Discrete unbounded sets in a finite dimensional space and quasicrystals

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Signed almost periodic discrete sets: zeros and poles of a certain class of meromorphic functions (F.Sunyer-i-Balaguer 1949, S.Yu.Favorov 2004).

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DEFINITION. $\mathrm{dist}(A, B) = \inf_\sigma \sup_{n\in\mathbb{N}} |a_n - b_{\sigma(n)}|$, infimum is taken over all bijections $\sigma : \mathbb{N} \to \mathbb{N}$.

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Examples

1. $A, B \subset \mathbb{R}, \quad A = \mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\}$,
   $B = \{3, 4, 8, 9, 10, 11, 12, \dots\} \Rightarrow \mathrm{dist}(A, B) = 5$.

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3. $A, B \subset \mathbb{R}^2$, $A = \mathbb{N}^2$, $B = \{(n, m) \in \mathbb{N}^2 : \min\{m, n\} \geq 5\}$ $\Rightarrow \operatorname{dist}(A, B) = 4\sqrt{2}$.

Two definitions

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DEFINITION *A multiple discrete set* is a sequence $A = (a_n)$ in $\mathbb{R}^p$ without finite limit points, in other words,

$$A = \{(x, m(x)) : m(x) > 0\},$$

where $m : \mathbb{R}^p \to \mathbb{N} \cup \{0\}$ is a mapping with a discrete support.

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DEFINITION A discrete set $A \subset \mathbb{R}^p$ is *an S- set* if

$$\forall\, t \in \mathbb{R}^p \qquad \mathrm{dist}(A, A + t) \leq \mathrm{const} < \infty.$$

Properties of S-sets

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Theorem (Kolbasina 2008)

1. Each S-set $A = (a_n)$ is translation–bounded, i.e.,

$$\forall\ c \in \mathbb{R}^p \qquad \#\{n : \ |a_n - c| < 1\} \leq \text{const} < \infty,$$

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2. has a uniform density $\Delta = \Delta(A)$, $0 < \Delta < \infty$, i.e.,

$$\exists \lim_{T\to\infty} \frac{\#\{n : |a_n - c| < T\}}{\omega_p T^p} = \Delta,$$

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3. is a bounded perturbation of the square lattice $\Delta^{-1/p}\mathbb{Z}^p$, i.e., a uniformly spread set in the sense of Laczkovich

$$a_k = \Delta^{-1/p}k + \varphi(k), \quad k \in \mathbb{Z}^p, \quad \varphi:\ \mathbb{Z}^p \to \mathbb{R}^p \text{ bounded}$$

   for a suitable indexing of the set A.

Transportation measure and distance

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Transportation measure between measures ν_1 and ν_2 is a measure γ such that

$$\int\int_{\mathbb{R}^p\times\mathbb{R}^p}\varphi(x)\ d\gamma(x,y)=\int_{\mathbb{R}^p}\varphi(x)\ d\nu_1(x),$$

$$\int\int_{\mathbb{R}^p\times\mathbb{R}^p}\varphi(y)\ d\gamma(x,y)=\int_{\mathbb{R}^p}\varphi(y)\ d\nu_2(y)$$

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The transportation distance $\mathrm{Tra}(\nu_1,\nu_2)$ is the number

$$\mathrm{Tra}(\nu_1,\nu_2) = \inf_\gamma \sup\{|x-y|:\ x,y \in \overline{\mathrm{supp}\gamma}\},$$

where the infimum is taken over all transportation measures (M. Sodin, B. Tsirelson 2009).

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$\mathrm{Tra}(\nu, \Delta^{-1/p} m_p) < \infty$, where m_p is the Lebesgue measure in $\mathbb{R}^p$.

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A continuous mapping $f : \mathbb{R}^p \to \mathbb{R}^l$ is *almost periodic*, if for any $\varepsilon > 0$ the set of ε-almost periods of f

$$\{\tau \in \mathbb{R}^p : \sup_{x \in \mathbb{R}^p} |f(x+\tau) - f(x)| < \varepsilon\}$$

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A discrete set $A = (a_n) \subset \mathbb{R}^p$ is *almost periodic*, if for any continuous function ϕ in $\mathbb{R}^p$ with a compact support the sum $\sum_n \phi(x + a_n)$ is an almost periodic function in $x \in \mathbb{R}^p$.

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Every discrete set of the form

$$a_k = \Delta^{-1/p} k + \varphi(k), \qquad k \in \mathbb{Z}^p,$$

with an almost periodic mapping $\varphi(x) : \mathbb{R}^p \to \mathbb{R}^p$ is almost periodic.
Conversely, for $p = 1$ every almost periodic discrete set has such form with an almost periodic mapping $\varphi : \mathbb{R} \to \mathbb{R}$. The last assertion does not valid for $p > 1$.

Almost periodic discrete sets and quasicrystals

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Theorem (Rashkovskii, Ronkin & me 1998)

A discrete set $A \subset \mathbb{R}^p$ is almost periodic if and only if for each $\varepsilon > 0$ the set of ε-almost periods of A

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Theorem (F. 2012)

If $A \subset \mathbb{R}^p$ is an almost periodic discrete set of a finite type, i.e., $A - A$ is discrete, then A is a pure crystal, i.e., there exists a lattice L and $c_1, c_2, \dots c_r \in \mathbb{R}^p$ such that $A = \cup_{j=1}^r (L + c_j)$.

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Theorem (Girya & me 2012)

If differences of zeros of $S(z)$ form a discrete set, then S(z) is periodic and, therefore,

$$S(z) = Ke^{i\beta z} \prod_{k=1}^{N} \sin(\omega z + b_k), \quad \omega, \beta \in \mathbb{R}, \quad K, b_k \in \mathbb{C}.$$

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The result is valid for entire almost periodic functions of exponential type with zeros in the horizontal strip of a finite width (so-called class Δ) as well.

Almost periodic signed multiple discrete sets

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A set $A = \{(x, m(x)) : m(x) \neq 0\}$, where $m : \mathbb{R}^p \to \mathbb{Z}$ is a mapping with a discrete support, is called *signed multiple discrete set*.

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A signed multiple discrete set A is *almost periodic*, if for any continuous function ϕ in $\mathbb{R}^p$ with a compact support the sum

$$\sum_{a \in \mathbb{R}^p} m(a)\phi(x + a) \qquad (*)$$

is an almost periodic function in $x \in \mathbb{R}^p$.

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Theorem (Kolbasina & me 2012)

There is a signed multiple discrete set such that $(*)$ is almost periodic for any test–function $\phi \in C^1$, but there is a continuous test–function ϕ such that $(*)$ is not almost periodic;

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$A_0 = \{x : m(x) \neq 0\}$ is not almost periodic.

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Thanks for your attention!