Restriction spaces of A^∞

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Basic Definitions

Let $\mathbb{D}$ be the open unit disc, $A^\infty := C^\infty(\overline{\mathbb{D}}) \cap H(\mathbb{D})$.

Let E be a proper compact subset of the unit circle. We will study the space

$$A_\infty(E) := A^\infty|_E = \{f|_E \,:\, f \in A^\infty\}$$

equipped with the quotient topology of the restriction map.

EQUIVALENTLY: $A^\infty = \{2\pi$-periodic C^∞-functions on $\mathbb{R}$ for which all negative Fourier coefficients vanish$\}$.

By $f \mapsto (a_0, a_1, \ldots)$, a_j Fourier coefficients, $A^\infty \cong s$, s the space of rapidly decreasing sequences.

$E \subset [0, 2\pi[$ compact, ε_n lengths of intervals in $[0, 2\pi[\setminus E$.

THEN, WITH THE EQUIVALENT DEFINITION:
$A_\infty(E) := A^\infty|_E = \{f|_E \,:\, f \in A^\infty\}$ equipped with the quotient topology of the restriction map.

We set $I_A(E) := \{f \in A^\infty \;:\; f|_E = 0\}$. Then

$$A_\infty(E) \cong A^\infty / I_A(E)$$

and it is a nuclear Fréchet space.

EXAMPLE: If E has not Lebesgue measure 0, then $I_A(E) = \{0\}$ hence $A_\infty(E) = A^\infty$.

DEFINITION: E is called a Carleson set if

$$\int_0^{2\pi} \log \frac{1}{d(x, E)}\, dx < +\infty.$$

Theorem (Taylor and Williams, Novinger): The following are equivalent:

1. $I_A(E) \neq \{0\}$.

2. There is $f \in A^\infty$ such that $\{t \in [0, 2\pi[\,:\; f(t) = 0\} = E$.

3. E is a Carleson set.

Lemma: If $\varepsilon_1, \varepsilon_2, \ldots$ denote the lengths of the disjoint intervals of which $[0, 2\pi[\backslash E$ consists, then E is a Carleson set if, and only if,

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} < +\infty.$$

Proof. For $0 \leq a < b$ we have

$$\int_a^b \log \frac{1}{d(x, \{a, b\})} dx = (b-a) \log \frac{1}{b-a} + (1 + \log 2)(b-a). \tag{1}$$

If $]a, b[$ is one of the disjoint intervals in $[0, 2\pi[\backslash E$, then $d(x, E) = d(x, \{a, b\})$, from where easily follows the equivalence. □

Examples

1ST CASE: E the classical Cantor set.

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \sum_{k=1}^{\infty} 2^{k-1} 3^{-k} \log 3^k = \frac{\log 3}{3} \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^{k-1} = 3 \log 3 < \infty.$$

2nd case: $E = \{x_1, x_2, \dots\} \cup \{0\}$, where $x_n \searrow 0$. Then $\varepsilon_n = x_n - x_{n+1}$ and we assume that there are $q \in \mathbb{N}$ and $C > 0$ such that for all $n \in \mathbb{N}$

$$x_n^q \leq C\varepsilon_n. \tag{2}$$

In this case

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} \leq 2\pi \log C + q \int_0^{2\pi} \log \frac{1}{x}\, dx < \infty.$$

Proposition: Both examples are Carleson sets.

Negative example: $x_n = \frac{1}{\log n}$ for $n = 2, 3, \dots$.

In this case $\frac{1}{n \log(n+1) \log n} \geq \varepsilon_n \geq \frac{1}{(n+1) \log(n+1) \log n}$, $\log \frac{1}{\varepsilon_n} \geq \log n$ for large n, from which the claim is easily derived.

The Problem

Claim (S. R. Patel 2011): For every Carleson set E the space $A_\infty(E)$ does not have a basis.

Recall: $e_1, e_2, \dots$ is a basis of X if every $x \in X$ has a unique expansion $\sum_{n=1}^\infty x_n e_n$.

Various examples of Fréchet without bases have been given by Mityagin and Zobin, Djakov and Mityagin, Djakov, Moscatelli, V. so solving in the negative a problem of Grothendieck.

Example: Set

$$M = \{(x, y) \in \mathbb{R}^2 \,:\, x \geq 0,\ |\sin y| \leq 2\, e^{-\frac{1}{x}}\}$$

and

$$E = \{f \in C^\infty(\mathbb{R}^2) \,:\, f|_M \in \mathscr{S}(M)\}$$

Then: E is a nuclear Fréchet space without basis.

Main Result

Theorem: For $E = \{2^{-n} : n \in \mathbb{N}\}$ the space $A_\infty(E)$ has a basis. The same holds for E the classical Cantor set.

Proof in several steps.

We set $C_\infty(E) = C^\infty(\mathbb{R})|_E = \{f|_E : f \in C^\infty(\mathbb{R})\}$.

We will proceed in two steps:

1. Study the space $C_\infty(E) := \{f|_E : f \in C^\infty(\mathbb{R})\}$.

2. Compare the spaces $C_\infty(E)$ and $A_\infty(E)$. Show that in our cases they coincide.

Reason: Almost no information on the structure of spaces $A_\infty(E)$. Spaces $C_\infty(E)$ are well investigated.

Structure of $C_\infty(E)$.

Characterization due to Whitney 1934 in terms of divided differences. Not suitable for our purposes.

We set $J(E) := \{f \in C^\infty(\mathbb{R}) \;:\; f|_E = 0\}$, that is, $J(E)$ is the ideal of E in $C^\infty(\mathbb{R})$. Then

$$C_\infty(E) = C^\infty(\mathbb{R})/J(E)$$

and $C_\infty(E)$ is a nuclear Fréchet space.

Lemma: If x_0 is an accumulation point of E and $f \in J(E)$ then $f^{(p)}(x_0) = 0$ for all $p \in \mathbb{N}_0$.

Proof. Repeated use of Rolle's Theorem. □

We set $J^\infty(E) := \{f \in C^\infty(\mathbb{R}) \;:\; f^{(p)}(x) = 0$ for all $p \in \mathbb{N}_0$ and $x \in E\}$. Then $\mathscr{E}(E) = C^\infty(\mathbb{R})/J^\infty(E)$ is the space of Whitney-jets on E.

Corollary: If E is perfect, then $J(E) = J^\infty(E)$.

Proposition: If E is perfect, then $C_\infty(E) = \mathscr{E}(E)$.

This applies, in particular, to the Cantor set.

We will use the following two theorems.

Theorem (Tidten): If E is the Cantor set, then $\mathscr{E}(E)$ is isomorphic to a complemented subspace of s.

Theorem (Aytuna, Krone, Terzioğlu): If X is a complemented subspace of s and $X \oplus X \cong X$, then $X \cong \Lambda_\infty(\alpha)$ for some α.

Here we define for any sequence $\alpha_1 \le \alpha_2 \le \cdots \nearrow +\infty$

$$\Lambda_\infty(\alpha) = \{\xi = (\xi_1, \xi_2, \dots) \, : \, \|\xi\|_p := \sum_n |\xi_n| e^{p\alpha_n} < +\infty \text{ for all } p \in \mathbb{N}_0\}.$$

Equipped with the norms $\|\cdot\|_p$ this is a Fréchet space.

Proposition: If E is the Cantor set, then $C_\infty(E)$ has a basis.

Proof. $C_\infty(E) \cong C_\infty(E \cap [0,1/3]) \oplus C_\infty(E \cap [2/3,1]) \cong C_\infty(E)^2$.

2ND CASE: $E = \{x_1, x_2, \dots\} \cup \{0\}$, where $x_n \searrow 0$. Then $\varepsilon_n = x_n - x_{n+1}$ and we make the

ASSUMPTIONS: Exist q, $C > 0$ such that $x_n^q \le C\varepsilon_n$, $\varepsilon_n \ge \varepsilon_{n+1}$ for all n.

REMARK: For any compact E we have: If $\varphi \in C_\infty(E)$, x_0 an accumulation point of E and $f \in C^\infty(\mathbb{R})$ an extension of φ then $f^{(p)}(x_0)$ is uniquely determined by φ for all $p \in \mathbb{N}_0$. We set $\varphi^{(p)}(x_0) := f^{(p)}(x_0)$.

Theorem: Let $\varphi \in C(E)$. Then $\varphi \in C_\infty(E)$ if and only if the following holds: there are numbers A_p, $p \in \mathbb{N}_0$, such that $A_0 = \varphi(0)$ and for all $p \in \mathbb{N}_0$ we have

$$A_{p+1} = \lim_{n\to\infty} \frac{(p+1)!}{x_n^{p+1}} \Big(\varphi(x_n) - \sum_{j=0}^{p} \frac{A_j}{j\,!}\, x_n^j\Big). \tag{3}$$

In this case $A_p = \varphi^{(p)}(0)$ for all p.

The family of norms
$\|\varphi\|_k := \max_{p=0,..,k} \Big\{ |\varphi^{(p)}(0)| + \sup_{n\in\mathbb{N}} \frac{p!}{x_n^p} \Big(\varphi(x_n) - \sum_{j=0}^{p} \frac{\varphi^{(j)}(0)}{j!} x_n^j\Big)\Big\}$, $k \in \mathbb{N}_0$,
is a fundamental system of seminorms in $C_\infty(E)$.

Theorem: If there exists $C > 0$ such that $x_n \leq Cx_{n+1}$ for all $n \in \mathbb{N}$, then $\|\cdot\|_k^2 \leq C_k \|\cdot\|_{k-1} \|\cdot\|_{k+1}$ for all k with suitable C_k, hence $C_\infty(E)$ is isomorphic to a subspace of s.

By definition $C_\infty(E)$ is isomorphic to a quotient of s and therefore (by a result of V.-Wagner) it is isomorphic to a complemented subspace of s.

For any complemented subspace X of s there is an associated power series space

$$\Lambda_\infty(\alpha) = \{x = (x_0, x_1, \dots) \, : \, \|x\|_t := \sum_n |x_n| e^{t\alpha_n} < \infty \text{ for all } t > 0\}.$$

Theorem (Aytuna, Krone, Terzioğlu): If $\sup \alpha_{2n}/\alpha_n < \infty$ then $X \cong \Lambda_\infty(\alpha)$.

We wish to calculate α. We set:

$$J_\infty(0) := \{\varphi \in C_\infty(E) \, : \, \varphi^{(p)}(0) = 0 \text{ for all } p \in \mathbb{N}_0\}.$$

Proposition: $\Phi : \varphi \mapsto (\varphi(x_n))_{n \in \mathbb{N}}$ maps $J_\infty(0)$ isomorphically onto $\Lambda_\infty(\beta)$ where $\beta_n = -\log x_n$.

We have an exact sequence

$$0 \longrightarrow J_\infty(0) \longrightarrow C_\infty(E) \xrightarrow{\Delta} \omega \longrightarrow 0$$

where $\Delta : \varphi \mapsto (\varphi^{(p)}(0))_{p\in\mathbb{N}_0}$.

Lemma: If β is shift-stable, then $\alpha \sim \beta$.

Shift-stable: $\limsup_n \frac{\beta_{n+1}}{\beta_n} < \infty$.

Corollary: If there is q such that $x_n^q \leq x_{n+1}$, then $\alpha_n \sim -\log x_n$.

Proposition: If there are q and C such that $x_n^q \leq \min(C\varepsilon_n, x_{2n})$ then $C^\infty(E) \cong \Lambda_\infty(-\log x_n)$.

Theorem: If $x_n = 2^{-n}$ then $C_\infty(E) \cong H(\mathbb{C})$.

Comparison of $A_\infty(E)$ and $C_\infty(E)$

Theorem (Alexander, Taylor, Williams): If there are constants C_1, C_2 such that

$$\frac{1}{b-a}\int_a^b \log\frac{1}{d(x,E)}\,dx \le C_1 \log\frac{1}{b-a} + C_2 \tag{4}$$

for all $0 \le a < b \le 2\pi$, then $A_\infty(E) = C_\infty(E)$.

1st case: $E = \{x_1, x_2, \dots\} \cup \{0\}$, where $x_n \searrow 0$. We set $\varepsilon_n = x_n - x_{n+1}$ and assume that $\varepsilon_n \ge \varepsilon_{n+1} > 0$ for all n.

Lemma: If $x_{k+1} \le \varepsilon_k$ (hence $x_k \le 2\varepsilon_k$) for all $k \in \mathbb{N}$ then

$$\frac{1}{b-a}\int_a^b \log\frac{1}{d(x,E)}\,dx \le \log\frac{1}{b-a} + 8 \tag{5}$$

for $0 \le a < b \le x_1$.

2ND CASE: E the classical Cantor set.

Lemma:

$$\frac{1}{b-a}\int_a^b \log\frac{1}{d(x,E)}\,dx \le 12\log\frac{1}{b-a} + 24 + \int_0^1 \frac{1}{d(x,E)}\,dx \tag{6}$$

Proposition: In both cases $A_\infty(E) = C_\infty(E)$.

Negative Example: We set

$$E = \{x_{m,k} = 2^{-m+1} - k2^{-m-m^2} \; : \; m \in \mathbb{N}_0,\ k = 0, .., 2^{m^2}\}.$$

Then E is a Carleson set which does not fulfill the condition in the A-T-W Theorem.

REMARK: The A-T-W Condition does also not hold if we restrict a to $a = 0$.

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A-T-W condition:

$$\frac{1}{b-a}\int_a^b \log\frac{1}{d(x,E)}\,dx \le C_1\,\log\frac{1}{b-a} + C_2$$

for all $0 \le a < b \le 2\pi$, then $A_\infty(E) = C_\infty(E)$.

Back to the main theorem

Proposition: 1. If there are q and C such that $x_n^q \leq \min(C\varepsilon_n, x_{2n})$ then $C^\infty(E) \cong \Lambda_\infty(-\log x_n)$.
2. If $2x_{n+1} \leq x_n$ and $x_n^q \leq x_{2n}$ for some q, then $A_\infty(E) \cong \Lambda_\infty(-\log x_n)$.

Theorem: If $x_n = 2^{-n}$ then $A_\infty(E) = C_\infty(E) \cong H(\mathbb{C})$.

Theorem: If E is the classical Cantor set, then $A_\infty(E) = C_\infty(E) = \mathscr{E}(E) \cong s$.

Proof. Return to previous proof:

$$C_\infty(E) \cong C_\infty(E \cap [0, 1/3]) \oplus C_\infty(E \cap [2/3, 1]) \cong C_\infty(E) \oplus C_\infty(E).$$

Lemma: If X is a complemented subspace of s and there is an isomorphism $\psi : X \oplus X \to X$ such that $\frac{1}{C_k}(\|x\|_k + \|y\|_k) \leq \|\psi(x \oplus y)\|_k \leq C_k(\|x\|_k + \|y\|_k)$ for all k, then $X \cong s$.

Thanks for your attention!