Discrete unbounded sets in a finite dimensional space and quasicrystals

S. FAVOROV

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Istanbul, March, 2013
The Problem

Discrete sets in the plane: zeros of entire functions.

Discrete sets in a finite dimensional space: Lacshkovich's uniformly spread discrete sets, mathematical models of quasicrystals.


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The distance between discrete sets

Let $A = \{a_n\}_{n \in \mathbb{N}}$ and $B = \{b_n\}_{n \in \mathbb{N}}$ be discrete sets in $\mathbb{R}^p$.

**DEFINITION.**

$$\text{dist}(A, B) = \inf \sigma \sup_{n \in \mathbb{N}} |a_n - b_{\sigma(n)}|,$$

infimum is taken over all bijections $\sigma : \mathbb{N} \to \mathbb{N}$.

**Examples**

1. $A, B \subset \mathbb{R}$, $A = \mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$, $B = \{3, 4, 8, 9, 10, 11, 12, \ldots\} \Rightarrow \text{dist}(A, B) = 5$.

2. $A, B \subset \mathbb{R}$, $A = \mathbb{N}$, $B = \{2, 4, 6, 8, 10, \ldots\} \Rightarrow \text{dist}(A, B) = \infty$.

3. $A, B \subset \mathbb{R}^2$, $A = \mathbb{N}^2$, $B = \{(n, m) \in \mathbb{N}^2 : \min\{m, n\} \geq 5\} \Rightarrow \text{dist}(A, B) = 4\sqrt{2}$. 

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Two definitions

DEFINITION
A multiple discrete set
$A = (a_n)$ in $\mathbb{R}^p$ without finite limit points, in other words,
$A = \{(x, m(x)) : m(x) > 0\}$,
where $m : \mathbb{R}^p \to \mathbb{N} \cup \{0\}$ is a mapping with a discrete support.

DEFINITION
A discrete set $A \subset \mathbb{R}^p$ is an S-set if $\forall t \in \mathbb{R}^p$ $\text{dist}(A, A + t) \leq \text{const} < \infty$. 
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Properties of $S$-sets

Theorem (Kolbasina 2008)

1. Each $S$-set $A = (a_n)$ is translation–bounded, i.e.,
   \[ \forall c \in \mathbb{R}^p \# \{n : |a_n - c| < 1 \} \leq \text{const} < \infty, \]

2. It has a uniform density $\Delta = \Delta(A)$, $0 < \Delta < \infty$, i.e.,
   \[ \exists \lim_{T \to \infty} \# \{n : |a_n - c| < T \} \approx \omega_p T^p = \Delta, \]
   uniformly in $c \in \mathbb{R}^p$, where $\omega_p$ is the volume of the unit ball,

3. It is a bounded perturbation of the square lattice $\Delta - 1/p \mathbb{Z}^p$, i.e., it has a uniformly spread set in the sense of Laczkovich:
   \[ a_k = \Delta - 1/p k + \varphi(k), \quad k \in \mathbb{Z}^p, \]
   $\varphi: \mathbb{Z}^p \to \mathbb{R}^p$ bounded for a suitable indexing of the set $A$. 
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   a_k = \Delta^{-1/p} k + \varphi(k), \quad k \in \mathbb{Z}^p, \quad \varphi : \mathbb{Z}^p \to \mathbb{R}^p \text{ bounded}
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   for a suitable indexing of the set $A$. 
Transportation measure and distance

Transportation measure and distance

between measures $\nu_1$ and $\nu_2$ is a measure $\gamma$ such that

$$\int \int_{R^p \times R^p} \phi(x) \, d\gamma(x, y) = \int_{R^p} \phi(x) \, d\nu_1(x),$$

$$\int \int_{R^p \times R^p} \phi(y) \, d\gamma(x, y) = \int_{R^p} \phi(y) \, d\nu_2(y),$$

for all continuous functions $\phi: R^p \to R$ with a compact support.

The transportation distance $\text{Tra}(\nu_1, \nu_2)$ is the number

$$\text{Tra}(\nu_1, \nu_2) = \inf_{\gamma} \sup \{ |x - y| : x, y \in \text{supp} \gamma \},$$

where the infimum is taken over all transportation measures (M. Sodin, B. Tsirelson 2009).

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**Transportation measure and distance**

*Transportation measure* between measures $\nu_1$ and $\nu_2$ is a measure $\gamma$ such that

$$\int \int_{\mathbb{R}^p \times \mathbb{R}^p} \varphi(x) \, d\gamma(x, y) = \int_{\mathbb{R}^p} \varphi(x) \, d\nu_1(x),$$

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for all continuous functions $\varphi : \mathbb{R}^p \to \mathbb{R}$ with a compact support.
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Theorem (Dudko & me 2011)

If a transportation distance between a measure $\nu \not\equiv 0$ and any its shift $\nu^z$ is uniformly bounded with respect to $z \in \mathbb{R}^p$, then
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If a transportation distance between a measure $\nu \not= 0$ and any its shift $\nu^z$ is uniformly bounded with respect to $z \in \mathbb{R}^p$, then

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$$\Delta = \lim_{T \to \infty} \nu(\{x : |x - c| < T\})(\omega_p T^p)^{-1}$$

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Theorem (Dudko & me 2011)

If a transportation distance between a measure $\nu \neq 0$ and any its shift $\nu^z$ is uniformly bounded with respect to $z \in \mathbb{R}^p$, then

a positive finite density

$$\Delta = \lim_{T \to \infty} \nu(\{x : |x - c| < T\})(\omega_p T^p)^{-1}$$

exists uniformly with respect to $c \in \mathbb{R}^p$,

$$\text{Tra}(\nu, \Delta^{-1/p} m_p) < \infty,$$ where $m_p$ is the Lebesgue measure in $\mathbb{R}^p$. 
Almost periodic discrete sets

A continuous mapping \( f: \mathbb{R}^p \rightarrow \mathbb{R}^l \) is almost periodic, if for any \( \varepsilon > 0 \) the set of \( \varepsilon \)-almost periods of \( f \) \( \{ \tau \in \mathbb{R}^p : \sup_{x \in \mathbb{R}^p} |f(x + \tau) - f(x)| < \varepsilon \} \) is a relatively dense set in \( \mathbb{R}^p \), i.e., there is \( r = r(\varepsilon) < \infty \) such that any ball of radius \( r \) contains an \( \varepsilon \)-almost period of \( f \).

A discrete set \( A = (a_n) \subset \mathbb{R}^p \) is almost periodic, if for any continuous function \( \phi \) in \( \mathbb{R}^p \) with a compact support the sum \( \sum_n \phi(x + a_n) \) is an almost periodic function in \( x \in \mathbb{R}^p \).

Every discrete set of the form \( a_k = \Delta - \frac{1}{p} k + \phi(k) \), \( k \in \mathbb{Z}^p \), with an almost periodic mapping \( \phi: \mathbb{R}^p \rightarrow \mathbb{R}^p \) is almost periodic.

Conversely, for \( p = 1 \) every almost periodic discrete set has such form with an almost periodic mapping \( \phi: \mathbb{R} \rightarrow \mathbb{R} \). The last assertion does not valid for \( p > 1 \).
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A continuous mapping $f : \mathbb{R}^p \to \mathbb{R}^l$ is *almost periodic*, if for any $\varepsilon > 0$ the set of $\varepsilon$-almost periods of $f$

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Every discrete set of the form

$$a_k = \Delta^{-1/p} k + \varphi(k), \quad k \in \mathbb{Z}^p,$$

with an almost periodic mapping $\varphi(x) : \mathbb{R}^p \to \mathbb{R}^p$ is almost periodic. Conversely, for $p = 1$ every almost periodic discrete set has such form with an almost periodic mapping $\varphi : \mathbb{R} \to \mathbb{R}$. The last assertion does not valid for $p > 1$. 
Almost periodic discrete sets and quasicrystals

Theorem (Rashkovskii, Ronkin & me 1998)

A discrete set $A \subset \mathbb{R}^p$ is almost periodic if and only if for each $\varepsilon > 0$ the set of $\varepsilon$-almost periods of $A$

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is relatively dense in $\mathbb{R}^p$.

Each almost periodic discrete set is an $S$-set, hence it is translation-bounded and has a uniform density $\Delta \in (0, \infty)$.

A positive answer on Lagarias' question (2000):

Theorem (F. 2012)

If $A \subset \mathbb{R}^p$ is an almost periodic discrete set of a finite type, i.e., $A - A$ is discrete, then $A$ is a pure crystal, i.e., there exists a lattice $L$ and $c_1, c_2, ..., c_r \in \mathbb{R}^p$ such that

$$A = \bigcup_{j=1}^{r} (L + c_j).$$
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Consider a finite exponential sum

\[ S(z) = \sum_{n} a_n e^{i\lambda_n z}, \quad \lambda_n \in \mathbb{R}, \quad a_n \in \mathbb{C}. \]

Theorem (Girya & me 2012)
If differences of zeros of \( S(z) \) form a discrete set, then \( S(z) \) is periodic and, therefore,

\[ S(z) = Ke^{i\beta z}N \prod_{k=1} e^{i(\omega z + b_k)}, \quad \omega, \beta \in \mathbb{R}, \quad K, b_k \in \mathbb{C}. \]

The result is valid for entire almost periodic functions of exponential type with zeros in the horizontal strip of a finite width (so-called class \( \Delta \)) as well.

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\[ S(z) = K e^{i\beta z} \prod_{k=1}^{N} \sin(\omega z + b_k), \quad \omega, \beta \in \mathbb{R}, \quad K, b_k \in \mathbb{C}. \]
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Almost periodic signed multiple discrete sets

A set \( A = \{ (x, m(x)) : m(x) \neq 0 \} \), where \( m : \mathbb{R}^p \rightarrow \mathbb{Z} \) is a mapping with a discrete support, is called a signed multiple discrete set.

A signed multiple discrete set \( A \) is almost periodic, if for any continuous function \( \phi \) in \( \mathbb{R}^p \) with a compact support the sum

\[
\sum_{a \in \mathbb{R}^p} m(a) \phi(x + a)
\]

is an almost periodic function in \( x \in \mathbb{R}^p \).

Theorem (Kolbasina & me 2012)

There is a signed multiple discrete set such that (*) is almost periodic for any test-function \( \phi \in C^1 \), but there is a continuous test-function \( \phi \) such that (*) is not almost periodic;

there is an almost periodic signed discrete set \( A = \{ (x, m(x)) : m(x) = \pm 1 \} \) such that the discrete set \( A_0 = \{ x : m(x) \neq 0 \} \) is not almost periodic.
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There is a signed multiple discrete set such that $(\ast)$ is almost periodic for any test–function $\phi \in C^1$, but there is a continuous test–function $\phi$ such that $(\ast)$ is not almost periodic; there is an almost periodic signed discrete set $A = \{(x, m(x)) : m(x) = \pm 1\}$ such that the discrete set $A_0 = \{x : m(x) \neq 0\}$ is not almost periodic.
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There is a signed multiple discrete set such that (\(*\)) is almost periodic for any test–function \( \phi \in \mathcal{C}^1 \), but there is a continuous test–function \( \phi \) such that (\(*\)) is not almost periodic; there is an almost periodic signed discrete set \( A = \{(x, m(x)) : m(x) = \pm 1\} \) such that the discrete set \( A_0 = \{x : m(x) \neq 0\} \) is not almost periodic.
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\sum_{a \in \mathbb{R}^p} m(a)\phi(x + a)
\]

\( (*) \)

is an almost periodic function in \( x \in \mathbb{R}^p \).

\textbf{Theorem (Kolbasina & me 2012)}

There is a signed multiple discrete set such that \( (*) \) is almost periodic for any test–function \( \phi \in C^1 \), but there is a continuous test–function \( \phi \) such that \( (*) \) is not almost periodic;
Almost periodic signed multiple discrete sets

A set \( A = \{(x, m(x)) : m(x) \neq 0\} \), where \( m : \mathbb{R}^p \to \mathbb{Z} \) is a mapping with a discrete support, is called signed multiple discrete set.

A signed multiple discrete set \( A \) is almost periodic, if for any continuous function \( \phi \) in \( \mathbb{R}^p \) with a compact support the sum

\[
\sum_{a \in \mathbb{R}^p} m(a) \phi(x + a)
\]

is an almost periodic function in \( x \in \mathbb{R}^p \).

\textbf{Theorem (Kolbasina & me 2012)}

There is a signed multiple discrete set such that (*) is almost periodic for any test–function \( \phi \in C^1 \), but there is a continuous test–function \( \phi \) such that (*) is not almost periodic;

there is an almost periodic signed discrete set

\( A = \{(x, m(x)) : m(x) = \pm1\} \) such that the discrete set

\( A_0 = \{x : m(x) \neq 0\} \) is not almost periodic.
Theorem (Kolbasina & me 2012)
Each almost periodic signed multiple discrete set
\[ A = \{ (x, m(x)) \} \]
is translation-bounded, i.e.,
\[ \sum_{x} |x - c| < 1 \]
\[ |m(x)| \leq \text{const} < \infty \]
uniformly in \( c \in \mathbb{R}^p \), it has a uniform density \( \Delta = \Delta(A) \), \( -\infty < \Delta < \infty \), i.e.,
\[ \exists \lim_{T \to \infty} \sum_{x} |x - c| < T m(x) \omega_p T^p = \Delta, \]
uniformly with respect to \( c \in \mathbb{R}^p \).
Theorem (Kolbasina & me 2012)

Each almost periodic signed multiple discrete set $A = \{(x, m(x))\}$ is translation-bounded, i.e.,

$$\sum_{x: |x-c|<1} |m(x)| \leq \text{const} < \infty$$

uniformly in $c \in \mathbb{R}^p$, 

Some properties
Some properties

Theorem (Kolbasina & me 2012)

Each almost periodic signed multiple discrete set \( A = \{(x, m(x))\} \) is translation-bounded, i.e.,

\[
\sum_{x:|x-c|<1} |m(x)| \leq \text{const} < \infty
\]

uniformly in \( c \in \mathbb{R}^p \),

it has a uniform density \( \Delta = \Delta(A) \), \(-\infty < \Delta < \infty\), i.e.,

\[
\exists \lim_{T \to \infty} \frac{\sum_{x:|x-c|<T} m(x)}{\omega_p T^p} = \Delta,
\]

uniformly with respect to \( c \in \mathbb{R}^p \).
Thanks for your attention!