On the structure of fractional spaces generated by the positive operators

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Abstract

The aim of this paper is to present the results on the structure of fractional spaces generated by the positive operators and to formulate the several open problems in this area.

1 Introduction

It is a well-known

KREIN S. G., *Linear Differential Equations in a Banach Space.* Moscow: Nauka, 1966. (Russian)

GRISVARD, P., *Elliptic Problems in Nonsmooth Domains*, Pitman Advanced Publishing Program – London, 1986–210p.

FATTORINI H.O., *Second Order Linear Differential Equations in Banach Spaces*, North-Holland: Mathematics Studies 108,1985–313p.

that various local and nonlocal boundary value problems for PDE can be reduced to the abstract boundary value problem for ODE in a Banach space E with an unbounded differential operator A. The study of the various properties of PDE is based on a positivity property of this differential operator in a Banach space. The positivity of the wider class of differential operators have been studied by Yosida K. (Japan), Kato T. (Japan, USA), Agmon S. (Israel), Friedman A. (USA), Solomyak M.Z. (USSR, Israel), Sobolevskii P.E. (USSR, Israel), Stewart H.B. (USA) et al.

AGMON S., *Lectures on Elliptic Boundary Value Problems*, D.Van Nostrand,Princeton,New Jersey,1965.

FRIEDMAN A., *Partial Differential Equations*, Holt,Rinehart and Winston, New York, 1969.

SOLOMYAK M.Z. *Analytic semigroups generated by elliptic operator in space L_p* , Dokl. Acad. Nauk SSSR **127** (1959), No. 1, 37—39. (Russian)

SOLOMYAK M.Z. *Estimation of norm of the resolvent of elliptic operator in spaces L_p*, Usp. Mat. Nauk **15** (1960), No.6, 141—148. (Russian)

Krasnosel'skii,M.A.,Zabreiko,P.P.,Pustyl'nik,E.I., and Sobolevkii,P.E., *Integral Operators in Spaces of Summable Functions,* Noordhoff, Leiden, 1976.

STEWART H.B. *Generation of analytic semigroups by strongly elliptic operators*, Trans.Amer.Math.Soc.**190** (1974), 141–162.

STEWART H.B. *Generation of analytic semigroups by strongly elliptic operators under general boundary conditions,* Trans.Amer.Math.Soc. **259** (1980),299–310.

Important progress has been made in the study of positive operators from the viewpoint of the stability analysis of high order accuracy difference schemes for PDE. It is a well-known the most useful methods for stability analysis of difference schemes are a difference analogue of maximum principle and energy method. The application of theory of positive difference operators permits us to investigate the stability and coercive stability properties of difference schemes in various norms for PDE specially when we can not able to use of a maximum principle and energy method. An excellent survey of works in this area is given in the books

ASHYRALYEV A. AND SOBOLEVSKII P.E., *Well-Posedness of Parabolic Difference Equations.* Birkhäuser Verlag, Basel, Boston, Berlin (1994), 349 p.

ASHYRALYEV A., *Method of Positive Operators of Investigations of High Order Difference Schemes for Parabolic and Elliptic Equations*, Doctor Sciences Thesis – Ashgabat, 1991, 312 p. (Russian)

The aim of this paper is to present the known facts, new results and open problems in this area.

2 Positive Operators

Definition. The operator A is said to be positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle $\varphi, 0 < 2\varphi < 2\pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1 = [\rho \exp(i\varphi) : 0 \leq \rho < \infty]$ and $S_2 = [\rho \exp(-i\varphi) : 0 \leq \rho < \infty]$,

and outside it the resolvent $(\lambda - A)^{-1}$ is subject to the bound

$$\|(\lambda - A)^{-1}\|_{E \to E} \leq \frac{M(\varphi)}{1 + |\lambda|}$$

The infimum of such angles φ is called the the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A, E)$.

Three cases arise,according with values of spectral angle:
i. $\varphi(A) < \frac{\pi}{2}$. An operator A is said to be strongly positive in a Banach space E.

ii. $\varphi(A) < \pi$. An operator A is said to be positive in a Banach space E.
iii. $\varphi(A) = \pi$. An operator A is said to be weakly positive in a Banach space E.
For example we consider $E = H$ is a Hilbert space, $A = A^* \geq \delta I \quad (\delta > 0)$ with dense domain $\bar{D}(A) = H$. Using the spectral representation of self-adjoint positive definite operators we can write

$$(\lambda - A)^{-1}u = \int_{\delta-}^{\infty} \frac{dE_\mu u}{\mu - \lambda}.$$

Therefore

$$\|(\lambda - A)^{-1}\|_{H \to H} \leq \sup_{\delta \leq \mu < \infty} \frac{1}{|\mu - \lambda|}.$$

We consider two cases: $\mathrm{Re}\lambda \leq \frac{\delta}{2}$ and $\mathrm{Re}\lambda > \frac{\delta}{2}$, $\varepsilon < \varphi < \frac{\pi}{2}$. In the case $\mathrm{Re}\lambda \leq \frac{\delta}{2}$ we have two estimates:

$$|\mu - \lambda| \geq \mu - \frac{\delta}{2} \geq \frac{\delta}{2},$$

$$|\mu - \lambda| = \sqrt{(\mu - 2\mathrm{Re}\lambda)\mu + |\lambda|^2} \geq |\lambda|.$$

Therefore we have that

$$|\mu - \lambda| \geq \frac{1}{2}(\frac{\delta}{2} + |\lambda|).$$

In the case $\mathrm{Re}\lambda > \frac{\delta}{2}$ we have that

$$|\mu - \lambda| = \sqrt{(\mu - |\lambda|\cos\varphi)^2 + |\lambda|^2 \sin(\varphi)} \geq |\lambda| \sin\varphi \geq |\lambda| \sin\epsilon.$$

and

$$|\mu - \lambda| \geq \frac{1}{2}(|\lambda| + \mathrm{Re}\lambda)\sin\epsilon \geq \frac{1}{2}(|\lambda| + \frac{\delta}{2})\sin\epsilon.$$

Therefore we have that

$$|\mu - \lambda| \geq \frac{\sin\epsilon}{2}(|\lambda| + \frac{\delta}{2}).$$

Thus we obtain that

$$\|(\lambda - A)^{-1}\|_{H \to H} \leq \frac{M(\epsilon)}{1 + |\lambda|},$$

and $\varphi(A) = 0$. So, the positivity of operators in a Banach space is the generalization of the notion of self-adjoint positive definite operators in a Hilbert space.

Now, we want to present the results of the theory of positive operators in a Banach space and to formulate several open problems in this area.

First, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbf{R}^n$ $(0 < x_k < 1,\ \ 1 \leq k \leq n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. We introduce the Banach spaces $C(\overline{\Omega})$, $C_{01}^{\beta}(\overline{\Omega})$ $(\beta = (\beta_1, ..., \beta_n),\ \ 0 < \beta_k < 1,\ \ k = 1, ..., n)$ of continuous functions for which the following norms are finite

$$\| f \|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|,$$

$$\| f \|_{C_{01}^{\beta}(\overline{\Omega})} = \| f \|_{C(\overline{\Omega})}$$
$$+ \sup_{x \in \overline{\Omega_h}} |f(x_1, ..., x_n) - f(x_1 + h_1, ..., x_n + h_n)| \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k}.$$

The differential expression

$$Av = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(x)}{\partial x^2} \tag{2.1}$$

defines a positive operator A acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S.

The strong positivity in the $C_{01}^{\beta}(\overline{\Omega_h})$ norms (difference analogues of weighted Hölder spaces) for an elliptic operator A_h of second order of accuracy that approximates this multidimensional elliptic operator without mixed derivatives was established for the first time in paper

Sobolevskii, P.E. *The coercive solvability of difference equations*, Dokl. Acad. Nauk SSSR **201** (1971), No. 5, 1063–1066. (Russian)

The strong positivity of the this second-order elliptic difference operator in the $L_p(\overline{\Omega}_h)-$ and $C(\overline{\Omega}_h)-$ norms was established in papers

Alibekov,Kh.A., and Sobolevskii, P.E. *Stability of difference schemes for parabolic equations*, Dokl. Acad. Nauk SSSR **232** (1977), No. 4, 737—740. (Russian)

Alibekov,Kh.A., and Sobolevskii, P.E. *Stability and convergence of difference schemes of a high order for parabolic differential equations*, Ukrain.Mat.Zh. **31** (1979), No. 6, 627—634. (Russian)

Problem 1. Find the strong positive difference operator A_h of the arbitrary order of accuracy that approximates this multidimensional elliptic operator without mixed derivatives under the Dirichlet or Neumann boundary conditions. Investigate the positivity property of A_h in $L_p(\overline{\Omega}_h)$, and $C(\overline{\Omega}_h)-$ norms.

Second,we introduce the Banach spaces $C(\mathbf{R}^n)$, $C^\beta(\mathbf{R}^n)(0<\beta<1)$ of all continuous bounded functions defined on $\mathbf{R}^n$ and for which the following norms are finite

$$\| f \|_{C(\mathbf{R}^n)}= \sup_{x\in\mathbf{R}^n} |f(x)|,$$

$$\| f \|_{C^\beta(\mathbf{R}^n)}=\| f \|_{C(\mathbf{R}^n)} + \sup_{x\neq y;x,y\in\mathbf{R}^n} |f(x)-f(y)||x-y|^{-\beta}.$$

We will assume that the symbol

$$B(\xi)=\sum_{|\tau|=2m} a_r(x)\,(i\xi)^{r_1}\dots(i\xi)^{r_n}$$

of the differential operator of the form

$$B=\sum_{|r|=2m} a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}$$

acting on functions defined on the space $\mathbf{R}^n$, satisfies the inequalities

$$0<M_1|\xi|^{2m}\leq(-1)^m B((\xi)\leq M_2|\xi|^{2m}<\infty.$$

for $\xi\neq 0$. For sufficiently large $\delta>0$

$$A=\sum_{|r|=2m} a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}+\delta I \tag{2.2}$$

defines a strongly positive operator in $C^\beta(\mathbf{R}^n)(0\leq\beta<1)$.

The strong positivity in the $L_p(\mathbf{R}_h^n)(1\leq p\leq\infty), C(\mathbf{R}_h^n)$ norms (difference analogues of $L_p(\mathbf{R}^n), C(\mathbf{R}^n,)$ spaces,respectively) for a difference elliptic operator A_h of an arbitrary high order of accuracy that approximates this multidimensional elliptic operator A was established in the papers

Smirnitskii Yu.A., and Sobolevskii, P.E. *Positivity of multidimensional difference operators in the $C-$norm*, Uspekhi.Mat.Nauk **36** (1981), No. 4, 202—203. (Russian)

Smirnitskii Yu.A., and Sobolevskii, P.E. *Positivity of difference operators*, in.Spline Methods,Novosibirsk (1981). (Russian)

Smirnitskii Yu.A., and Sobolevskii, P.E. *Pointwise estimates of the Green function of a difference elliptic operator*, in: Vychisl. Methody Mekh.Sploshn.Sredy **15** (1982),No.4,129—142. (Russian)

Smirnitskii Yu.A., and Sobolevskii, P.E. *Pointwise estimates of the Green function of the resolvent of a difference elliptic operator with variable coefficients in R^n*, Voronezh.Gosud.Uni.1982,32p.,Deposited VINITI 5.2.1982,No1519.(Russian)

Later, the strong positivity in the $W_p^\beta(\mathbf{R}_h^n)$ $(1\leq p\leq\infty,\ 0<\beta<1)$,

$C^\beta(\mathbf{R}_h^n)$ $(0<\beta<1)$ norms for the same difference elliptic operator A_h of an arbitrary high order of accuracy is established in

Ashyralyev A. and Sobolevskii P.E., *Well-Posedness of Parabolic Difference Equations.* Birkhäuser Verlag, Basel, Boston, Berlin (1994), 349 p.

Ashyralyev A. and Sobolevskii P. E. *Theory of the interpolation of the linear operators and the stability of the difference schemes*, Dokl. Akad. Nauk SSSR **275**(1984), No 6, pp. 1289—1291 (Russian).

Here $W_p^\beta(\mathbf{R}_h^n)$ is the space of grid functions u^h defined by the norm

$$\|u^h\|_{W_p^\beta(\mathbf{R}_h^n)} = \left(\sum_{x\in\mathbf{R}_h^n} \sum_{z\in\mathbf{R}_h^n, z\neq 0} \frac{|u^h(x) - u^h(x+z)|^p h^{2n}}{|z|^{n+\beta p}} + \sum_{x\in\mathbf{R}_h^n} |u^h(x)|^p h^n \right)^{\frac{1}{p}}.$$

Now,we introduce the Banach spaces $C([0,\infty)\times\mathbf{R}^n)$, $C^\beta([0,\infty)\times\mathbf{R}^n)(0<\beta<1)$ of all continuous bounded functions defined on $[0,\infty)\times$ $\mathbf{R}^n$ and for which the following norms are finite

$$\| f \|_{C([0,\infty)\times\mathbf{R}^n)} = \sup_{x\in[0,\infty)\times\mathbf{R}^n} |f(x)|,$$

$$\| f \|_{C^\beta([0,\infty)\times\mathbf{R}^n)} = \| f \|_{C([0,\infty)\times\mathbf{R}^n)} + \sup_{x\neq y; x,y\in[0,\infty)\times\mathbf{R}^n} |f(x)-f(y)||x-y|^{-\beta}.$$

We will assume that the symbol

$$B(\xi) = \sum_{|\tau|=2m} a_r(x)\,(i\xi)^{r_1} \ldots (i\xi)^{r_n}$$

of the differential operator of the form

$$B = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}$$

acting on functions defined on the space $[0,\infty)\times$ $\mathbf{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \leq (-1)^m B((\xi) \leq M_2|\xi|^{2m} < \infty.$$

for $\xi\neq 0$. For sufficiently large $\delta>0$ and $a(x)>0$,

the differential operator A of the form

$$A = (-1)^m a(x) \frac{\partial^{2m}\cdot}{\partial x_{n+1}^{2m}} + \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}\cdot}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}} \tag{2.3}$$

with

D(A) $=$ $\{u : Au(x_{n+1},x)\epsilon C([0,\infty)\times\mathbf{R}^n), u(0,x)=0, \frac{\partial u(0,x)}{\partial x_{n+1}}=0,\cdots\frac{\partial^{m-1}u(0,x)}{\partial x_{n+1}^{m-1}} = 0, x\epsilon R^n\}$ defines a strongly positive operator in $C^\beta([0,\infty)\times\mathbf{R}^n)(0\leq\beta<1)$.

The strong positivity in the $C([0,\infty)_h\times\mathbf{R}_h^n)$ norms (difference analogues of , $C([0,\infty)\times$ $\mathbf{R}^n$,) spaces,respectively) for a difference elliptic operator A_h of an arbitrary high order of accuracy that approximates this multidimensional elliptic operator A was established in the papers

Danelich, S. I., *Positive difference operators in R_{h1}*, Voronezh. Gosud. Univ. 1987, 13p. Deposited VINITI 3. 18. 1987, No. 1936-B87. (Russian).

Danelich, S. I.,*Positive difference operators with constant coefficients in half-space*, Voronezh. Gosud. Univ. 1987, 56p. Deposited VINITI 11. 5. 1987, No. 7747-B87. (Russian).

Danelich, S. I., *Positive difference operators with variable coefficients on the half-line*, Voronezh. Gosud. Univ. 1987, 16p. Deposited VINITI 11. 9. 1987, No. 7713-B87. (Russian).

Problem 2. Investigate the positivity property of A_h in $L_p([0,\infty)_h \times \mathbf{R}_h^n)(1 \leq p \leq \infty)$ norm.

Third, we introduce the Banach space $C^\beta[0,1]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ for which the following norm is finite

$$\| \varphi \|_{C^\beta[0,1]} = \| \varphi \|_{C[0,1]} + \sup_{0 \leq x \leq x+\tau \leq 1} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^\beta},$$

where $C[0,1]$ is the space of all continuous functions $\varphi(x)$ defined on [0,1] with the usual norm

$$\| \varphi \|_{C[0,1]} = \max_{0 \leq x \leq 1} |\varphi(x)|.$$

For $\delta > 0, a(x) \geq 0$ the differential expression

$$Av = -a(x)v''(x) + \delta v(x) \tag{2.4}$$

defines a positive operator A acting in $C^\beta[0,1]$ with domain $C^{\beta+2}[0,1]$ and satisfying the condition $v(0) = v(1),\ v_x(0) = v_x(1)$.

Now, we consider the difference operator A_h^x a first order of approximation of the differential operator A^x,defined by the formula

$$A_h^x u^h = \left\{ -a(x_k) \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{M-1},\ u_h = \{u_k\}_0^M, Mh = 1$$

with $u_0 = u_M$ and $u_1 - u_0 = u_M - u_{M-1}$ and the difference operator A_h^x a second order of approximation of the differential operator A^x,defined by the formula

$$A_h^x u^h = \left\{ -a(x_k) \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{M-1},\ u_h = \{u_k\}_0^M, Mh = 1$$

with $u_0 = u_M$ and $-u_2 + 4u_1 - 3u_0 = u_{M-2} - 4u_{M-1} + 3u_M$.

The strong positivity in the $C[0,1]_h,\ C^\beta[0,1]_h$ $(0 < \beta < 1)$ norms (difference analogues of $C[0,1], C^\beta[0,1]$) spaces, respectively) for these difference operators A_h of first and second order of approximation of this differential operator A was established in papers

Ashyralyev, A. and Kendirli, B., *Positivity in C_h of one-dimensional difference operators with nonlocal boundary conditions*, in: Some Problems of Applied Mathematics, Fatih University, Istanbul, 2000, 56–70.

Ashyralyev, A. and Kendirli, B., *Positivity in Hölder norms of one-dimensional difference operators with nonlocal boundary conditions*, in: Application of Mathematics in Engineering and Economics-26, Heron Press and Technical University of Sofia(2001), 134-137.

Ashyralyev, A. and Yenial-Altay N., *Positivity of difference operators generated by the nonlocal boundary conditions.* 5-10 July- 2004, Antalya, Turkey-Dynamical Systems and Applications, Proceedings,113-135.

Problem 3. Find the strong positive difference operator A_h of the arbitrary order of accuracy that approximates this one-dimensional differential operator (2.10) under the nonlocal boundary conditions. Investigate the positivity property of A_h in $L_p[0,1]_h, C^\beta[0,1]_h$ and $C[0,1]_h-$ norms. Find the strong positive difference operator A_h of the approximation of the strong positive multidimensional differential operator under the nonlocal boundary conditions.

3 The Structure of Fractional Spaces Generated by Positive Operators

Let A be a strongly positive operator. With the help of A we introduce the fractional spaces $E_{\alpha,p} = E_{\alpha,p}(A,E)$ $(0 < \alpha < 1, 1 \leq p \leq \infty)$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{\alpha,p} = \left(\int_0^\infty \|\lambda^\alpha A(\lambda + A)^{-1} v\|_E^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < \infty,$$
$$\|v\|_\alpha = \sup_{\lambda > 0} \|\lambda^\alpha A(\lambda + A)^{-1} v\|_E \quad \text{if} \quad p = \infty.$$

Spaces $E_{\alpha,p}$ arise in the theory of interpolation of linear operators (see

Triebel,H., *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, New York, 1978.)

We are interested in the structure of fractional spaces $E_{\alpha,p}$. First, we consider the difference operator A_h of an arbitrary high order of accuracy that approximates a multidimensional elliptic operator A of $2m$-th order.In

Ashyralyev A. and Sobolevskii P.E., *Well-Posedness of Parabolic Difference Equations.* Birkhäuser Verlag, Basel, Boston, Berlin (1994), 349 p.

Ashyralyev A. and Sobolevskii P. E. *Theory of the interpolation of the linear operators and the stability of the difference schemes*, Dokl. Akad. Nauk SSSR **275**(1984), No 6, pp. 1289—1291 (Russian).

it was established that for any $0 < \alpha < \frac{1}{2m}$ and $1 \leq p \leq \infty$ the norms in the spaces $E_{\alpha,p}(A_h, L_p(\mathbf{R}_h^n)$ and $W_p^{2m\alpha}(\mathbf{R}_h^n)$ are equivalent uniformly in h. This fact corresponds to the following equality, known in interpolation theory

Triebel,H., *Interpolation Theory, Function Spaces,Differential Operators,* North-Holland, Amsterdam, New York, 1978.

$E_{\alpha,p}(A, L_p(\mathbf{R}^n)$ and $W_p^{2m\alpha}(\mathbf{R}^n)$, $0 < \alpha < \frac{1}{2m}$, $1 < p < \infty$, are equivalent which in turn follows from the equality $D(A) = W_p^{2m}(\mathbf{R}^n)$ for an $2m$-th order elliptic operator A in $L_p(\mathbf{R}^n)$, $1 < p < \infty$, via the real interpolation method. The alternative method of investigation adopted in based on estimates of fundamental solution of the resolvent equation for the operator A_h, allows us to consider also the cases $p = 1$ and $p = \infty$.

Problem 4. Investigate the structure of fractional spaces $E_{\alpha,p}(A_h, L_p(\overline{\Omega}_h)$ generated by difference operator A_h of the arbitrary order of accuracy that approximates this multidimensional elliptic operator without mixed derivatives under the Dirichlet or Neumann boundary conditions. Investigate the positivity property of A_h in $W_p^\beta(\overline{\Omega_h})-$ norm.

Problem 5. Investigate the structure of fractional spaces $E_{\alpha,p}(A_h, L_p[0,1]_h)$ generated by difference operator A_h of the arbitrary order of accuracy that approximates this one-dimensional elliptic operator under the nonlocal boundary conditions. Investigate the positivity property of A_h in $W_p^\beta[0,1]_h-$ norm.

Problem 6. Investigate the structure of fractional spaces $E_{\alpha,p}(A_h, L_p(\overline{\Omega}_h)$ generated by difference operator A_h that approximates this multidimensional elliptic operator under the nonlocal boundary conditions.Investigate the positivity property of A_h in $W_p^\beta(\overline{\Omega_h})-$ norm.

Problem 7. Investigate the structure of fractional spaces $E_{\alpha,p}(A_h, L_p([0,\infty) \times \mathbf{R}_h^n)$ generated by difference operator A_h that approximates this multidimensional elliptic operator (2.9).Investigate the positivity property of A_h in $W_p^\beta([0,\infty)_h \times \mathbf{R}_h^n)$ $(1 \le p \le \infty,\ 0 < \beta < 1)$, $C^\beta([0,\infty)_h \times \mathbf{R}_h^n)$ $(0 < \beta < 1)$ norms.

Here $W_p^\beta([0,\infty)_h \times \mathbf{R}_h^n)$ is the space of grid functions u^h defined by the norm

$$\|u^h\|_{W_p^\beta([0,\infty)\times\mathbf{R}_h^n)} = \left(\sum_{x\in[0,\infty)\times\mathbf{R}_h^n} \sum_{z\in[0,\infty)\times\mathbf{R}_h^n, z\neq 0} \frac{|u^h(x) - u^h(x+z)|^p h^{2n}}{|z|^{n+\beta p}} + \sum_{x\in[0,\infty)\times\mathbf{R}_h^n} |u^h(x)|^p h^n \right)^{\frac{1}{p}}.$$