Multidimensional spectral order

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- By an *operator* in a complex Hilbert space $\mathcal{H}$ we understand a linear mapping $A\colon \mathcal{H} \supseteq \mathscr{D}(A) \to \mathcal{H}$ defined on a linear subspace $\mathscr{D}(A)$ of $\mathcal{H}$, called the *domain* of A.

Definitions

- Denote by $\boldsymbol{B}(\mathcal{H})$ the C^*-algebra of all bounded operators A in $\mathcal{H}$ with $\mathscr{D}(A) = \mathcal{H}$. As usual, $I = I_{\mathcal{H}}$ stands for the identity operator on $\mathcal{H}$.

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- $\boldsymbol{B}_s(\mathcal{H}) = \{A \in \boldsymbol{B}(\mathcal{H}) \colon A = A^*\}$
- Given two selfadjoint operators $A, B \in \boldsymbol{B}(\mathcal{H})$, we write $A \leqslant B$ whenever $\langle Ah, h\rangle \leqslant \langle Bh, h\rangle$ for all $h \in \mathcal{H}$.

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- A densely defined operator A in $\mathcal{H}$ is said to be *selfadjoint* if $A = A^*$ and *positive* if $\langle Ah, h\rangle \geqslant 0$ for all $h \in \mathscr{D}(A)$.

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- If A and B are positive selfadjoint operators in $\mathcal{H}$ such that $\mathscr{D}(B^{1/2}) \subseteq \mathscr{D}(A^{1/2})$ and $\|A^{1/2}h\| \leqslant \|B^{1/2}h\|$ for all $h \in \mathscr{D}(B^{1/2})$, then we write $A \leqslant B$.

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- The last definition of $\leqslant$ is easily seen to be consistent with that for bounded operators.

Remark

In general inequality $0 \leqslant A \leqslant B$, where $A, B \in \boldsymbol{B}(\mathcal{H})$, may not imply $A^n \leqslant B^n$, where $n \in \mathbb{N}$.

Theorem (M. P. Olson, A. P., J. Stochel)

Let A and B be positive selfadjoint operators in $\mathcal{H}$. Then the following conditions are equivalent:

(i) $A^n \leqslant B^n$ *for all* $n \in \mathbb{N}$,

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(ii) $\{n \in \mathbb{N} \colon A^n \leqslant B^n\}$ *is infinite,*

(iii) $A \preccurlyeq B$.

The definition of spectral order

- Let $A, B \in \boldsymbol{B}_s(\mathcal{H})$ with spectral measure E_A and E_B, respectively. we write $A \preccurlyeq B$ if $E_B((-\infty, x]) \leqslant E_A((-\infty, x])$ for all $x \in \mathbb{R}$.

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- The relation $\preccurlyeq$ is a partial order in the set of all selfadjoint operators in $\mathcal{H}$, but it is not a vector order!
- This definition was introduced in 1971 by Olson.

Lattices

- Kadison (1951): $(\boldsymbol{B}_s(\mathcal{H}), \leqslant)$ is an anti-lattice, i.e., for any $A, B \in \boldsymbol{B}_s(\mathcal{H})$, the supremum of the set $\{A, B\}$ exists if and only if A, B are comparable (either $A \leqslant B$ or $B \leqslant A$).

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- Sherman (1951): If the set of all selfadjoint elements of a C^*-algebra $\mathcal{A}$ with the usual order forms a lattice, then $\mathcal{A}$ is commutative.
- Olson (1971): If $\mathcal{S}$ is the set of all selfadjoint elements of a von Neumann algebra $\mathscr{V}$ in $\boldsymbol{B}(\mathcal{H})$ then, $(\mathcal{S}, \preccurlyeq)$ is a conditionally complete lattice.

Theorem (T. Kato)

If $A_1, \ldots, A_k \in \boldsymbol{B}(\mathcal{H})$ *are positive, then*

$$\lim_{n\to\infty} (A_1^n + \ldots + A_k^n)^{\frac{1}{n}} h = \left(\bigvee_{j=1}^{k} A_j \right) h$$

for every $h \in \mathcal{H}$.

The definition of spectral order for unbounded operators

Given two selfadjoint operators A and B in $\mathcal{H}$ with spectral measure E_A and E_B, respectively, we write $A \preccurlyeq B$ if $E_B((-\infty, x]) \leqslant E_A((-\infty, x])$ for all $x \in \mathbb{R}$.

In the case of unbounded operators closed supports of E_A and E_B are not compact.

Proposition

Let A and B be selfadjoint operators in $\mathcal{H}$ such that $A \preccurlyeq B$. Then $\langle Ah, h\rangle \leqslant \langle Bh, h\rangle$ for all $h \in \mathscr{D}(A) \cap \mathscr{D}(B)$. Moreover, if A and B are bounded from below, then $\mathscr{D}(B) \subseteq \mathscr{D}(A)$.

Corollary (Olson)

If $A, B \in \boldsymbol{B}_s(\mathcal{H})$, then $A \preccurlyeq B \Rightarrow A \leqslant B$.

Remark

In general, the relation $A \preccurlyeq B$ implies neither $\mathscr{D}(B) \subseteq \mathscr{D}(A)$ nor $\mathscr{D}(A) \subseteq \mathscr{D}(B)$. It is even possible to find operators A and B such that $A \preccurlyeq B$ and $\mathscr{D}(A) \cap \mathscr{D}(B) = \{0\} \neq \mathcal{H}$.

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Example

- Let A_1 be the selfadjoint operator of multiplication by a Borel function $\phi\colon \mathbb{R} \to \mathbb{R}$ in $L^2(\mathbb{R})$ and let A_2 be the selfadjoint operator of multiplication by the identity function on $\mathbb{R}$ in $L^2(\mathbb{R})$.

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- Consider the continuous function ϕ:

$$\phi(x) = \begin{cases} -x^2 & \text{if } x \in (-\infty, -1], \\ x & \text{if } x \in (-1, 1], \\ \sqrt{x} & \text{if } x \in [1, \infty). \end{cases}$$

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- Note that the spectral measure E_2 of A_2 is given by $E_2(\sigma)h = \chi_\sigma h$ for $h \in L^2(\mathbb{R})$.
- In turn, the spectral measure E_1 of A_1 takes the form $E_1(\sigma) = E_2(\phi^{-1}(\sigma))$ for Borel subsets σ of $\mathbb{R}$.

Example

- Since ϕ is strictly increasing and $\phi(x) \leqslant x$ for all $x \in \mathbb{R}$, we see that

$$E_2((-\infty, x]) = E_1((-\infty, \phi(x)]) \leqslant E_1((-\infty, x]), \quad x \in \mathbb{R},$$

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- Define two functions $h_1, h_2 \in L^2(\mathbb{R})$ by

$$h_1(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 1], \\ x^{-3/2} & \text{if } x \in (1, \infty), \end{cases}$$

$$h_2(x) = \begin{cases} |x|^{-5/2} & \text{if } x \in (-\infty, -1], \\ 0 & \text{if } x \in (-1, \infty). \end{cases}$$

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- Then $h_1 \in \mathscr{D}(A_1) \setminus \mathscr{D}(A_2)$ and $h_2 \in \mathscr{D}(A_2) \setminus \mathscr{D}(A_1)$.

Theorem (M. P. Olson, M. Fujii, I. Kasahara, A. P., J. Stochel)

If A and B are selfadjoint operators in $\mathcal{H}$, then the following conditions are equivalent:

(i) $A \preccurlyeq B$,

(ii) $f(A) \leqslant f(B)$ *for each bounded continuous monotonically increasing function* $f: \mathbb{R} \to [0, \infty)$,

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Definitions

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- An element of

$$\mathscr{B}(A) = \bigcup_{a>0} \{h \in \mathscr{D}^\infty(A) \colon \exists_{c>0} \forall_{n \geqslant 0} \|A^n h\| \leqslant c a^n\}$$

is called a *bounded vector* of A.

Theorem

If A and B are positive selfadjoint operators in $\mathcal{H}$, then the following conditions are equivalent:

(i) $A \preccurlyeq B$,

(ii) $\mathscr{D}^\infty(B) \subseteq \mathscr{D}^\infty(A)$ *and the set* $\mathscr{I}_{A,B}(h)$ *is unbounded for all* $h \in \mathscr{D}^\infty(B)$,

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(iv) $\mathscr{B}(B) \subseteq \mathscr{B}(A)$ *and the set* $\mathscr{I}_{A,B}(h)$ *is unbounded for all* $h \in \mathscr{B}(B)$,

where $\mathscr{I}_{A,B}(h) := \{s \in [0,\infty) \colon \langle A^s h, h\rangle \leqslant \langle B^s h, h\rangle\}$ *for* $h \in \mathscr{D}^\infty(A) \cap \mathscr{D}^\infty(B)$.

Corollary

Let A_1 and A_2 be positive selfadjoint operators in $\mathcal{H}$. Assume that $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ and $\{r_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ are sequences such that $\lim_{n\to\infty} k_n = \infty$ and $\liminf_{n\to\infty} \sqrt[k_n]{r_n} \leqslant 1$. Then the following conditions are equivalent:

(i) $A_1 \preccurlyeq A_2$,

(ii) $A_1^n \preccurlyeq A_2^n$ *for all* $n \geqslant 0$,

(iii) $A_1^n \leqslant A_2^n$ *for all* $n \geqslant 0$,

(iv) $A_1^{k_n} \leqslant r_n A_2^{k_n}$ *for all* $n \geqslant 1$.

Let us consider two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$.Let A and B_θ be the matrices given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B_\theta = \begin{bmatrix} 2 & 1 \\ 1 & \theta \end{bmatrix} \text{ for } \theta \in [1, \infty). \tag{1}$$

Clearly, $A \geqslant 0$ and $B_\theta \geqslant 0$.

Proposition

Let A and B_θ be as in (1). Then for every positive integer k there exists $\theta_k \in (2, \infty)$ such that for all $\theta \in [\theta_k, \infty)$,

(i) $A^n \leqslant B_\theta^n$ *for all* $n = 0, \ldots, k$,

(ii) $A \not\preccurlyeq B_\theta$.

Recall that due to Stone's theorem the infinitesimal generator of a C_0-semigroup of bounded selfadjoint operators on $\mathcal{H}$ is always selfadjoint.

Theorem

Let $\{T_j(t)\}_{t\geqslant 0} \subseteq \boldsymbol{B}(\mathcal{H})$ be a C_0-semigroup of selfadjoint operators and A_j be its infinitesimal generator, $j = 1, 2$. Then the following conditions are equivalent:

(i) $A_1 \preccurlyeq A_2$,

(ii) $T_1(t) \preccurlyeq T_2(t)$ *for some* $t > 0$,

(iii) $T_1(t) \preccurlyeq T_2(t)$ *for every* $t > 0$,

(iv) $T_1(t) \leqslant T_2(t)$ *for some* $t > 0$ *and* $E_A((-\infty, x])E_B((-\infty, x]) = E_B((-\infty, x])E_A((-\infty, x])$ *for every* $x \in \mathbb{R}$,

(v) $T_1(nt) \leqslant T_2(nt)$ *for some* $t > 0$ *and for infinitely many* $n \in \mathbb{N}$.

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- $E_{\mathbf{A}}$-joint spectral measure of $\mathbf{A} = (A_1, \ldots, A_\kappa)$,

Definition

Let $\mathbf{A} = (A_1, \ldots, A_\kappa)$ and $\mathbf{B} = (B_1, \ldots, B_\kappa)$ be a κ-tuples of commuting selfadjoint operators in $\mathcal{H}$. We write $\mathbf{A} \preccurlyeq \mathbf{B}$ if $E_{\mathbf{B}}((-\infty, x]) \leqslant E_{\mathbf{A}}((-\infty, x])$ for every $x = (x_1, \ldots, x_\kappa) \in \mathbb{R}^\kappa$, where $(-\infty, x] := (-\infty, x_1] \times \ldots \times (-\infty, x_\kappa]$.

Notation and definitions

- $S(\mathbb{R}^\kappa, E)$ - the set of all E - a.e. finite Borel function $f\colon \mathbb{R}^\kappa \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$,

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- $|\alpha| := \alpha_1 + \ldots + \alpha_\kappa$ for $\alpha = (\alpha_1, \ldots, \alpha_\kappa) \in [0, \infty)^\kappa$,
- $x^\alpha := x_1^{\alpha_1} \ldots x_\kappa^{\alpha_\kappa}$ for $x = (x_1, \ldots, x_\kappa)$ and $\alpha = (\alpha_1, \ldots, \alpha_\kappa)$.

Definitions

- Let $\iota \in \{1, \ldots, \kappa\}$. We define a relation $\leqslant_\iota$ on $\mathbb{R}^\kappa$ requiring that $a \leqslant_\iota b$ if $a_j \leqslant b_j$ for $j = 1, \ldots, \iota$ and $a_j = b_j$ for $j = \iota + 1, \ldots, \kappa$. If $\iota = \kappa$ we write $\leqslant$ instead of $\leqslant_\kappa$.

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- Let $\kappa \in \mathbb{N}_*$, $\Omega \subset \mathbb{R}^\kappa$ and $\varphi \colon \Omega \to \overline{\mathbb{R}}$. We say, that φ is a separately increasing function if

$$x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y),$$

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$$x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y),$$

  for every $x, y \in \Omega$.
- Let $(X, \leq)$ be a partially ordered set. A set $S \subset X$ is called a lower set in X if

$$(y \leq x \wedge x \in S) \Rightarrow y \in S,$$

  for every $x, y \in X$.

Lemma

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *be a commuting* κ*-tuple of selfadjoint operators in* $\mathcal{H}$ *and let* $\iota \in \{1, \ldots, \kappa\}$*. Assume that* $A_j = B_j$ *for every* $j = \iota + 1, \ldots, \kappa$*. If* $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$*, then*

$$E_{\boldsymbol{B}}(\Omega) \leqslant E_{\boldsymbol{A}}(\Omega) \tag{2}$$

for every $\Omega \in \mathfrak{B}(\mathbb{R}^\kappa)$*, which is a lower set in* $(\mathbb{R}^\kappa, \leqslant_\iota)$*.*

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ and $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ be commuting κ-tuple of selfadjoint operators in $\mathcal{H}$. Then the following conditions are equivalent:

(i) $\mathbf{A} \preccurlyeq \mathbf{B}$,

(ii) $\varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B})$ *for every separately increasing function* $\varphi \in S(\mathbb{R}^\kappa, E_{\boldsymbol{A}}) \cap S(\mathbb{R}^\kappa, E_{\boldsymbol{B}})$,

(iii) $A_j \preccurlyeq B_j$ *for every* $j = 1, \ldots, \kappa$.

Remark

Suppose that $\dim \mathcal{H} \geqslant 1$. Then each Borel function $\varphi\colon \mathbb{R}^\kappa \to \mathbb{R}$ satisfying condition

$$\mathbf{A} \preccurlyeq \mathbf{B} \implies \varphi(\mathbf{A}) \preccurlyeq \varphi(\mathbf{B}) \tag{3}$$

for every $\mathbf{A}$, $\mathbf{B}$ κ-tuples of commuting selfadjoint operators, has to be separately monotonically increasing.

Corollary

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* κ*-tuples of commuting selfadjoint operators. Then the following conditions are equivalent:*

(i) $\mathbf{A} \preccurlyeq \mathbf{B}$,

(ii) $\varphi(\mathbf{A}) \leqslant \varphi(\mathbf{B})$ *for every separately monotonically increasing bounded continuous function* $\varphi\colon \mathbb{R}^\kappa \to \mathbb{R}$,

(iii) $\varphi(\mathbf{A}) \leqslant \varphi(\mathbf{B})$ *for every separately monotonically increasing bounded Borel function* $\varphi\colon \mathbb{R}^\kappa \to \mathbb{R}$.

Remark

Olson showed that $(\boldsymbol{B}_s(\mathcal{H}), \preccurlyeq)$ is not a vector ordered space. The missing property is that $A \preccurlyeq B$ does not imply $A + C \preccurlyeq B + C$ for every $A, B, C \in \boldsymbol{B}_s(\mathcal{H})$.

Example (Olson)

Let $\mathcal{H} = \mathbb{C}^2$ with orthonormal basis $\{(1,0),(0,1)\}$. Define three selfadjoint operators A, B and C by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \text{ i } C = \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}.$$

Then $A \preccurlyeq B$ and $A + C \not\preccurlyeq B + C$.

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Example (Olson)

Let $\mathcal{H} = \mathbb{C}^2$ with orthonormal basis $\{(1,0),(0,1)\}$. Define three selfadjoint operators A, B and C by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \text{ i } C = \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}.$$

Then $A \preccurlyeq B$ and $A + C \not\preccurlyeq B + C$.

Remark

Nevertheless spectral order has some traces of vector order properties.

Corollary

Let (A_1, A_2) and (B_1, B_2) be a commuting pair of selfadjoint operators in $\mathcal{H}$. If $A_1 \preccurlyeq B_1$ and $A_2 \preccurlyeq B_2$, then [a]

$$\overline{A_1 + A_2} \preccurlyeq \overline{B_1 + B_2}. \tag{4}$$

[a] If A_1 and A_2 are not bounded, then it may happen, that $A_1 + A_2 \subsetneq \overline{A_1 + A_2}$.

Remark

If $A, C \in \boldsymbol{B}_s(\mathcal{H})$, then $AC \in \boldsymbol{B}_s(\mathcal{H})$ if and only if A and C commute.

Corollary

Suppose that A, B, C are selfadjoint operators in $\mathcal{H}$. Assume also that C is positive and commutes with A and B. Then inequality $A \preccurlyeq B$ implies that

$$\overline{AC} \preccurlyeq \overline{BC}.$$

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Let

$$\mathbf{A}^\alpha = \int_{\mathbb{R}^\kappa} x^\alpha dE_{\mathbf{A}}(x) = \overline{A_1^{\alpha_1} \dots A_\kappa^{\alpha_\kappa}},$$

for $\alpha \in \mathbb{N}^\kappa$.

What are the connections between the domains of operators $\mathbf{A}^\alpha$ and $\mathbf{B}^\alpha$, if $\mathbf{A} \preccurlyeq \mathbf{B}$?

Let

$$\mathbf{C}_\epsilon := ((C_1)_{\epsilon_1}, \ldots, (C_\kappa)_{\epsilon_\kappa}),$$

for $\mathbf{C} = (C_1, \ldots, C_\kappa)$ - κ-tuples of commuting selfadjoint operators in $\mathcal{H}$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\kappa) \in \{-,+\}^\kappa$, where $C_\pm := \int_{\mathbb{R}} x^\pm dE_C(x)$.

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ and $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ be a κ-tuples of commuting selfadjoint operators such that $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ and $\alpha \in \mathbb{N}^\kappa$. If

$$\boldsymbol{A}_\epsilon^\alpha \in \boldsymbol{B}(\mathcal{H}), \quad \epsilon \in \{-,+\}^\kappa \setminus \{(+, \ldots, +)\},$$

then

$$\mathscr{D}(\boldsymbol{B}^\alpha) \subset \mathscr{D}(\boldsymbol{A}^\alpha).$$

Condition

$$\mathbf{A}_\epsilon^\alpha \in \boldsymbol{B}(\mathcal{H}), \quad \epsilon \in \{-,+\}^\kappa \setminus \{(+,\dots,+)\},$$

can't be weakened.

Example

For every $\epsilon \neq (+,\dots,+)$ we can find **A** and **B** such that $\mathbf{A} \preccurlyeq \mathbf{B}$ and

1. $\mathbf{A}_\delta^\alpha \in \boldsymbol{B}(\mathcal{H})$ for every $\delta \in \{-,+\}^\kappa \setminus \{\epsilon\}$ and $\alpha \in \mathbb{N}_*^\kappa$,
2. $\mathscr{D}(\mathbf{B}^\alpha) \not\subset \mathscr{D}(\mathbf{A}^\alpha)$ for every $\alpha \in \mathbb{N}_*^\kappa$.

Let $\mathbf{A} = (A_1, \dots, A_\kappa)$ and $\mathbf{B} = (B_1, \dots, B_\kappa)$ be κ- tuples of commuting positive selfadjoint operators in $\mathcal{H}$. Define the set

$$\Lambda(\mathbf{A}, \mathbf{B}) := \{\alpha \in [0, \infty)^\kappa \colon \mathbf{A}^\alpha \leqslant \mathbf{B}^\alpha\}.$$

We know that relation $\mathbf{A} \preccurlyeq \mathbf{B}$ implies the equality $\Lambda(\mathbf{A}, \mathbf{B}) = [0, \infty)^\kappa$.
What should be assumed about $\Lambda(\mathbf{A}, \mathbf{B})$ to have the reverse implication?

Without any additional informations about **A** and **B** we can formulate the following

Proposition

If $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *are* κ*-tuples of commuting positive selfadjoint operators in* $\mathcal{H}$*, then the following conditions are equivalent*

(i) $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$,

(ii) *for every* $j = 1, \ldots, \kappa$ *the set* $\Lambda(\boldsymbol{A}, \boldsymbol{B}) \cap \{se_j : s \in [0, \infty)\}$*, where* $e_j = (0 \ldots, \underbrace{1}_{j}, \ldots, 0)$*, is unbounded.*

Theorem

Let $\boldsymbol{A} = (A_1, \ldots, A_\kappa)$ *and* $\boldsymbol{B} = (B_1, \ldots, B_\kappa)$ *be a commuting* κ*-tuple of positive selfadjoint operators. Assume that* $\mathscr{N}(A_j) = \{0\}$ *for* $j = 1, \ldots, \kappa$*. Then the following conditions are equivalent:*

(i) $$\sup_{\alpha \in \Lambda(\boldsymbol{A},\boldsymbol{B})} \frac{\alpha_j}{1 + |\alpha| - \alpha_j} = \infty, \text{ for all } j = 1, \ldots, \kappa,$$

(i') $$\sup_{\alpha \in \Lambda(\boldsymbol{A},\boldsymbol{B})} \frac{\alpha_j}{1 + |\alpha|} = 1, \text{ for all } j = 1, \ldots, \kappa,$$

(ii) $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$.

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