S-Parabolic manifolds

A. Aytuna and A. Sadullaev

Abstract. A Stein manifold is called $S-$*parabolic* in case there exits a special plurisubharmonic exhaustion function that is maximal outside a compact set. If a continuous special plurisubharmonic exits then we will call the manifold S^*-*parabolic*. In one dimensional case these notions are equivalent. However in several variables the question as to weather these notions coincide seems open. In this note we establish an interrelation between these two notions.

1. Introduction

In this note we establish an interrelation between two notions of paraboliticity in several complex variables that exit in the literature. We start by giving the relevant definitions.

Definition 1. *A Stein manifold X of dimension n is called $S-$parabolic in case there exits a special plurisubharmonic function $\rho \in PSH(X)$ with the properties:*

a) The set $\{z \in X : \rho(z) \leq C\} \subset\subset X$ is relatively compact in X, for every $C \in \mathbb{R}$. That is ρ is an exhaustion,

b) $(dd^c\rho)^n = 0$ off a compact set, i.e. ρ is maximal outside a compact set.([**6**])

In the previous papers on parabolic manifolds (see for example [**8**],[**9**],[**4**]) authors usually required the conditions of continuity or $C^\infty-$ smoothness of ρ in the above definition. In this note we distinguish the case of continuity and call a complex manifold S^*-*parabolic*, in case it possesses a *continuous* plurisubharmonic exhaustion that is maximal outside a compact subset.

We note, that without maximality condition b) an exhaustion function always exist for any Stein manifold X, since in general Stein manifolds can be properly embedded into $\mathbb{C}^N_w$ for some large N and one take for ρ the restriction of $\ln|w|$ to X.

1991 *Mathematics Subject Classification.* Primary 32U05, 46A61, 32U15 ; Secondary 46A63, 32U15.

Key words and phrases. Parabolic manifolds, plurisubharmonic exhaustion functions, maximal plurisubharmonic functions.

The second author is partially supported by Khorezm Mamun Academy, Grant ΦA-Φ1- Φ024 The first author is partially supported by a grant from Sabanci University.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Special plurisubharmonic exhaustion functions on parabolic manifolds X play a key role in Navanlinna's value distribution theory of holomorphic maps from X into projective spaces (see for example [**4**], [**5**]).

Stein manifolds, on which every bounded above plurisubharmonic function reduces to a constant play a role in the study of the Fréchet spaces of analytic functions on Stein manifolds, the bases on them and in finding continuous extension operators for analytic functions from complex submanifolds (see for example [**1**],[**2**]). Such spaces will be called *parabolic* in this paper.

It is not difficult to see that $S-parabolic$ manifolds are *parabolic*.In particular there are no bounded non constant analytic functions on such manifolds.

The most important example of a $S^{*}-parabolic$ manifold is $\mathbb{C}_z^n$ with the special plurisubharmonic function $\ln|z|$. Affine algebraic manifolds, parabolic Riemann surfaces also among the examples of $S^{*}-parabolic$ manifolds

$S^{*}-$ *parabolic* manifolds (also $S^{*}-$ *parabolic* Stein spaces) and the structure of certain plurisubharmonic functions and currents on them where studied in detail by Demailly([**3**]), and Zeriahi ([**12**]),([**13**]). Moreover on such manifolds one can define extremal Green functions and apply it to the pluripotential theory on such manifolds.

Let us fix an $S-$ *parabolic* manifold X and a special exhaustion function ρ on it. Let

$$L_\rho=\left\{u\ \in PSH\left(X\right):u\left(z\right)\leq\rho^{+}\left(z\right)+C,\text{ where }\rho^{+}=\max\left(\rho,0\right),\ C=C\left(\rho\right)\epsilon\mathbb{R}\right\}$$

be the Lelong class of plurisubharmonic functions and for a compact set $K\subseteq X$, and set

$$L_\rho\left(K\right)=\left\{u\ \in\ L_\rho:u|K\leq 0\right\}.$$

Definition 2. *Let X and K be as above, the upper regularization $V^{*}\left(z,K\right)=\limsup_{w\to z}V\left(z,K\right)$ of $V\left(z,K\right)=\sup\left(u\left(z\right):u\ \in L_\rho\left(K\right)\right)$ is called the Green function of K.*

We note that the Green function is either $+\infty$ (K is pluripolar) or belongs to the class L_ρ. An analytic function f on X will be called a $\rho-polynomial$ in case $\frac{\ln|f|}{d}$ belongs to L_ρ for some integer d. The minimal such d is called the degree of the polynomial. The space of all ρ-polynomials of degree less than or equal to d, $\mathbb{P}_\rho^d\left(X\right)$, is a finite dimensional space and dim $\mathbb{P}_\rho^d\left(X\right)\leq\binom{n+dn}{n}$. This result was proved in[**12**] for $S^{*}-$ *parabolic* manifolds but the same proof also works for $S-$ *parabolic* manifolds. . In the special case of affine algebraic manifolds a detailed analysis of these generalized polynomials is given in [**13**].

In the one dimensional case the notions of $S-$ *parabolicity*, $S^{*}-$ *parabolicity* and *parabolicity* coincide, ([**7**]). However in several variables the question as to weather these notions coincide seems open.

The aim of this short note is establish an interrelation between $S-$ *parabolicity* and $S^{*}-$ *parabolicity.*

2. Results

Let X be a $S-$ *parabolic* manifold and choose a special plurisubharmonic exhaustion function ρ. If ρ is not continuous it is of interest to examine the jumps at its discontinuities. Note that for any plurisubharmonic function σ and any point z in its domain of definition , $\sigma^*(z) = \limsup_{w\to z} \sigma(w) = \sigma(z)$. For a given point z in the domain of definition of the function we set $\sigma_*(z) = \liminf_{w\to z} \sigma(w)$. If $\sigma_*(z) < \sigma(z)$ then we have a jump at z.

Definition 3. *Let λ be a plurisubharmonic function exhaustion of a complex manifold. We say that λ is strongly continuous at the point at infinity in case*

$$\lim_{\lambda(z)\to\infty} \frac{\lambda(z)}{\lambda_*(z)} = 1.$$

Lemma 1. *Let X be a $S-$ parabolic manifold and ρ its special exhaustion function. Then the following are equivalent:*

a) The function ρ is strongly continuous at the point at infinity

b) The Green function $V^*(z, K)$ corresponding to ρ is strongly continuous at the point at infinity for any nonpluripolar compact set K.

Proof. Fix a nonpluripolar compact set $K \subset X$. There are positive constants C_1 and C_2 such that

$$\rho(z) - C_1 \le V^*(z, K) \le \rho(z) + C_2 \quad .$$

The first inequality is by definition of the Green function and the second follows from the remarks given in section 1. The Lemma follows easily from these inequalities. □

Now we can state our result.

Theorem 1. *Let X be a $S-$parabolic manifold. Then X is S^*-parabolic if and only if there is an plurisubharmonic exhaustion function on X that is maximal outside a compact set and strongly continuous at the point at infinity.*

Proof. Lets assume that there exits a plurisubharmonic exhaustion function on X that is maximal outside a compact set and is strongly continuous at the point at infinity. We fix a big pluriregular compact set $K \subset X$. Let us denote Green function $V^*(z, K)$ corresponding to this compact set by $v(z)$. Then in view of the lemma $v(z)$ is strongly continuous at the point at infinity. Using the approximation theorem given in [**10**] (this fact was also proved independently, but later by the second author see [**11**]) we can find a sequence of plurisubharmonic functions $v_j(z) \in PSH(X) \cap C^\infty(X)$, $v_j(z) \downarrow v(z) \quad \forall z \in X$.

Since K is pluriregular, $v|K = 0$. In view of Hartog's theorem for an arbitrary $\epsilon > 0$, we can find a j_0 such that $v_j < \epsilon$ uniformly on K, $j \ge j_0$.

Since v is strongly continuous at infinity, there is an $R > 0$ such that

$$v(z) \le v_*(z) + \epsilon v_*(z) \quad \text{for } z \notin B_R \text{ where } B_R = \{z \,\varepsilon X : v(z) < R\}.$$

In particular we have $v|\partial B_R \le (1+\epsilon) R$. Applying again Hartog's theorem we can find an j_1 so that

$$v_j(z) \le (1 + 2\epsilon) R, \quad j > j_1 \ge j_0,\ z \,\varepsilon \partial B_R.$$

For $j > j_1$ we put:

$$w(z) = \begin{matrix} \max\{v_j(z), (1+3\epsilon) v(z) - \epsilon R\} \;\; if\;\; z \in B_R \\ (1+3\epsilon) v(z) - \epsilon R \;\; if \;\; z \notin B_R \end{matrix} \quad .$$

Since for $z \in B_R$ we have $w(z) = (1+3\epsilon)v(z) - \epsilon R \geq (1+2\epsilon)R \geq v_j(z)$ the function $w(z)$ is plurisubharmonic on X. It follows that the function $\frac{1}{1+3\epsilon}[w(z) - \epsilon]$ belongs to the Lelong class $L_\rho(K)$. So we have

$$\frac{1}{1+3\epsilon}[w(z) - \epsilon] \leq V^*(z, K) = v(z).$$

In particular $v_j(z) \leq (1+3\epsilon)V^*(z, K) + \epsilon$. Since $v_j(z) \geq V^*(z, K)$ the continuity of $V^*(z, K)$ follows. □

COROLLARY 1. *If the "jumps" of the special exhaustion function ρ of an $S-$parabolic manifold X satisfy*

$$\rho(z) - \rho_*(z) = o(\rho_*(z)),$$

then X is S^*-*parabolic.*

References

[1] Aytuna A., Stein Spaces M for which O(M) is isomorphic to a power series space, Advances in the theory of Fréchet spaces, Kluwer Academic Publishers, Dodrecht-Boston-London,1988, 115-154

[2] Aytuna A., Linear tame extension operator from closed subvarieties of, Proc. Amer. Math. Soc., 123:3 (1995),759-763

[3] Demailly J.P., Mesures de Monge - Ampère et caractèrisation gèomètrique des variétés algébriques affines, Memoires de la societe mathematique de France.V.19(1985),1-125

[4] Griffiths P. and King J.,Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Mathematica,V.130 (1973), 145-220

[5] Sadullaev A., Valiron's defect divisors for holomorphic map,Mathem. Sbornic,V 108:4 (1979), 567-580.=Math. USSR -Sb.V.36(1980),535-547

[6] Sadullaev A., Plurisubharmonic measure and capacity on complex manifolds, Uspehi Math.Nauk, Moscow, V.36 N4, (220) (1981) , 53-105 = Russian Math.Surveys V.36 (1981), 61-119

[7] Sario,L-M.Nakai, *Classification theory of Riemann surfaces*, Springer, Berlin-Heidelberg-New York-1970, pp.216.

[8] Stoll W., *Value distribution of holomorphic maps into compact complex manifolds*, Lecture notes,N 135,Springer,Berlin-Heidelberg-New York-1970

[9] Stoll W.,*Value distribution on parabolic spaces,* Lecture notes, 600, Springer,Berlin-Heidelberg-New York-1977

[10] Fornaess J.E. and Narasimhan R., The Levi problem on complex spaces with singularities, Math.Ann., 248 (1980), 47-72

[11] Sadullaev A., Continuation plurisubharmonic functions from submanifold,Dokl.AN Uzbekistan,N 5(1982), 3

[12] Zeriahi A., Fonction de Green pluricomplex a pole a l'infini sur un espace de Stein parabolique, Math.Scand., V.69 (1991), 89-126

[13] Zeriahi A., Approximation polynomial et extension holomorphe avec croissance sur une variete algebrique, Ann.Polon.Math. V.63:1, (1996), 35-50

FENS, SABANCI UNIVERSITY, 34956 TUZLA, ISTANBUL, TURKEY
E-mail address: aytuna@sabanciuniv.edu

Current address: MATHEMATICS DEPARTMENT, NATIONAL UNIVERSITY OF UZBEKISTAN VUZ GORODOK, 700174, TASHKENT, UZBEKISTAN
E-mail address: sadullaev@mail.ru