Invariants of a mathematical life
Zakharyuta's contributions to Functional Analysis

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Colloquium in honour of
Professor Vyacheslaw Pavlovich Zakharyuta

- Complex Analysis, Pluripotential Theory

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  - e.g. Invariants and applications, quasi-equivalence, isomorphic classification of Fréchet spaces, automatic compactness.

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Theorem of Dynin-Mityagin: E nuclear Fréchet space, then $\lambda_e = \{\xi = (\xi_0, \xi_1, \dots) : \|\xi\|_k = \sum_j |\xi_j| \, \|e_j\|_k < \infty$ for all $k \in \mathbb{N}\}$ and Φ_e is a topological isomorphism.

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Criterion: (Grothendieck-Pietsch)
$\lambda(A)$ nuclear $\Leftrightarrow \forall k \exists p : \sum_j a_{j,k}/a_{j,k+p} \leq \infty$

Classical Problems

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- Isomorphic classification.
  - By topological linear invariants.
    - Separating spaces.
    - Giving a complete identification, at least among spaces of a subclass.
  - By properties of $L(E, F)$, making isomorphisms impossible.
- Existence of a basis.
  - Yes or no?
  - If yes determine the coordinate space.
- Quasi-equivalence.
  - General problem (unsolved!!!)
  - Under certain assumptions.
  - Quasi-diagonal classification by topological linear invariants.

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Notation: Two bases are quasi-equivalent if their coordinate spaces are quasi-equivalent. E has the quasi-equivalence property if all bases are quasi-equivalent.

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Important class of regular spaces: power series spaces.

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REMARK: This and BESSAGA'S Theorem also show quasi-equivalence.

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(d$_1$) if $\exists p \, \forall q \, \exists r, C > 0 \, \forall j : a_{j,q}^2 \leq C \, a_{j,p} a_{j,r}$.

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Theorem (DRAGILEV): If $\lambda(A)$ is regular, nuclear and in class (d$_1$) or (d$_2$) then it has the quasi-equivalence property.

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Lemma (Douady): Let $T : X_1 \times X_2 \to Y_1 \times Y_2$ be an isomorphism given by the matrix $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ with inverse $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$. If T_{21} and S_{21} are compact, then T_{11} and T_{22} are Fredholm.

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Theorem (Zakharyuta): If $T : X_1 \times X_2 \to Y_1 \times Y_2$ is a isomorphism, where all spaces are shift-stable Montel spaces and $(X_1, Y_2) \in \mathfrak{R}$, $(Y_1, X_2) \in \mathfrak{R}$ then $X_1 \cong Y_1$, $X_2 \cong Y_2$.

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EXAMPLE: The isomorphy class of $H(\mathbb{D}^n) \times H(\mathbb{C}^m)$ determines n and m.

Isomorphisms of Cartesian products

RECALL: A space E is called shift-stable if $\mathbb{K} \times E \cong E$.

Corollary: If in Douady's Lemma all spaces are shift-stable then $X_1 \cong Y_1$ and $X_2 \cong Y_2$.

NOTATION: $(E, F) \in \mathfrak{R}$ iff $L(E, F) = K(E, F)$.

Theorem (ZAKHARYUTA): If $T : X_1 \times X_2 \to Y_1 \times Y_2$ is a isomorphism, where all spaces are shift-stable Montel spaces and $(X_1, Y_2) \in \mathfrak{R}$, $(Y_1, X_2) \in \mathfrak{R}$ then $X_1 \cong Y_1$, $X_2 \cong Y_2$.

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Theorem (ZAKHARYUTA): Spaces $\Lambda_0(\alpha) \times \Lambda_\infty(\beta)$, as above, have the quasi-equivalence property.

Diametral dimension

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SOME NOTATION: $U, V \subset X$ absolutely convex zero neighborhoods. KOLMOGOROV diameters:

$$d_n(V, U) := \inf_{\dim L \leq n} \inf\{\delta > 0 : V \subset \delta U + L\}.$$

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Diametral dimension of X:

$$\begin{aligned} \Gamma(X) &= \{\gamma = (\gamma_n) \, : \, \forall U \, \exists V : \gamma_n d_n(V, U) \to 0\} \\ \Gamma'(X) &= \{\gamma = (\gamma_n) \, : \, \exists U \, \forall V : \gamma_n / d_n(V, U) \to 0\} \end{aligned}$$

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Theorem: 1. Both invariants distinguish power series spaces.
2. $\Gamma'(X)$ distinguishes regular spaces.

Explanation of problem

REMARK: If $\lambda(A)$ is regular then $d_n(U_q, U_p) = \frac{a_{n,p}}{a_{n,q}}$. If not then comes in a permutation depending on p and q.

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- In the regular case $\Gamma'(\lambda(A))$ is the union of countably many diagonal transforms of $\lambda(A)$.
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- Regular case $\Rightarrow$ Crone-Robinson.
- General case: what information remains?

Counting invariants

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Inverse diametral dimension for $X = \lambda(A)$:

$$\gamma(X) = \{\varphi \,:\, \exists p \,\forall q \,\exists c \; |\varphi(t)| \lesssim |\{j \,:\, a_{j,q}/a_{j,p} \leq c\, e^t\}|\}.$$

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EXAMPLES: φ denotes real valued function on $\mathbb{R}_+$.

$$\begin{aligned}
\gamma(\Lambda_0(\alpha)) &= \{\varphi \,:\, \exists A\ |\varphi(t)| \lesssim m_\alpha(At)\} \\
\gamma(\Lambda_\infty(\alpha)) &= \{\varphi \,:\, \forall \varepsilon\ |\varphi(t)| \lesssim m_\alpha(\varepsilon t)\} \\
\gamma(\Lambda_0(\alpha) \times \Lambda_\infty(\beta)) &= \{\varphi \,:\, \exists A\, \forall \varepsilon\ |\varphi(t)| \lesssim m_\alpha(At) + m_\beta(\varepsilon t)\} \\
\gamma(\Lambda_0(\alpha) \otimes \Lambda_\infty(\beta)) &= \{\varphi \,:\, \exists A\, \forall \varepsilon\ |\varphi(t)| \lesssim m_\alpha(At) \cdot m_\beta(\varepsilon t)\}
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$\Leftrightarrow \varepsilon\alpha_\nu + \beta_\mu \geq \varepsilon\alpha_n + \beta_m$ for all $\varepsilon > 0 \Rightarrow \beta_\mu \geq \beta_m$.

EXAMPLE: $\alpha_j = j^{1/n}$, $\beta_j = j^{1/m}$ then $m_\alpha(t) \sim t^n$, $m_\beta(t) \sim t^m$
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Theorem (ZAKHARYUTA): If $X = \Lambda_0(\alpha) \otimes \Lambda_\infty(\beta)$,
$Y = \Lambda_0(\alpha') \otimes \Lambda_\infty(\beta')$ and all spaces are stable, i.e. α_j/α_{2j} etc. bounded, then:

$$X \cong Y \Leftrightarrow \gamma(X) = \gamma(Y).$$

First type power spaces

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Definition: $\mathcal{E}$ is the class of all Köthe spaces $E(\lambda, a) = \lambda(A)$ where A has the form

$$a_{j,p} = e^{(-\frac{1}{p}+\lambda_j p)a_j}$$

where $a_j > 0$ and $0 < \lambda_j \leq 1$ for all j.
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One distinguishes the cases:

1. $\lambda_j \to 0$ (finite type; $\mathcal{E}_1$)
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2. $T\Lambda_0 = E(\lambda, a)$ with $a_{n,\nu} = n\nu^{1/n}$, $\lambda_{n,\nu} = \nu^{-1/n}$, $T\Lambda_0 \in \mathcal{E}_3$,
$T\Lambda_0 \cong T(H(\mathbb{D}^d))$ for all dimensions d.

Two-sided counting functions

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Lemma: If $E(\lambda, a) \cong E(\mu, b)$, then $\forall\, B \;\exists\, A \;\forall\, \delta \;\exists\, \varepsilon$:

(a) $|\{j : \mu_j > \delta,\, t/B < b_j \leq Bt\}| \leq |\{j : \lambda_j > \varepsilon,\, t/A < a_j \leq At\}|$

(b) $|\{j : \mu_j < \varepsilon,\, t/B < b_j \leq Bt\}| \leq |\{j : \lambda_j < \delta,\, t/A < a_j \leq At\}|$

(c) $\forall\, \varepsilon' \;\exists\, \delta'$: $|\{j : \varepsilon' < \mu_j \leq \varepsilon,\, t/B < b_j \leq Bt\}| \leq |\{j : \delta' < \lambda_j \leq \delta,\, t/A < a_j \leq At\}|$

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and the same relations hold with spaces interchanged.

Theorem: If $X = E(\lambda, a),\ Y = E(\mu, b)$ and $X \overset{p}{\simeq} X^2$ then: $X \cong Y \Leftrightarrow X \overset{p}{\simeq} Y$.

Quasi-equivalence

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Theorem (Dragilev-Bessaga): X nuclear Fréchet spaces, e, f bases. Then there is a sequence $j_k \to \infty$ of indices and a sequence $\gamma_k > 0$ such that $\lambda(\|e_k\|_p) = \lambda(\gamma_k \|f_{j_k}\|_p)$.

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Example: $H(\mathbb{D}^n \times \mathbb{R}^m)$ has the quasi-equivalence property for all n, m.

Back to tensor products

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Theorem: $X = \Lambda_0(j^{\alpha}) \otimes \Lambda_\infty(j^{\beta}),\ Y = \Lambda_0(j^{\alpha'}) \otimes \Lambda_\infty(j^{\beta'})$. Tfae:

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1. $X \cong Y \Leftrightarrow X = Y$
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Multirectangular invariants

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Consider $X = E(\lambda, a)$ with $a_j > 1$, $0 < \lambda_j \leq 1$ for all j and likewise $Y = E(\mu, b)$.

Definition: For $m \in \mathbb{N}$ the m-regular characteristic of $X = E(\lambda, a)$ is given by

$$\mu_m^X(\delta, \varepsilon; \tau, t) := \left| \bigcup_{k=1}^{m} \{ j \,:\, \delta_k < \lambda_j \leq \varepsilon_k,\ \tau_k < a_j \leq t_k \} \right|,$$

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Definition: $(\mu_m^X) \approx (\mu_m^Y)$ if there is a bijection $\varphi : [0, 2] \to [0, 1]$ and a constant Δ such that

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for all parameters and symmetric condition.

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Theorem (Chalov-Zakharyuta): $X \overset{qd}{\simeq} Y \Leftrightarrow (\mu_m^X) \approx (\mu_m^Y)$.

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J.W. Goethe

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The difficulties grow, the closer you come to your task.

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Results not touched in this lecture:

- Second type power spaces:
  $a_{j,p} = \exp(-\frac{1}{p} + \min(\lambda_j p))a_j$, $1 \leq \lambda_j$.
- Tensor products of (F)- and (DF)-spaces.
- Gradually relaxing the assumptions (non-nuclear, non-Schwartz).
- Classification of function spaces in Real and Complex Analysis.
- Spectral theory in locally convex spaces.

Good luck Slava
have a happy time
in your new home!!