; a, b, c) = E(a, b, c)$ the set of points in $\mathbb{R}^n_+$ defined by the following conditions: there exists a function $f = f(\cdot, \lambda) \in X$ so that

$$T_{\mu_0} f = \sum_{j=1}^{n-1} \lambda_j T_{\mu_j} f + \lambda_n f, \tag{20}$$

$$\int |f|^2 d\mu_j \geq a_j,\ j = 0, ..., n-1 \tag{21}$$

$$\|f\| \leq b \tag{22}$$

$$\lambda_n \geq c. \tag{23}$$

Proposition

$E(a, b, c)$ is compact.

Main Lemma

For $\lambda = (\lambda_1, ..., \lambda_n) \in E(a, b, c)$ let us denote by S_λ the set of all real numbers $\tau > 1$ such that there exists a function $f(\cdot; \lambda, \tau)$ which satisfies equation (20) with $\tau\lambda_j$ instead of λ_j, and define τ_λ by

$$\tau_\lambda = \begin{cases} \inf\{\tau \in S_\lambda\} & \text{if } S_\lambda \text{ is not empty,} \\ \infty & \text{otherwise.} \end{cases}$$

We now set

$$\tau(\mu_0, ..., \mu_{n-1}; a, b, c) = \tau(a, b, c) = \inf\{\tau_\lambda : \lambda \in E(\mu_0, ..., \mu_{n-1}; a, b, c)\}$$

Lemma

We have for every a, b, c and positive compactly supported measures $\mu_0, ..., \mu_{n-1}$
(a) $\tau(\mu_0, ..., \mu_{n-1}; a, b, c) > 1$.
(b) $\tau(\mu_0, ..., \mu_{n-1}; a, b, c)$ is a lower semi-continuous function of measures $\mu_0, ..., \mu_{n-1}$ (we define the distance between μ and $\widetilde{\mu}$ as $\|T_\mu - T_{\widetilde{\mu}}\|$).

Prof of Lemma, part (a)

We will give a sketch of the proof of (a).
1. We prove that $\sigma(T_{\mu_0}, \cdots, T_{\mu_{n-1}}, I) \cap \{Im(\lambda_j) = 0,\ j = 1, ..., n\}$ is a real analytic set outside of $\{\lambda_n = 0\}$.

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2. We prove that for every smooth path $\lambda(t) = (\lambda_1(t), \cdots, \lambda_n(t)),\ 0 \leq t \leq 1$ in

$$\sigma(T_{\mu_0}, \cdots, T_{\mu_{n-1}}, I) \cap \{Im(\lambda_j) = 0,\ j = 1, ..., n\}$$

we can find a path of corresponding spectral functions $f(\cdot; t)$ which is differentiable with respect to t, so that

$$T_{\mu_0} f(\cdot; t) = \sum_{j=1}^{n-1} \lambda_j(t) T_{\mu_j} f(\cdot; t) + \lambda_n(t) f(\cdot; t). \tag{24}$$

Proof of Lemma, part (a) continued

3. Suppose that $\tau(\mu_0, ..., \mu_{n-1}; a, b, c) = 1$. Then it is possible to show that there exist paths $\lambda(t) = (\lambda_1(t), \cdots, \lambda_n(t))$, $f(\cdot; t) \subset E(a, b, c)$ and $g(\cdot; t)$, and a strictly decreasing function $\tau(t)$ $(0 \leq t \leq 1)$ such that

- $T_{\mu_0} f(\cdot; t) = \sum_{j=1}^{n-1} \lambda_j(t) T_{\mu_j} f(\cdot; t) + \lambda_n(t) f(\cdot; t)$,

Proof of Lemma, part (a) continued

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- $T_{\mu_0} g(\cdot; t) = \tau(t) \left(\sum_{j=1}^{n-1} \lambda_j(t) T_{\mu_j} g(\cdot; t) + \lambda_n(t) g(\cdot; t) \right)$,

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- $\tau(1) = 1$,
- $f(\cdot; 1) = g(\cdot; 1) = \varphi(\cdot)$,

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- $\tau(1) = 1$,
- $f(\cdot; 1) = g(\cdot; 1) = \varphi(\cdot)$,
- $f(\cdot; t)$, $g(\cdot; t)$ and $\tau(t)$ are differentiable in t (left differentiable at $t = 1$), and $\tau'(1) \neq 0$.

Proof of Lemma, part (a) continued

Then

$$T_{\mu_0}(g(\cdot;t) - f(\cdot;t)) = (\tau(t) - 1)\left[\sum_{j=1}^{n-1} \lambda_j(t) T_{\mu_j}(g(\cdot;t)) + \lambda_n(t) g(\cdot;t)\right]$$
$$+ \left[\sum_{j=1}^{n-1} \lambda_j(t) T_{\mu_j}(g(\cdot;t) - f(\cdot;t)) + \lambda_n(t)(g(\cdot;t) - f(\cdot;t))\right].$$

Proof of Lemma, part (a) continued

Then

$$T_{\mu_0}(g(\cdot;t)-f(\cdot;t)) = (\tau(t)-1)\left[\sum_{j=1}^{n-1}\lambda_j(t)T_{\mu_j}(g(\cdot;t))+\lambda_n(t)g(\cdot;t)\right]$$

$$+\left[\sum_{j=1}^{n-1}\lambda_j(t)T_{\mu_j}(g(\cdot;t)-f(\cdot;t))+\lambda_n(t)(g(\cdot;t)-f(\cdot;t))\right].$$

Obviously, there is a limit (a derivative)

$$h(\cdot)=\lim_{t\to 1}\frac{g(\cdot;t)-f(\cdot;t)}{\tau(t)-1}.$$

Write $R = T_{\mu_0}-\sum_{j=1}^{n-1}\lambda_j(1)T_{\mu_j}$. Dividing the last equation by $(\tau(t)-1)$ and passing to the limit as $t\to 1$ gives:

$$(R-\lambda_n(1)I)h=\sum_{j=1}^{n-1}\lambda_j(1)T_{\mu_j}\varphi+\lambda_n(1)\varphi = T_{\mu_0}\varphi.$$

Proof of Lemma, part (a) complete

Since R is self-adjoint, and $\lambda_n(1)$ is an eigenvalue of R, and φ is an eigenfunction with this eigenvalue, we have for every ψ

$$\langle (R - \lambda_n(1)I)\psi, \varphi \rangle = 0.$$

Proof of Lemma, part (a) complete

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Thus, the last relation gives

$$\int |\varphi(z)|^2 d\mu_0(z) = \langle T_{\mu_0}\varphi, \varphi \rangle = 0,$$

a contradiction.

Proof of Lemma, part (b)

To prove part (b) we first note that compactness of $E(a,b,c)$ implies that there exists $\lambda \in E(a,b,c)$ such that $\tau(a,b,c)\lambda \in S_\lambda$.

Now, suppose that $\mu_j^k \to \mu_j,\ j = 0, ..., n-1$ in the sense that $\| T_{\mu_j^k} - T_{\mu_j} \| \to 0$ as $k \to \infty$. Assume that for some $\epsilon > 0$

$$\tau(\mu_0^k, ..., \mu_{n-1}^k; a, b, c) \le \tau(\mu_0, ..., \mu_{n-1}; a, b, c) - \epsilon \qquad (25)$$

for all k.

Proof of Lemma, part (b)

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for all k.
Let $\lambda^{(k)} \in E(\mu_0^K, ..., \mu_{n-1}^k; a, b, c)$ and f_k be such that

$$T_{\mu_0^k} f_k = \tau(\mu_0^k, ..., \mu_{n-1}^k; a, b, c) \left(\sum_{j=1}^{n-1} \lambda_j^{(k)} T_{\mu_j^k} f_k + \lambda_n^{(k)} f_k \right), \tag{26}$$

and let

$$\tau(\mu_0^k, ..., \mu_{n-1}^k; a, b, c) \to \tau,\ \text{as } k \to \infty.$$

Proof of Lemma, part (b), complete

It is not difficult to prove that $\tau > 1$. Clearly (25) implies

$$\tau < \tau(\mu_0, ..., \mu_{n-1}; a, b, c).$$

Proof of Lemma, part (b), complete

It is not difficult to prove that $\tau > 1$. Clearly (25) implies

$$\tau < \tau(\mu_0, ..., \mu_{n-1}; a, b, c).$$

The sequence $\{\lambda^{(k)}\}$ is bounded (the proof, in fact follows from the proof of Proposition 3), so without loss of generality we may assume that $\lambda^{(k)}$ converges to λ as $k \to \infty$. It suffices to show that $\lambda \in E(\mu_0, ..., \mu_{n-1}; a, b, c)$ and $\tau \in S_\lambda$ for this would contradict the definition of $\tau(\mu_0, ..., \mu_{n-1}; a, b, c)$.

Next, we show that there is a subsequence (and we think that it is the sequence itself) which converges to a function f, that this function f satisfies (21) and (22), and f and λ satisfy (20). We are done.

Proof of Stability Theorem, 1

Our proof consists of three steps.
I. Let $\epsilon = \max\{\| T_{\mu_j} - T_{\widetilde{\mu}_j} \|, \ j = 0, ..., n-1\}$. If $\widetilde{f}$ is an admissible function for problem $(\widetilde{2})$, then

$$f = \widetilde{f} \min\{\frac{\delta_j}{\delta_j + \epsilon} : j = 1, ..., n-1\}$$

is admissible for problem (2). Hence, if $\widetilde{d}$ is the value of problem $(\widetilde{2})$ and d is the value of problem (2), we have

$$d \geq \widetilde{d} \min\{\frac{\delta_j}{\delta_j + \epsilon} : j = 1, ..., n-1\}.$$

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A similar argument shows that

$$\widetilde{d} \min\{\frac{\delta_j}{\delta_j + \epsilon} : j = 1, ..., n-1\} \leq d \leq \widetilde{d} \max\{\frac{\delta_j + \epsilon}{\delta_j},\ j = 1, ..., n-1\}. \quad (27)$$

Proof of Stability Theorem, 2

II. Next we show that $Extr(\widetilde{\mu}_0, ..., \widetilde{\mu}_{n-1})$ converges to $Extr(\mu_0, ..., \mu_{n-1})$ as $\epsilon \to 0$ in the sense that a convergent sequence $\lambda^{(k)} \in Extr(\widetilde{\mu}_0^k, ..., \widetilde{\mu}_{n-1}^k)$ converges to a point in $Extr(\mu_0, ..., \mu_{n-1})$.

Proof of Stability Theorem, 2

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Let f_k be the sequence of corresponding spectral functions (for problem $(\widetilde{2}_k)$). Since $\{f_k\}$ are uniformly bounded, there is a subsequence which converges weakly. Again, we assume that $f_k \to f$ weakly as $k \to \infty$. Then (20) implies that $T_{\mu_j^k} f_k$ converges to $T_{\mu_j} f$ in norm. This implies that f is admissible for problem (2) and is extremal. Further, we note that $\lambda_n \neq 0$, for otherwise, (20) for f and λ is satisfied, and, consequently, $\lambda \in Extr(\mu_0, ..., \mu_{n-1})$, so λ_n can not be equal zero. Since $\lambda_n \neq 0$, the same way as above we conclude that f_k converges to f in norm, and, therefore, $\lambda \in Extr(\mu_0, ..., \mu_{n-1})$.

Proof of Stability Theorem, 3

III. Now we use the main Lemma to finish the proof.
WLOG we assume that $\lambda_j \neq 0,\ j = 1, ..., n$. Set

$$a = (d/2, \delta_1^2/2, ..., \delta_{n-1}^2/2),\ b = 1,$$
$$c = \frac{1}{2}\min\{\lambda_n :\ \lambda \in Extr(\mu_0, ..., \mu_{n-1})\}.$$

It follows from what was proved in steps I and II that if ϵ is small enough, then every $\widetilde{\lambda} \in Extr(\widetilde{\mu}_0, ..., \widetilde{\mu}_{n-1})$ belongs to $E(\widetilde{\mu}_0, ..., \widetilde{\mu}_{n-1}; a, b, c)$.

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By Lemma (lower semicontinuity), if ϵ is small enough

$$\tau(\widetilde{\mu}_0, ..., \widetilde{\mu}_{n-1}; a, b, c) > 1 + \frac{\tau(\mu_0, ..., \mu_{n-1}; a, b, c) - 1}{2}. \quad (28)$$

Let $\widetilde{\lambda} \in Extr(\widetilde{\mu}_0, ..., \widetilde{\mu}_{n-1})$. Step II implies that there exists $\lambda \in Extr(\mu_0, ..., \mu_{n-1})$ close to $\widetilde{\lambda}$. By step I the values of problems $(\widetilde{6_{\widetilde{\lambda}}})$ and (6_λ) are close, and the same is true for the values of problems $(\widetilde{2})$ and (2).

Proof of Stability Theorem, 4

Since problem (2) is regular, the values of problems (2) and (6_λ) are the same, and, therefore, the values of problems $(\widetilde{2})$ and $(\widetilde{6_{\widetilde{\lambda}}})$ are close. Let $\widetilde{f}$ be an extremal function for problem $(\widetilde{6_{\widetilde{\lambda}}})$.

By Theorem 1 there exists $\tau > 0$ such that

$$T_{\widetilde{\mu}_0}\widetilde{f} = \tau\left(\sum_{j=1}^{n-1}\widetilde{\lambda}_j T_{\widetilde{\mu}_j}\widetilde{f} + \widetilde{\mu}_n\widetilde{f}\right). \tag{29}$$

By Theorem 2 the values of problems $(\widetilde{2})$ and $(\widetilde{6_{\widetilde{\lambda}}})$ are equal to

$$\widetilde{d} = \sum_{j=1}^{n-1}\widetilde{\lambda}_j\delta_j^2 + \widetilde{\lambda}_n, \text{ and } \widetilde{d}_{\widetilde{\lambda}} = \tau\left(\sum_{j=1}^{n-1}\widetilde{\lambda}_j\delta_j^2 + \widetilde{\lambda}_n\right)$$

respectively.

Proof of Stability Theorem, 5

Since every extremal function of problem $(\widetilde{2})$ is admissible for problem $(\widetilde{6_{\widetilde{\lambda}}})$, $\tau \geq 1$. If $\widetilde{d}_{\widetilde{\lambda}} > \widetilde{d}$, then $\tau > 1$, and, hence, (29) shows that $\tau \in S_{\widetilde{\lambda}}$. Thus, $\tau \geq \tau(\widetilde{\mu}_0, ..., \widetilde{\mu}_{n-1}; a, b, c)$. Now, equation (28) shows that $\widetilde{d}$ and $\widetilde{d}_{\widetilde{\lambda}}$ can not be arbitrary close, a contradiction. Thus $\tau = 1$ and problem $(\widetilde{2})$ is regular. The proof is complete.

Commuting operators

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Let $T_1, ..., T_n$ be self-adjoint compact operators. It is very easy to see that if these operators commute, then the joint spectrum $\sigma(T_1, .., T_n, I)$ consists of a locally finite countable union of planes. Of course, if more than one of them is not compact, this is not true, but instead of joint spectrum we might talk about joint point spectrum. It natural to ask whether the converse is true, that is given that the joint spectrum of self-adjoint operators consists of a locally finite countable union of planes, does this imply that the operators commute?

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Recently the affirmative answer was obtained in some cases. The following result (Chagouel, Stessin, 2013) holds.

Joint spectra of commuting operators

Theorem

Suppose that T_1, T_2 are self-adjoint bounded operators on a separable Hilbert space X, and let T_1 be compact. Then, $\sigma(T_1, T_2, I)$ is a locally finite union of $(n-1)$-dimensional planes if and only if T_1 and T_n commute.

Corollary

Let $T_1, ..., T_n$ be self-adjoint bounded operators on a Hilbert space X, and let T_1 be compact with $dim(ker(T_1)) < \infty$. If $\sigma_p(T_1, ..., T_n, I)$ is a countable union of planes, then $T_1, ..., T_n$ mutually commute.