INVARIANT SUBSPACES OF WEAKLY COMPACT-FRIENDLY OPERATORS

MERT ÇAĞLAR AND TUNÇ MISIRLIOĞLU

Abstract. We prove that if a non-zero weakly compact-friendly operator B on a Banach lattice with topologically full center is locally quasi-nilpotent, then the super right-commutant $[B\rangle$ of B has a non-trivial closed invariant ideal. An example of a weakly compact-friendly operator which is not compact-friendly is also provided.

1. introduction

Weakly compact-friendly operators have been defined in [3] as a natural extension of compact-friendly operators. Therein, it was shown [3, Theorem 2.3], among others, that a locally quasi-nilpotent weakly compact-friendly operator on a Banach lattice has a non-trivial closed invariant ideal. The purpose of this note is to extend some results in [1] and [4] in the setting of weakly compact-friendly operators on Banach lattices with topologically full center. In doing so, we also provide an example of a weakly compact-friendly operator which is not compact-friendly.

Throughout the paper E denotes an infinite-dimensional Banach lattice. As usual, $\mathcal{L}(E)$ and $\mathcal{L}(E)^+$ stand, respectively, for the algebra of all bounded linear operators and the collection of all positive operators on E. For a positive operator B on a Banach lattice E, the ***super right-commutant*** $[B\rangle$ of B is defined by

$$[B\rangle := \{A \in \mathcal{L}(E)^+ \mid AB - BA \geq 0\}.$$

A subspace V of a Banach space X is called ***non-trivial*** if $\{0\} \neq V \neq X$. If V is a subspace of a Banach lattice and if $v \in V$ and $|u| \leq |v|$ imply $u \in V$, then V is called an ***ideal***. A subspace V of a Banach space X for which $TV \subseteq V$ for a bounded operator T on X is called an ***invariant subspace*** for T or a T***-invariant subspace***.

An operator T on E is said to be ***dominated*** by a positive operator B on E, denoted by $T \prec B$, provided $|Tx| \leq B|x|$ for each $x \in E$. An operator on E which is dominated by a multiple of the identity operator is called a ***central operator***. The

2000 *Mathematics Subject Classification.* Primary 47A15, 47B60, 47B65, 46B42.

Key words and phrases. Banach lattice, topologically full center, invariant subspace, weakly compact-friendly.

collection of all central operators on E is denoted by $Z(E)$ and is referred to as the ***center*** of the Banach lattice E. A positive operator $B : E \to F$ between two Banach lattices is said to be a ***lattice homomorphism*** if $B(x \vee y) = Bx \vee By$ for all $x, y \in E$. Every positive central operator is a lattice homomorphism. A positive operator B on E is said to be ***compact-friendly*** [1] if there exist three non-zero operators R, K, and C on E with R, K positive and K compact such that R and B commute, and C is dominated by both R and K. It is worth mentioning that the notion of compact-friendliness is of substance only on infinite-dimensional Banach lattices, since every positive operator on a finite-dimensional Banach lattice is compact. Also, if B is compact, letting $R = K = C = B$ in the definition, it is seen that compact operators are compact-friendly, but the converse is not true as the identity operator on an infinite-dimensional space shows. Furthermore, it is straightforward to observe that any power (even every polynomial with non-negative coefficients) of a compact-friendly operator is also compact-friendly. A fairly complete treatment of compact-friendly operators is given in [1]. Lastly, an operator from a Banach lattice to a Banach space is ***AM-compact*** if it takes order intervals into relatively compact sets. Clearly, each compact operator is necessarily AM-compact.

For all unexplained notation and terminology, we refer to [1, 2].

Definition 1.1. *A positive operator $B \in \mathcal{L}(E)$ is called* ***weakly compact-friendly*** *if there exist three non-zero opeators R, K, and C on E with R, K positive and K compact such that $R \in [B\rangle$, and C is dominated by both R and K.*

Let us start by recalling some more terminology. A continuous function $\varphi : \Omega \to \mathbb{R}$, where Ω is a topological space, has a ***flat*** if there exists a non-empty open set Ω_0 in Ω such that φ is constant on Ω_0. If Ω is a compact Hausdorff space and $\varphi : \Omega \to \mathbb{R}$ is a continuous function, then $M_\varphi : C(\Omega) \to C(\Omega)$ denotes the ***multiplication operator*** generated by φ, i.e., for each function $f \in C(\Omega)$ and each $\omega \in \Omega$ we have $(M_\varphi f)(\omega) := \varphi(\omega) f(\omega)$, or briefly $M_\varphi f = \varphi f$. The function φ is called the ***multiplier***. It is straightforward to check that a multiplication operator M_φ is positive if and only if the multiplier φ is positive.

The following result, which is Theorem 10.65 in [1], characterizes compact-friendly multiplication operators on $C(\Omega)$-spaces.

Theorem 1.2. *A positive multiplication operator M_φ on a $C(\Omega)$-space, where Ω is a compact Hausdorff space, is compact-friendly if and only if the multiplier φ has a flat.*

Unlike Theorem 1.2, the multiplier of a positive multiplication operator having a flat is not necessary for the multiplication operator to be weakly compact-friendly. This fact, which is the subject matter of the following example, also shows that there are weakly compact-friendly operators that are not compact-friendly.

Example 1.3. Consider the space E of all continuous functions $f : [0, 1/2] \to \mathbb{R}$ equipped with the usual uniform norm. The multiplication operator $M_\varphi : E \to E$ with the multiplier φ defined by $\varphi(\omega) := 1 - 2\omega$ for all $\omega \in [0, 1/2]$ is not compact-friendly by Theorem 1.2, since φ has no flats. To see that M_φ is weakly compact-friendly, choose $R = C = K$ as the required three operators for the weak compact-friendliness of M_φ, where K is the rank-one (and hence, compact) operator on E defined by $(Kf)(\omega) := (1 - 2\omega)f(0)$ for all $f \in E$ and $\omega \in [0, 1/2]$.

2. Invariant subspaces of weakly compact-friendly operators

We start this section, in which the main results of the present note are provided, with the notion of topological fullness of the center of a Banach lattice.

Definition 2.1. *The center $Z(E)$ of a Banach lattice E is called **topologically full** if whenever $x, y \in E$ with $0 \le x \le y$ one can find a sequence $(T_n)_{n\in\mathbb{N}}$ in $Z(E)$ such that $\|T_n y - x\| \to 0$.*

Banach lattices with topologically full center were initiated in [5]. Spaces of this kind are quite large and contain, for instance, Banach lattices with quasi-interior points and Dedekind σ-complete Banach lattices (see [5, 6] for details).

Before proceeding, let us first observe that [6] if $0 \le x \le y$ and $T_n y \to x$, then one has $(T_n^+ \wedge I)y = (T_n y)^+ \wedge y \to x \wedge y = x$, so we may assume that $0 \le T_n \le I$ for all $n \in \mathbb{N}$. Set $Z(E)_{1+} := \{T \in Z(E) \mid 0 \le T \le I\}$.

It is shown in [4, Theorem 3.10] that for a locally quasi-nilpotent positive operator B on a Banach lattice E with a quasi-interior point for which $[B\rangle$ contains an operator which dominates a non-zero AM-compact operator, $[B\rangle$ has an invariant closed ideal. The following result extends this to positive operators on a Banach lattice with topologically full center, following similar lines of thought.

Theorem 2.2. *Suppose that B is a positive operator on a Banach lattice E with topologically full center such that*

(i) *B is locally quasi-nilpotent at some $x_0 > 0$, and*
(ii) *there is $S \in [B\rangle$ such that S dominates a non-zero AM-compact operator K.*

Then $[B\rangle$ has an invariant closed ideal.

Proof. Since the null ideal N_B of B is $[B\rangle$-invariant, we may assume that $N_B = \{0\}$. Let $z \in E$ such that $Kz \neq 0$. This means that at least one of the vectors $(Kz)^+$ and $(Kz)^-$ is non-zero. Suppose $(Kz)^+ \neq 0$. Then, by topological fullness of $Z(E)$, there exists an operator $M \in Z(E)_{1+}$ such that $M|Kz| \neq 0$. Indeed, otherwise for all $M \in Z(E)_{1+}$ we would have $M|Kz| = 0$. But then, for the sequence $(T_n)_{n\in\mathbb{N}}$ in $Z(E)_{1+}$ with $T_n(|Kz|) \to (Kz)^+$ in norm, we would have $(Kz)^+ = 0$, which is a contradiction. Suppose now that there exists $M \in Z(E)_{1+}$ such that $M|Kz| \neq 0$. From $M((Kz)^+) + M((Kz)^-) \neq 0$ it follows that $M((Kz)^+) \neq 0$ or $M((Kz)^-) \neq 0$. Suppose that $M((Kz)^+) \neq 0$. But since M is a lattice homomorphism, we have $(MKz)^+ \neq 0$, and so it follows from $M((Kz)^+) \wedge M((Kz)^-) = (MKz)^+ \wedge (MKz)^- = 0$ that $(MKz)^- = 0$ and $MKz > 0$. Put $K_1 := MK$. It follows from $N_B = \{0\}$ that $BK_1z \neq 0$, hence $BK_1 \neq 0$. It is also clear that BK_1 is AM-compact and is dominated by BS.

Let $\mathcal{J}$ be the semigroup ideal in $[B\rangle$ generated by BS, that is,

$$\mathcal{J} = \{A_1BSA_2 \mid A_1, A_2 \in [B\rangle\}.$$

It can be verified directly that $\mathcal{J}$ is finitely quasi-nilpotent at x_0. Since $BS \in \mathcal{J}$ and BS dominates a non-zero AM-compact operator, $\mathcal{J}$ has an invariant closed ideal by [1, Theorem 10.44]. Now [1, Theorem 10.49] yields that $[B\rangle$ has an invariant closed ideal. □

The next result is a generalization of [1, Theorem 10.57] which states that if a non-zero compact-friendly operator B on a Dedekind-complete Banach lattice E is locally quasi-nilpotent, then there exists a non-trivial closed ideal that is invariant under $[B\rangle$. We show that Dedekind completeness and compact-friendliness are not needed and that E having topologically full center and B being weakly compact-friendly are sufficient. The proof is a modification of the proof of Theorem 10.57 in [1] and uses Theorem 2.2.

Theorem 2.3. *Let E be a Banach lattice with topologically full center. If B is a locally quasi-nilpotent weakly compact-friendly operator on E, then $[B\rangle$ has a non-trivial closed invariant ideal.*

Proof. For each $x > 0$, denote by J_x the ideal generated by the orbit $[B\rangle x$; that is

$$J_x := \{y \in E \mid |y| \leq Ax \text{ for some } A \in [B\rangle\}.$$

Since the identity operator belongs to $[B\rangle$, we have that $x \in J_x$, so this is a non-zero ideal. Note that J_x is $[B\rangle$-invariant: because, if $y \in J_x$, then $|y| \leq Ax$ for some $A \in [B\rangle$ and hence for any $A_1 \in [B\rangle$ we have

$$|A_1 y| \leq A_1|y| \leq A_1 A x,$$

yielding that $A_1 y \in J_x$ since the operator $A_1 A$ belongs to $[B\rangle$ which is a multiplicative semigroup. Therefore, in case where there exists a positive $x \in E$ such that the ideal J_x is not norm-dense in E, the proof is complete. So, suppose that $\overline{J_x} = E$ for each $x > 0$.

Fix three non-zero operators with R, K positive, K compact, and satifying

$$BR \leq RB, \quad |Cx| \leq C|x|, \quad \text{and} \quad |Cx| \leq K|x| \quad \text{for each } x \in E.$$

Since $C \neq 0$ there exists some $x_1 > 0$ such that $Cx_1 \neq 0$. This means that at least one of the vectors $(Cx_1)^+$ and $(Cx_1)^-$ is non-zero. Suppose $(Cx_1)^+ \neq 0$. Then, by topological fullness of $Z(E)$, there exists an operator $M_1 \in Z(E)_{1+}$ such that $M_1|Cx_1| \neq 0$. Indeed, otherwise for all $M_1 \in Z(E)_{1+}$ we would have $M_1|Cx_1| = 0$. But then, for the sequence $(T_n)_{n\in\mathbb{N}}$ in $Z(E)_{1+}$ with $T_n(|Cx_1|) \to (Cx_1)^+$ in norm, we would have $(Cx_1)^+ = 0$, which is a contradiction. Suppose now that there exists $M_1 \in Z(E)_{1+}$ such that $M_1|Cx_1| \neq 0$. From $M_1((Cx_1)^+) + M_1((Cx_1)^-) \neq 0$ it follows that $M_1((Cx_1)^+) \neq 0$ or $M_1((Cx_1)^-) \neq 0$. Suppose that $M_1((Cx_1)^+) \neq 0$. But since M_1 is a lattice homomorphism, we have $(M_1Cx_1)^+ \neq 0$, and so it follows from $M_1((Cx_1)^+) \wedge M_1((Cx_1)^-) = (M_1Cx_1)^+ \wedge (M_1Cx_1)^- = 0$ that $(M_1Cx_1)^- = 0$ and $M_1Cx_1 > 0$. Let $x_2 := M_1Cx_1 > 0$ and $\pi_1 := M_1C$. Note that π_1 is dominated by R and K.

Now we have $\overline{J_{x_2}} = E$, and since $C \neq 0$ there exists some $y \in J_{x_2}$ and an operator $A_1 \in [B\rangle$ such that $0 < y \leq A_1x_2$ and $Cy \neq 0$. We claim that there exists $M \in Z(E)_{1+}$ such that $CMA_1x_2 \neq 0$. Otherwise, if $CMA_1x_2 = 0$ for all $M \in Z(E)_{1+}$, we would have $CT_nA_1x_2$ for each $n \in \mathbb{N}$ for the sequence $(T_n)_{n\in\mathbb{N}}$ for which $T_nA_1x_2 \to y$. This would yield $CT_nA_1x_2 \to Cy$ and $Cy = 0$, which is a contradiction. Since $CMA_1x_2 \neq 0$, one has $|CMA_1x_2| \neq 0$. Suppose $(CMA_1x_2)^+ \neq 0$. By topological fullness of $Z(E)$, there exists a sequence $(T_n)_{n\in\mathbb{N}}$ in $Z(E)_{1+}$ such that $T_n(|CMA_1x_2|) \to (CMA_1x_2)^+$. Since $(CMA_1x_2)^+ \neq 0$, not all $T_n(|CMA_1x_2|)$ are zero, and we can choose $M_2 \in Z(E)_{1+}$ with $M_2|CMA_1x_2| \neq 0$. Notice that $M_2((CMA_1x_2)^+) \wedge M_2((CMA_1x_2)^-) = 0$. Since $M_2((CMA_1x_2)^+) \neq 0$, we have $M_2((CMA_1x_2)^-) = (M_2CMA_1x_2)^- = 0$, which yields $M_2CMA_1x_2 > 0$. Put $x_3 := M_2CMA_1x_2 > 0$ and $\pi_2 := M_2CMA_1$ and observe that π_2 is dominated by RA_1 and KA_1. Repeating once more the preceding argument with x_2 replaced by x_3, we then obtain an operator $A_2 \in [B\rangle$ and an operator $\pi_3 : E \to E$ such that $\pi_3 x_3 > 0$ and π_3 is dominated by RA_2 and KA_2. From $\pi_3\pi_2\pi_1 x_1 = \pi_3 x_3 > 0$, we see that $\pi_3\pi_2\pi_1 \neq 0$.

Set $S := RA_2RA_1R \geq 0$. Since $|\pi_3\pi_2\pi_1 x| \leq S|x|$ for each $x \in E$, it follows that $S \neq 0$. Moreover, since each $\pi_i\,(i = 1, 2, 3)$ is dominated by a compact operator, we have by [2, Theorem 5.14] that $\pi_3\pi_2\pi_1$ is compact. Moreover, because R, A_1, and A_2 belong to $[B\rangle$, so does S. Thus, $[B\rangle$ contains a non-zero positive operator which dominates a compact operator. Now, invoke Theorem 2.2 to complete the proof. □

References

[1] Y.A. Abramovich & C.D. Aliprantis, *An Invitation to Operator Theory*, Graduate Studies in Mathematics, Vol. 50, American Mathematical Society, Providence, RI, 2002.

[2] C.D. Aliprantis & O. Burkinshaw, *Positive Operators*, Springer, The Netherlands, 2006.

[3] M. Çağlar & T. Mısırlıoğlu, "Weakly compact-friendly operators," *Vladikavkaz Mat. Zh.* **11** (2009), no. 2, 27-30.

[4] J. Flores, P. Tradacete & V.G. Troitsky, "Invariant subspaces of positive strictly singular operators on Banach lattices," *J. Math. Anal. Appl.* **343** (2008), no. 2, 743-751.

[5] A.W. Wickstead, "Extremal structure of cones of operators," *Quart. J. Math. Oxford* **32** (1981), no. 2, 239-253.

[6] A.W. Wickstead, "Banach lattices with topologically full centre," *Vladikavkaz Mat. Zh.* **11** (2009), no. 2, 50-60.

(Mert Çağlar) Department of Mathematics and Computer Science, İstanbul Kültür University, Bakırköy 34156, İstanbul, Turkey
E-mail address: `m.caglar@iku.edu.tr`

(Tunç Mısırlıoğlu) Department of Mathematics and Computer Science, İstanbul Kültür University, Bakırköy 34156, İstanbul, Turkey
E-mail address: `t.misirlioglu@iku.edu.tr`