Mean ergodic operators on Banach and Fréchet lattices

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Mini-course
Introduction to Dynamics of Linear Operators
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Outline

Fix a Banach space X. A non-empty subset $\mathcal{A} \subseteq \mathcal{L}(X)$ is called a **semigroup** if TS belongs to $\mathcal{A}$ whenever T and S do for every T and S in $\mathcal{A}$. Our emphasis will be on operator semigroups indexed by $\mathbb{N} \cup \{0\}$ or $\mathbb{R}_+ := [0, \infty)$, called **one-parameter semigroups**.

Notation

- $(T_t)_{t \geqslant 0}$: the continuous parameter case
- $(T^n)_{n=0}^{\infty}$: the **discrete semigroup** generated bt T
- $\mathcal{T} := (T_j)_{j \in J}$, where $J = \mathbb{R}_+$ or $J = \mathbb{N} \cup \{0\}$

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A semigroup $\mathcal{T}$ is called **bounded** if

$$M_{\mathcal{T}} := \sup\{\|T\| \mid T \in \mathcal{T}\} < \infty.$$

An operator T is called:

- **power-bounded** if the semigroup $(T^n)_{n=0}^{\infty}$ is bounded,
- **doubly power-bounded** if T is invertible and $\sup\{\|T^n\| \mid n \in \mathbb{Z}\} < \infty$.

Given a bounded semigroup $\mathcal{T}$, the equivalent norm

$$\|x\|_{\mathcal{T}} := \sup\{\|Tx\| \mid T \in \mathcal{T}\},$$

where $x \in X$, makes $\mathcal{T}$ into a contractive one.

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For a one-parameter semigroup $\mathcal{T}$ in $\mathcal{L}(X)$, the **Cesàro means** of $\mathcal{T}$ are defined by

$$\mathcal{A}_\tau^{\mathcal{T}} = \mathcal{A}_\tau^{T} := \frac{1}{\tau}\sum_{k=0}^{\tau-1} T^k, \quad \text{whenever } \mathcal{T} = (T^n)_{n=0}^{\infty},$$

$$\mathcal{A}_\tau^{\mathcal{T}} := \frac{1}{\tau}\int_0^\tau T_t\,dt, \quad \text{whenever } \mathcal{T} = (T_t)_{t\geqslant 0}.$$

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Theorem (The Mean Ergodic Theorem of W.F. Eberlein)

Let $\mathcal{T}$ be a one-parameter Cesàro bounded semigroup in a Banach space X. Then, for any $x \in X$ satisfying

$$\lim_{t\to\infty} \|t^{-1} T_t x\| = 0 \tag{1}$$

and for any $y \in X$, the following are equivalent:

(i) $\mathcal{T} y = y$ *and* $y \in \overline{co}\{T_t x \mid t \in J\}$;

(ii) $y = \lim_{t\to\infty} \mathcal{A}_t^{\mathcal{T}} x$;

(iii) y *is a weak cluster point of the net* $(\mathcal{A}_t^{\mathcal{T}} x)_{t \in J}$.

Mean ergodicity

A one-parameter semigroup $\mathcal{T}$ is called **mean ergodic** if the norm limit $\lim_{\tau\to\infty} \mathcal{A}_\tau^{\mathcal{T}} x$ exists for all $x \in X$. An operator T is **mean ergodic** if the semigroup $\mathcal{T} = (T^n)_{n=0}^\infty$ is mean ergodic.

J. von Neumann proved in 1931 that unitary operators in a Hilbert space are mean ergodic.
Any one-parameter semigroup which is mean ergodic and Cesàro bounded satisfies (1).

For $\mathcal{T} = (T^n)_{n=0}^\infty$, define:

$$X_{\mathrm{me}}(T) := \left\{x \in X \mid \lim_{n\to\infty} \mathcal{A}_n^T x \text{ exists}\right\}, \quad N(T) := (I - T)X,$$

$$\mathrm{Fix}\,(T) := \{x \in X \mid Tx = x\}, \quad \mathrm{Fix}\,(T^*) := \{y \in X^* \mid T^* y = y\}.$$

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Theorem (K. Yosida)

Let T be a Cesàro bounded operator on a Banach space X which satisfies (1) for all $x \in X$. Then $X_{me}(T) = Fix(T) \oplus \overline{N(T)}$, and the operator $P : X_{me}(T) \to X$ defined for all $x \in X_{me}(T)$ as

$$Px := \lim_{n\to\infty} \mathcal{A}_n^T x$$

is a projection of $X_{me}(T)$ onto $Fix(T)$ satisfying $P = T \circ P = P \circ T$. Moreover, for any $x \in X$, the assertions

(i) $\lim_{n\to\infty} \mathcal{A}_n^T x = 0$,

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are equivalent.

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Let T be a Cesàro bounded operator on X which satisfies (1). Then T is mean ergodic if and only if $X = Fix(T) \oplus \overline{N(T)}$.

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Theorem

If X is a reflexive Banach space, then every power-bounded operator $T : X \to X$ is mean ergodic.

Problem (L. Sucheston–1975)

Let X be a Banach space on which every power-bounded operator is mean ergodic. Is X reflexive?

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Let E be a Banach lattice and $T \in \mathcal{L}(E)$. The operator T is called **power order-bounded** if for every $x \in E^+$ there exists $z \in E^+$ such that $T^n([-x, x]) \subseteq [-z, z]$ for all $n \in \mathbb{N}$. The norm of E is **order-continuous** if every net in E decreasing to zero is norm convergent to zero.

Theorem (E. Emel'yanov & M. Wolff–1999)

Let E be a Banach lattice. Then the following assertions are equivalent:

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(i) *the norm of E is order-continuous;*

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Proof. (i) $\Rightarrow$ (ii): Notice that every power order-bounded operator, in particular, T, is power bounded by the UBP. Let $x \in X$ be arbitrary. Then there exists $u \geqslant 0$ such that $T^n x \in [-u, u]$ for all n. Since the order interval is $[-u, u]$ is weakly compact, the assertion follows from Eberlein's Theorem.

(ii) $\Rightarrow$ (i): Assume that the norm on E is not order continuous.

- There exists a disjoint order-bounded sequence $(e_n)_{n=1}^{\infty}$ in E_+ with $e_n \nrightarrow 0$.
- Assume, w.l.o.g., that $\|e_n\| = 1$ and $e_n \leqslant u$ for some u and all n.
- There exists a disjoint normalized sequence $(\psi_n)_{n=1}^{\infty}$ in E_+^* such that $\psi_n(e_m) = 0$ for $m \neq n$ and $\psi_n(e_n) \geqslant 1/2$.
- Set

$$\varphi_n := \frac{\psi_n}{\psi_n(e_n)}$$

and observe that $\|\varphi_n\| \leqslant 2$ and $\varphi_n(e_m) = \delta_{n,m}$.

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- Since $\sum_{k=1}^{n} e_k = \sup\{e_1, \ldots, e_n\} \leqslant u$, the map $U : c_0 \to E$ given by
$$U(f) := \sum_{n=1}^{\infty} f(n) e_n$$
is a well-defined topological lattice isomorphism into E; denote its range by E_0.
- Define $V : E \to \ell_\infty$ by $V(x)(n) := \varphi_n(x)$ and observe that $\|V\| \leqslant 2$ and $V \circ U = I_{c_0}$, that $V(u) \geqslant V(e_n)$ for every n, whence $V(u) \geqslant \mathbf{1}$. In particular, $V(u) \notin c_0$.
- Let $h \in c_0$ satisfy $0 < h(n) \leqslant 1$ for every n. Then the multiplication operator S_h in ℓ_∞ given by $f \mapsto S_h(f) := hf$ maps ℓ_∞ into c_0 and is compact, since the unit ball of ℓ_∞ (which coincides with the order interval $[-\mathbf{1}, \mathbf{1}]$) is mapped into the order interval $[-h, h]$, which is known to be compact—here, $\mathbf{1}$ denotes the constant function $n \mapsto 1$.

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- Set $A := U \circ S_h \circ V$, which is well-defined since $S_h(\ell_\infty) \subseteq c_0$.
- Moreover, A is compact and positive since S_h is.
- Set $T := I_E - A$, and observe that

$$T^r = I_E - U \circ \sum_{k=1}^{r} \binom{r}{k} (-1)^{k-1} S_h^k \circ V.$$

- Define R_r on ℓ_∞ by

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- T is power order-bounded: more precisely, for $|y| \leqslant x$ and all r, one has $|T^r y| \leqslant x + 2\|x\|\,u$, since

$$|T^r y| \leqslant x + U \circ R_r \circ V(x).$$

- T is *not* mean ergodic: if it were, then

$$E = \ker(A) \oplus \overline{A(E)}$$

would be true by Yosida's Theorem, which would imply $E = \ker(V) \oplus E_0$ since U and S_h are injective, which in turn would imply that $u = v + w$ where $V(v) = 0$ and $V(u) = V(w) \in c_0$, a contradiction. □

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$$|T^r y| \leqslant x + U \circ R_r \circ V(x).$$

- T is *not* mean ergodic: if it were, then

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Theorem (E. Emel'yanov & M. Wolff−1999)

Let E be a Banach space on which every power-bounded operator is mean ergodic. Then E does not contain a lattice isomorphic copy of c_0.

Proof. Assume that there exists a topological isomorphism of c_0 into E: this means, since c_0 and c are topologically isomorphic, that there exists an isomorphism U of c into E as such whose dual U^* maps E^* topologically onto $c^* = \ell_1(\mathbb{N}_0)$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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- Define $e_n \in \ell_1(\mathbb{N}_0)$ for all $f \in c$ by

$$e_n(f) := \begin{cases} f(n), & \text{if } n \geqslant 1; \\ \lim_{k\to\infty} f(k), & \text{if } n = 0. \end{cases}$$

- Since the operator U^* is open, there exist $M > 0$ and, to each $n \geqslant 1$, a linear functional $\psi_n \in E^*$ satisfying $\|\psi_n\| \leqslant M$ and $U^*\psi_n = e_n$.
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- Set $T := I - A$.
- Arguments similar to those of the previous theorem allow one to infer that T is well-defined, power bounded, and not mean ergodic. □

Note: The proof of the existence of a power-bounded operator which is not mean ergodic in a *separable* Banach space X containing c_0 is a consequence of Sobczyk's Theorem that provides a bounded projection of X onto c_0.

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Theorem (R. Zaharopol–1986)

Let E be a Dedekind σ-complete Banach lattice. If every positive power-bounded operator $T : E \to E$ is mean ergodic, then E is reflexive.

Proof. By next-to-last theorem, the norm of E is order-continuous. It is known that a Banach lattice whose norm is order-continuous is reflexive if and only if it contains neither a lattice isomorphic copy of ℓ_1 nor of c_0. By the previous theorem, E does not contain a copy of c_0.

- Assume that there is a Banach sublattice E_0 of E which is isomorphic to ℓ_1, with an ℓ_1-basis $(x_n)_{n=1}^{\infty}$ and the basis constant γ so that

$$\|x\| \leqslant \gamma \cdot \sum_{n=1}^{\infty} |\alpha_n|$$

for every $x = \sum_{n=1}^{\infty} \alpha_n x_n \in E_0$.

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- Let $(x'_n)_{n=1}^{\infty} \subseteq E^*$ be a dual basis of $(x_n)_{n=1}^{\infty}$ with $\langle x_n, x'_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.
- Define $T : E \to E$ as

$$T(x) := \sum_{n=1}^{\infty} \langle x_n, x'_n \rangle x_{n+1}.$$

Then, T is well-defined, $Tx \in E_0$ for every $x \in E$, and T is power-bounded since

$$\|T^n x\| \leqslant \rho \cdot \|x\| \cdot \gamma$$

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- Since the right shift

$$(a_n)_{n=1}^{\infty} \mapsto (0, a_1, a_2, a_3, \ldots)$$

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Lemma

Let E be a Banach lattice that fails to be Dedekind σ-complete. Then there exists an order-bounded countable set $X \subseteq E$ of pairwise disjoint positive elements which has no supremeum, and $\|e\| \geqslant 1$ for every $e \in X$.

Lemma

Let E be a Banach lattice. Then, for every order-bounded sequence $(e_n)_{n=1}^{\infty}$ of pairwise disjoint positive elements in E and for every real sequence $(a_n)_{n=1}^{\infty}$ converging to zero, the series $\sum_{n=1}^{\infty} a_n e_n$ converges in norm.

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Theorem (E. Emel'yanov−1997)

Let E be a Banach lattice that fails to be Dedekind σ-complete. Then there exists a positive compact operator $A : E \to E$ such that the operator $T := I_E - A$ is power-bounded and not mean ergodic.

Proof. According to the first Lemma, in the Banach lattice E, there exists an order-bounded family $\{e_n \mid n \geqslant 0\}$ of positive pairwise disjoint elements without supremum and such that $\|e_n\| \geqslant 1$ for all n. Take an element $u \in E_+$ satisfying $\{e_n \mid n \geqslant 0\} \subseteq [0, u]$.

- For every $n \geqslant 0$, take a functional f_n defined on the one-dimensional subspace $\{\lambda e_n \mid \lambda \in \mathbb{R}\}$ of E by $f_n(\lambda e_n) := \lambda$.
- Extend, using the Hahn-Banach Theorem, f_n to $\overline{f_n}$ preserving its norm.
- Define the functional ξ_n on E as follows:

$$\xi_n(x) := \sup_k \overline{f_n}_+(x \wedge k e_n)$$

for $x \geqslant 0$ and $\xi_n(x) := \xi_n(x_+) - \xi_n(x_-)$ for arbitrary $x \in E$.

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- Put
$$\varphi_n(x) := \xi_n(x) + \frac{\|u\| - \xi_n(u)}{\xi_0(u)}\,\xi_0(x)$$
for all $x \in E$ and $n \geqslant 0$.
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$$\varphi_n \geqslant 0, \quad \varphi_n(u) = \|u\|, \quad \varphi_n(e_n) = 1, \quad \|\varphi_n\| \leqslant 1 + \|u\|$$
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- Define $A : E \to E$ as

$$Ax := \sum_{n=1}^{\infty} \alpha_n \varphi_n(x) e_n$$

for all $x \in E$.

- Then, A is well-defined, positive, and compact.
- Set $T := I_E - A$, and observe that

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- It then follows that

$$\|T^k y\| \leqslant (1 + \|u\| + \|u\|^2) \cdot \|y\|$$

for all $k \in \mathbb{N}$ and $y \in E$, whence T is power-bounded.

- To show that T is *not* mean ergodic, assume the contrary. Let T be mean ergodic. Then $\mathcal{A}_n^T x \to \overline{x} \in E$ for all $x \in E$. In particular, the norm limit $v := \lim_{n\to\infty} \mathcal{A}_n^T u$ exists, and $T^k u \downarrow$. Furthermore,

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Combining theorems of Zaharopol and Emel'yanov, one gets the following characterization of reflexivity, thereby establishing an affirmative answer to Sucheston's question for Banach lattices.

Theorem (E. Emel'yanov−1997)

For every Banach lattice E, the following conditions are equivalent:

(i) *every power-bounded operator $T : E \to E$ is mean ergodic;*

(ii) *every power-bounded regular operator $T : E \to E$ is mean ergodic;*

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For a locally-convex Hausdorff space E with $\mathcal{L}(E)$ being the space of all continuous linear operators on E, power-boundedness of an operator $T \in \mathcal{L}(E)$ is defined as the set $\{T^n \mid n \geqslant 0\}$ being an equicontinuous subset of $\mathcal{L}(E)$, and mean ergodicity of an operator, as defined before, makes perfectly good sense.

A locally convex-solid Riesz space which is metrizable and complete is called a **Fréchet lattice**.

The set E', equipped with the strong topology $\beta(E', E)$, is denoted by E'_β.

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Theorem (J. Bonet, B. de Pagter & Werner J. Ricker–2011)

If E is a Fréchet lattice, then the following assertions are equivalent:

(i) *E is reflexive;*

(ii) *E is mean ergodic;*

(iii) *E is Dedekind σ-complete and every positive power-bounded operator on E is mean ergodic;*

(iv) *every positive, power bounded operator on E'_β is mean ergodic.*

The following is crucial for establishing the above result.

Theorem (J. Bonet, B. de Pagter & Werner J. Ricker–2011)

If E is a Dedekind σ-complete locally convex-solid Riesz space which is complete and $\aleph_0$-barrelled, then the following assertions are equivalent:

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The relevance of the Banach lattices c_0, ℓ_1, and ℓ_∞ is that each one admits a positive power bounded operator which *fails* to be mean ergodic.

Denoting the elements in the sequence by $x := (x_1, x_2, \ldots)$, one gets that the operators

$$T_0 : x \mapsto (x_1, x_1, x_2, x_3, \ldots), \qquad x \in c_0,$$

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SOME OPEN PROBLEMS

(I)

An old result due to I.M. Gelfand states that if $T \in \mathcal{L}(X)$ is a doubly power-bounded operator with $\sigma(T) = \{1\}$, then $T = I$.

W. Arendt, H.H. Schaefer and M.P.H. Wolff proved that if T is a Riesz homomorphism on a Banach lattice with $\sigma(T) = \{1\}$, then $T = I$.

- Let T be a positive operator on a Banach lattice such that $\sigma(T) = \{1\}$. Does $T \geqslant I$ hold?

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(II)

A one-parameter semigroup $\mathcal{T} = (T_t)_{t \geqslant 0}$ in X is called **strongly stable** if the norm limit $\lim_{t \to \infty} T_t x$ exists for all $x \in X$.

An operator $T \in \mathcal{L}(X)$ is called **strongly stable** if the semigroup $\mathcal{T} = (T^n)_{n=0}^{\infty}$ is strongly stable.

The **peripheral spectrum** of an operator $T \in \mathcal{L}(X)$ is defined as

$$\sigma_\pi(T) := \{\lambda \in \sigma(T) \mid |\lambda| = r(T)\},$$

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- Let E be a Banach lattice. Are the following conditions equivalent:

  (i) E is reflexive;
  (ii) every positive power-bounded operator on E is mean ergodic;
  (iii) every power-bounded operator T on E such that $\sigma_\pi(T) = \{1\}$ is strongly stable?

(III)

It is known that every Montel-Fréchet lattice is discrete.

J. Bonet, B. de Pagter and Werner J. Ricker proved that a Fréchet lattice E is Montel if and only if E is discrete and every power-bounded operator on E is mean ergodic.

- Is every Fréchet lattice on which every power-bounded operator is mean ergodic discrete (and hence, Montel)?

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