# Weighted Approximation by Videnskii and Lupas Operators

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# Abstract

Weighted modifications of generalized Bernstein operators in rational functions (Videnskii operators) are introduced. Their convergence in weighted spaces is studied.

The Bernstein polynomials

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{1}$$

associated with a function $f$ defined on $[0,1]$ have been the subject of much recent research and have been generalized in many directions.(see for instance [1],[2],[3]) .

# Introduction

In 1966 J. P. King [4] introduced the following generalization of the Bernstein polynomials

$$L_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) u_{nk}(x), \tag{2}$$

where $u_{nk}(x)$ are given by the generating function

$$g_n(x,y) = \prod_{i=1}^{n} \left(h_{ni}(x)\,y + (1 - h_{ni}(x))\right) = \sum_{k=0}^{n} u_{nk}(x)\,y^k, \tag{3}$$

and $h_{ni}(x) = h_i(x)$ is a sequence of continuous functions defined on $[0,1]$, $0 \le h_i(x) \le 1$.

# Introduction

King's operators converge to the approximated function if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}h_i\left(x\right)=x. \tag{4}$$

Now let $x_{ni}$ be fixed poles $x_{ni}=1+\rho_{ni}$, $\rho_{ni}>0$ and

$$h_{ni}\left(x\right)=\frac{\rho_{ni}x}{1+\rho_{ni}-x}. \tag{5}$$

Put

$$\phi_n\left(x\right)=\frac{1}{n}\sum_{k=1}^{n}h_{nk}\left(x\right).$$

Observe that $\phi_n\left(x\right)$ is strictly increasing from 0 to 1 on the interval $\left[0,1\right]$. The nodes $\tau_{nk}$ are well-defined by $\phi_n\left(\tau_{nk}\right)=\frac{k}{n}$, $\left(k=0,1,...,n\right)$.

In 1979 V. S. Videnskii [5] introduced another generalization of Bernstein operators for approximation by rational functions with fixed poles

$$\sum_{k=0}^{n} V_n(f,x) = f(\tau_{nk})\, u_{nk}(x). \tag{6}$$

Later V. S Videnskii considered more general case of the operators (6), where $u_{nk}$ are defined for arbitrary increasing functions $h_{ni}(x)$. The main difference between those families of operators is in nodes. The advantage of Videnskii's operators can be easily seen from the conditions for their convergence. Namely V. S Videnskii (see for instance ([6] th. 3.1)) proved that sequence $V_n(f,x)$ uniformly converges to arbitrary $f \in C[0,1]$ if and only if

$$\lim_{n\to\infty} S_n = \infty \tag{7}$$

where $S_n = \sum_{i=1}^{n} \frac{\rho_{ni}}{1+\rho_{ni}}$.

A simple example $(\rho_{ni} = \rho_n = 1)$ shows that condition $(4)$ is essentially more restrictive than $(7)$ . Later V. S. Videnskii [7] considered arbitrary matrices of nodes $\xi_{nk}$ instead of $\tau_{nk}$ and proved the convergence results for those operators $V_n^{\xi}(f, x)$ . Note that for $\xi_{nk} = \frac{k}{n}$ we recover King's operators. Moreover recently he observed [8] that another well-known generalization of Bernstein polynomials, namely Lupaș operators, (see, for example [9]) can be considered as a particular case of the operators $V_n^{\xi}(f, x)$ , too.
Recently many authors pay attention to weighted approximation by classical polynomial operators and to construction of their weighted modifications. The reason is that usual operators are not always suitable for approximating functions with singularities in weighted spaces.

# Introduction

For instance, the sequence of classical Bernstein operators (1) is not bounded in the space

$$C_w = \left\{ f \in C\left((0,1)\right) : \lim_{x \to 0} \left(wf\right)(x) = \lim_{x \to 1} \left(wf\right)(x) = 0 \right\},$$

where

$$\|f\|_{C_w} := \|wf\| = \sup_{x \in [0,1]} \left(wf\right)(x),$$

$$w(x) = x^{\alpha}(1-x)^{\beta}, \alpha, \beta \geq 0, \alpha + \beta > 0, 0 \leq x \leq 1,$$

but it's slight modification

$$B_n^*(f,x) = (1-x)^n \left[2f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right] + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) p_{nk}(x) + x^n \left[2f\left(1-\frac{1}{n}\right) - f\left(1-\frac{2}{n}\right)\right] \quad (8)$$

is bounded. One can consult papers [10], [11] and [12] which contain these and other deep results in this direction.

Following [10] we consider the Sobolev type space $W_{\omega}^{2}$ defined as

$$W_{\omega}^{2} := \left\{ f \in C_w : f' \in AC\left((0,1)\right), \left\| f^{''} \omega \varphi^2 \right\| < \infty \right\}$$

where $\varphi(x) = \sqrt{x(1-x)}$ and $AC(I)$ is the set of absolutely continuous functions in $I$.

Observe also that modification (8) is not a positive operator, so general results about weighted approximation by linear positive operators on a real interval (see, for instance, [13] and references therein) are not applicable here.The main goal of the research is to investigate approximation properties of Videnskii operators in the norm of $C_w$ under some restrictions on the sequence of denominators.In the following C denotes a positive constant which may assume different values in different formulas. Moreover we write $v \sim u$ for two quantities $v$ and $u$ depending on some parameters, if $\left|\frac{v}{u}\right|^{\pm 1} \leq C$ with $C$ independent of the parameters.

Note that operators (6) as well as the Bernstein operators are not bounded (in fact even not defined) in $C_w$.
Here we consider modifications of the Videnskii operators similar to (8):

$$V_n^*(f,x)=\sum_{k=1}^{n-1} f(\tau_{nk})\,u_{nk}(x)+u_{n0}(x)\left[2f(\tau_{n1})-f(\tau_{n2})\right] \qquad (9)$$
$$+u_{nn}(x)\left[2f(\tau_{n\,n-1})-f(\tau_{n\,n-2})\right].$$

# Introduction

The main result of the research is

**Theorem**

*Suppose that* $\rho_{ni}$ *satisfy* $\rho_{ni} > C > 0$ *and* $\sum_{i=1}^{n} \frac{1}{\rho_{ni}} \leq C$, *then*

*a)*

$$\left\| V_n^* (f) \right\|_{C_w} \leq C \left\| f \right\|_{C_w}$$

*b)*

$$\left\| \left[ f - V_n^* (f) \right] \right\|_{C_w} \leq \frac{C}{n} \left\| \varphi^2 f'' \right\|_{C_w} \text{ if } f \in W_\omega^2.$$

# Videnskii operators and their properties

Now we can give some basic facts about Videnskii's operators (6) from [8]. Put

$$P_n(x) = \prod_{i=1}^{n} (1 + \rho_{ni} - x) = \prod_{i=1}^{n} (\rho_{ni} x + (1 + \rho_{ni})(1 - x))$$
$$= \sum_{k=0}^{n} \alpha_{nk} x^k (1 - x)^{n-k}$$

then we can write basic functions $u_{nk}(x)$ for Videnskii operators as

$$u_{nk}(x) = \alpha_{nk} \frac{x^k (1 - x)^{n-k}}{\prod_{i=1}^{n} |x - x_{ni}|}, \ \alpha_{nk} > 0 \tag{10}$$

It is clear that $u_{nk}$ satisfy the equality

$$\sum_{k=0}^{n} u_{nk}(x) = 1. \tag{11}$$

Differentiate in $y$

$$\frac{\partial\left(g_n\left(x,y\right)\right)}{\partial y}=g_n\left(x,y\right)\sum_{i=0}^{n}\frac{h_{ni}\left(x\right)}{h_{ni}\left(x\right)y+1-h_{ni}\left(x\right)}=\sum_{k=0}^{n}ku_{nk}\left(x\right)y^{k-1}$$

and put $y=1$

$$\phi_n\left(x\right)=\sum_{k=0}^{n}\frac{k}{n}u_{nk}\left(x\right). \qquad (12)$$

Note that the functions 1, $\phi_n\left(x\right)$ play role of the fixed functions $f_0, f_1$ for generalized Bernstein operators in the sense of [14] for the system of rational functions of degree $n$ with denominator $P_n\left(x\right)$.
Rewrite (12) in the form

$$\sum_{k=0}^{n}\left(\phi_n\left(\tau_{nk}\right)-\phi_n\left(x\right)\right)u_{nk}\left(x\right)=0. \qquad (13)$$

# Videnskii operators and their properties

Differentiation of $\ln u_{nk}(x)$ $(0 < x < 1)$ gives

$$u'_{nk}(x) = \frac{n}{x(1-x)}\left(\phi_n(\tau_{nk}) - \phi_n(x)\right) u_{nk}(x). \tag{14}$$

Formula (14) shows by the way that the point $\tau_{nk}$ $(1 \leq k \leq n-1)$ is the unique point of maximum of the function $u_{nk}$ in the interval $[0,1]$. That is a reason why the Videnskii operators can be considered as a natural analogue of the Bernstein operators for rational functions. Derivative of (13) with taking into account (14) gives

$$\sum_{k=0}^{n}\left(\phi_n(\tau_{nk}) - \phi_n(x)\right)^2 u_{nk}(x) = \frac{x(1-x)}{n}\phi'_n(x). \tag{15}$$

Next assertions are not given in [8] but are necessary for the following. First we give a lemma which can be considered as an excercise for a calculus textbook.

### Lemma

*If $n, m \in \mathbb{N}$ , $f_i \in C^n([a,b])$ , $i = 1, ..., m$ then the following equality holds*

$$\frac{d^n}{dy^n}\left(\prod_{i=1}^{m} f_i(y)\right) = \sum_{\substack{j_1+j_2+...+j_m=n \\ j_1\geq 0,..., j_m\geq 0}} \frac{n!}{j_1!j_2!...j_m!}\frac{d^{j_1}}{dy^{j_1}}(f_1(y))\dots\frac{d^{j_m}}{dy^{j_m}}(f_m(y)). \tag{16}$$

### Corollary

*If $h_{ni}$ is defined as in (5) then*

$$u_{nk}(x) = \frac{1}{k!}\sum_{\substack{j_1+j_2+...+j_n=k \\ 0\leq j_i\leq 1}} \frac{k!}{j_1!j_2!...j_n!}\prod_{i=1}^{n}\left(1 - j_i - (-1)^{j_i} h_{ni}(x)\right) \tag{17}$$

Differentiate (3) and use Lemma 2.

## Lemma

*Under suppositions of Theorem 1,*
*for any* $x \in (0,1)$

$$C \leq \frac{\phi_n(x)}{x} \leq 1 \tag{18}$$

*and*

$$1 \leq \frac{1-\phi_n(x)}{1-x} \leq C. \tag{19}$$

## Proof.

Firstly

$$\phi_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{\rho_{ni}x}{1+\rho_{ni}-x} \geq x\frac{1}{n}\sum_{i=1}^{n}\frac{\rho_{ni}}{1+\rho_{ni}} \geq Cx$$

and

$$\phi_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{\rho_{ni}x}{1+\rho_{ni}-x} \leq x.$$

## Corollary

*Suppose that* $\rho_{ni}$ *satisfy suppositions of Theorem 1 then* $w(x) \sim w\left(\phi_n^{-1}(x)\right)$ *and* $\varphi(x) \sim \varphi\left(\phi_n^{-1}(x)\right)$
*Observe also that from definition of* $u_{nk}(x)$ *it follows immediately that* $0 \leq u_{nk}(x) \leq 1 \;\; k = 0, ..., n \; ; \; n = 1, ....$

Using $(5), (17)$ we get

$$u_{nk}(x) = \frac{\displaystyle\sum_{\substack{j_1+j_2+...+j_n=k \\ 0\leq j_i \leq 1}} x^k (1-x)^{n-k} \prod_{i=1}^{n} (1+\rho_{ni}-j_i)}{\displaystyle\prod_{i=1}^{n} (1+\rho_{ni}-x)},$$

and we can write down an explicit formula for the coefficients $\alpha_{nk}$ from $(10)$ :

$$\alpha_{nk} = \sum_{\substack{j_1+j_2+...+j_n=k \\ j_i \in \{0,1\}}}^{n} \prod_{i=1}^{n} (\rho_{ni} + 1 - j_i)\,. \qquad (20)$$

## Lemma

*If $\rho_{ni}$ satisfy the suppositions of Theorem 1 then*

$$\frac{\alpha_{nk}}{\binom{n}{k} P_n(x)} \leq C$$

## Proof.

Firstly $\ln \prod_{i=1}^{n} \left(1 + \frac{1}{\rho_{ni}}\right) \leq \sum_{i=1}^{n} \frac{1}{\rho_{ni}} \leq C$

then

$$= \frac{\alpha_{nk}}{\binom{n}{k} P_n(x)} = \frac{1}{\binom{n}{k}} \sum_{\substack{j_1+j_2+\ldots+j_n=k \\ j_i \in \{0,1\}}}^{n} \prod_{i=1}^{n} \frac{\rho_{ni} + 1 - j_i}{\rho_{ni} + 1 - x}$$

$$\leq \frac{1}{\binom{n}{k}} \sum_{\substack{j_1+j_2+\ldots+j_n=k \\ j_i \in \{0,1\}}}^{n} \prod_{i=1}^{n} \frac{\rho_{ni} + 1 - j_i}{\rho_{ni}} \leq \prod_{i=1}^{n} \frac{\rho_{ni} + 1}{\rho_{ni}} \leq C. \qquad (21)$$

# Proof of Theorem 1

We start with

a)

$$w(x)\left|\sum_{k=1}^{n-1} f(\tau_{nk})\, u_{nk}(x)\right| = w(x)\left|\sum_{k=1}^{n-1} \frac{\alpha_{nk}\binom{n}{k}x^k(1-x)^{n-k}}{\binom{n}{k}P_n(x)} f(\tau_{nk})\right|$$

$$= w\left(\phi_n^{-1}(x)\right)\left|\sum_{k=1}^{n-1} \frac{\alpha_{nk}.p_{nk}(x)\, f(\tau_{nk})\, w(\tau_{nk})}{\binom{n}{k}P_n(x)\, w(\tau_{nk})}\right|$$

$$\leq \|wf\| \sum_{k=1}^{n-1} \frac{\alpha_{nk}.p_{nk}(x)\, w\left(\phi_n^{-1}(x)\right)}{\binom{n}{k}P_n(x)\, w\left(\phi_n^{-1}\left(\frac{k}{n}\right)\right)}.$$

The proof of $\sum_{k=1}^{n-1} p_{nk}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} \leq C$ is contained in [10], p.30. Analogously other terms in $V_n^*(f,x)$ are considered.

The Corollary 5 and Lemma 6 finish the proof of Part a).

# Proof of Theorem 1

## Lemma

*Under suppositions of Theorem 1 the inequalities* $\phi_n'(x) \sim 1$ *and* $\left\| \left(\phi_n^{-1}\right)'' \right\| \leq C$ *hold.*

## Proof.

We start with

$$\phi_n'(x) \geq \frac{1}{n} \sum_{k=1}^{n} \frac{\rho_{nk}}{(1+\rho_{nk})} \geq C,$$

$$\phi_n'(x) \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1+\rho_{nk}}{\rho_{nk}} \leq C.$$

□

# Proof of Theorem 1

**Proof.**

Put $t = \phi_n(x)$. Then

$$\left|\left(\phi_n^{-1}\right)''(t)\right| = \left|\left(\frac{1}{\phi_n'\left(\phi_n^{-1}(t)\right)}\right)'\right|$$

$$= \frac{1}{\left(\phi_n'\left(\phi_n^{-1}(t)\right)\right)^2}\left|\phi_n''\left(\phi_n^{-1}(t)\right)\left(\phi_n^{-1}\right)'(t)\right|$$

$$\phi_n''(x) = \frac{2}{n}\sum_{k=1}^{n}\frac{\rho_{nk}\left(1+\rho_{nk}\right)}{\left(1+\rho_{nk}-x\right)^3} \leq \frac{2}{n}\sum_{k=1}^{n}\frac{1+\rho_{nk}}{\rho_{nk}^2} \leq C.$$

we prove the lemma. □

# Proof of Theorem 1

## Lemma

*If* $f \in W_w^2$ *then* $f \circ \phi_n^{-1} \in W_w^2$.

## Proof.

We start with

$$\left(f\left(\phi_n^{-1}(t)\right)\right)'' = \left(f'\left(\phi_n^{-1}(t)\right)\left(\phi_n^{-1}(t)\right)'\right)'$$
$$= f''\left(\phi_n^{-1}(t)\right)\left(\left(\phi_n^{-1}(t)\right)'\right)^2 + f'\left(\phi_n^{-1}(t)\right)\left(\phi_n^{-1}(t)\right)''.$$

Consider firtsly $0 \leq t \leq \frac{1}{2}$ .Then

$$f'(t)\,\varphi^2(t)\,w(t) = \int_{\frac{1}{2}}^{t} f''(x)\,dx\varphi(t)^2\,w(t) + f'\left(\frac{1}{2}\right)\varphi^2(t)\,w(t)$$

**Proof.**

and

$$\left|\int_{\frac{1}{2}}^{t} f''(x)\,dx\varphi(t)^2 w(t)\right| \leq \left\|f''\varphi^2 w\right\| \int_t^{\frac{1}{2}} \frac{dx}{\varphi^2(x)\,w(x)}\varphi^2(t)\,w(t)$$

$$\leq C\left\|f''\varphi^2 w\right\| \left[x^{-\alpha}\right]_t^{\frac{1}{2}} \varphi^2(t)\,w(t) \leq C\left\|f''\varphi^2 w\right\|$$

The case $\frac{1}{2} \leq t \leq 1$ is analogous. Hence by Corollary 5 and Lemma 7 , the lemma is proved. □

## Lemma

*f* $\alpha, \beta > 0$, $0 \le x \le 1$ *then*

$$D_n(x) = w(x) \sum_{\left|\frac{k}{n} - \phi_n(x)\right| \ge \frac{\phi_n(x)}{2}} u_{nk}(x) \left| \int_{\frac{k}{n}}^{\phi_n(x)} \frac{\left|\xi - \frac{k}{n}\right|}{\varphi^2(\xi)\, w(\xi)} d\xi \right| \le \frac{C}{n}$$

## Proof.

Firstly let us assume that $0 \le x \le \frac{1}{2}$.Then the restriction $\left|\frac{k}{n} - \phi_n(x)\right| \ge \frac{\phi_n(x)}{2}$ splits in to either

$$\frac{k}{n} - \phi_n(x) \le -\frac{\phi_n(x)}{2}$$

i.e

$$\frac{k}{n} \le \frac{\phi_n(x)}{2}$$

## Proof.

$$D_{n1}(x) = Cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{\phi_n(x)}{2}} u_{nk}(x) \int_{\frac{k}{n}}^{\phi_n(x)} \frac{\left|\xi - \frac{k}{n}\right|}{\varphi^2(\xi)\, w(\xi)} d\xi \leq Cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{\phi_n(x)}{2}} u_{nk}(x) \int_{\frac{k}{n}}^{\phi_n}$$

$$\leq Cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{x}{2}} p_{nk}(x) \int_{\frac{k}{n}}^{x} \xi^{-\alpha} d\xi \leq \frac{C}{n}$$

by lemma 6 and $[[10], (18)]$.
For estimating $D_{n2}(x)$

$$D_{n2}(x) \leq w(x) \sum_{\frac{3\phi_n(x)}{2} \leq \frac{k}{n}} u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\frac{k}{n} - \xi}{\varphi^2(\xi)\, w(\xi)} d\xi$$

$$= w(x) \left\{ \sum_{\frac{3\phi_n(x)}{2} \leq \frac{k}{n} \leq \frac{2}{3}} + \sum_{\frac{k}{n} \geq \frac{2}{3}} \right\} u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\frac{k}{n} - \xi}{\varphi^2(\xi)\, w(\xi)} d\xi$$

$$= D^{(1)} + D^{(2)}$$

# Proof of Theorem 1

## Proof.

Note that one of $D_{n2}^{(1)}$ or $D_{n2}^{(2)}$ may be absent.
For $D_{n2}^{(1)}$ we can write using Lemma 7, (15) and (18).

$$D_{n2}^{(1)} \leq Cx^{\alpha} \sum_{\frac{3\phi_n(x)}{2} \leq \frac{k}{n} \leq \frac{2}{3}} u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\frac{k}{n} - \xi}{\xi^{\alpha+1}} d\xi$$

$$\leq \frac{C}{\phi_n(x)} \sum_{k=0} u_{nk}(x) \left( \frac{k}{n} - \phi_n(x) \right)^2 \leq \frac{C}{n}.$$

For $D_{n2}^{(2)}$ we have

$$D_{n2}^{(2)} = w(x) \sum_{\frac{k}{n} \geq \frac{2}{3}} u_{nk}(x) \left( \int_{\phi_n(x)}^{\frac{2}{3}} \frac{\frac{k}{n} - \xi}{\xi^{\alpha+1}(1-\xi)^{\beta+1}} d\xi + \int_{\frac{2}{3}}^{\frac{k}{n}} \frac{\frac{k}{n} - \xi}{\xi^{\alpha+1}(1-\xi)^{\beta+1}} \right.$$

## Proof.

$$\leq w(x) \sum_{\frac{k}{n} \geq \frac{2}{3}} u_{nk}(x) \left( \frac{C}{x^{\alpha}} + \int_0^{\frac{k}{n}} \frac{d\xi}{(1-\xi)^{\beta}} \right)$$

$$\leq Cx^{-1} \sum_{\frac{k}{n} \geq \frac{2}{3}} p_{nk}(x) \int_0^{\frac{k}{n}} \frac{d(\xi)}{(1-\xi)^{\beta}} \leq Cx^{-1} p_{n,\left[\frac{2n}{3}\right]}(x)\, n$$

$$= Cx^{\left[\frac{2n}{3}\right]-1} (1-x)^{n-\left[\frac{2n}{3}\right]} \binom{n}{\left[\frac{2n}{3}\right]} n \leq C \left( \frac{3\sqrt{3}}{4\sqrt{2}} \right)^{\frac{2n}{3}}$$

$$\binom{n}{\frac{2n}{3}} \leq \left( \frac{3}{2}\sqrt{3} \right)^{\frac{2n}{3}}$$

$$x^{\left[\frac{2n}{3}\right]-1} (1-x)^{n-\left[\frac{2n}{3}\right]} \leq x^{\frac{n}{3}-1} x^{\frac{n}{3}} (1-x)^{\frac{n}{3}}$$

$$\leq \left( \frac{1}{2} \right)^{\frac{n}{3}-1} \left( \frac{1}{4} \right)^{\frac{n}{3}}.$$

## Definition

Let $\psi(x)$, $P_1(f)$, $P_2(f)$ and $F_n(f\,;x)$ be defined as in [10],

$$\psi(x) = \begin{pmatrix} 10x^3 - 15x^4 + 6x^5, & \text{if } \ 0 \le x \le 1 \\ 0, & \text{if } \ 0 \le x \\ 1, & \text{if } \ x \ge 1. \end{pmatrix}$$

$P_1(f)$ and $P_2(f)$ are the linear functions interpolating $f$ at the points $\frac{1}{n}, \frac{2}{n}$ and $1-\frac{1}{n}$ and $1-\frac{2}{n}$ respectively

$$P_1(f,x) := (2-nx)\, f\left(\frac{1}{n}\right) + (nx-1)\, f\left(\frac{2}{n}\right)$$

$$P_2(f,x) := [2-n(1-x)]\, f\left(\frac{n-1}{n}\right) + [n(1-x)-1]\, f\left(\frac{n-2}{n}\right).$$

$$\begin{aligned} F_n(f,x) := {} & (1-\psi(nx-1))\, P_1(f,x) + \\ & (1-\psi(nx-n+2))\, \psi(nx-1)\, f(x) + \\ & \psi(nx-n+2)\, P_2(f,x) \end{aligned}$$

## Lemma

*If $f \in W_{\omega}^{2}$ then for $F_n = F_n\left(f \circ \phi_n^{-1}\right)$ and for all $\alpha, \beta \geq 0$*

$$\left\|\left(F_n \circ \phi_n - V_n^{*}\left(f\right)\right) w\right\| \leq \frac{C}{n}\left\|F_n'' \varphi^2 w\right\|.$$

## Proof.

First consider

$$\sum_{k=0}^{n} u_{nk}\left(x\right) \int_{x}^{\tau_{nk}} \left(\phi_n\left(t\right) - \phi_n\left(\tau_{nk}\right)\right) F_n''\left(\phi_n\left(t\right)\right) \phi_n'\left(t\right) d\left(t\right)$$

$$= -\sum_{k=0}^{n} F_n\left(\frac{k}{n}\right) u_{nk}\left(x\right) + \sum_{k=0}^{n} F_n\left(\phi_n\left(x\right)\right) u_{nk}\left(x\right)$$

$$= -\sum_{k=1}^{n-1} f \circ \phi_n^{-1}\left(\frac{k}{n}\right) u_{nk}\left(x\right) - \left[2f \circ \phi_n^{-1}\left(\frac{1}{n}\right) - f \circ \phi_n^{-1}\left(\frac{2}{n}\right)\right] u_{n,0}\left(x\right)$$

□

## Proof.

$$= -\left[2f\circ\phi_n^{-1}\left(\frac{n-1}{n}\right) - f\circ\phi_n^{-1}\left(\frac{n-2}{n}\right)\right]u_{n,n}(x) + F_n(\phi_n(x))$$
$$= F_n(\phi_n(x)) - V_n^*(f,x).$$

Hence

$$\left|F_n(\phi_n(x)) - V_n^*(f,x)\right| w(x) \le w(x)\sum_{k=0}^{n} u_{nk}(x)\int_{\phi_n(x)}^{\frac{k}{n}}\left|\xi-\frac{k}{n}\right|\left|F_n''(\xi)\right| d($$

$$\sum_{\left|\frac{k}{n}-\phi_n(x)\right|\le\frac{\phi_n(x)}{2}} + \sum_{\left|\frac{k}{n}-\phi_n(x)\right|\ge\frac{\phi_n(x)}{2}} := E_1(x) + E_2(x).$$

Firstly suppose that $0\le x\le\frac{1}{2}$. For $E_1(x)$ as in $[[10],\ p.27]$, $E_1(x)=0$ for $0\le x\le\frac{1}{n}$. Now if $\frac{1}{n}\le x\le\frac{1}{2}$ then $1\le k\le n-1$ and □

## Proof.

$$E_1(x) \leq \sum_{\left|\frac{k}{n}-\phi_n(x)\right| \leq \frac{\phi_n(x)}{2}} w(x)\, u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\left|\xi - \frac{k}{n}\right| \left|F_n''(\xi)\, w(\xi)\, \varphi^2(\xi)\right|}{w(\xi)\, \varphi^2(\xi)} d(\xi$$

$$\leq C \frac{\left\|F_n'' \varphi^2 w\right\|}{(1-x)\,x} \sum_{\left|\frac{k}{n}-\phi_n(x)\right| \leq \frac{\phi_n(x)}{2}} u_{nk}(x) \left(\frac{k}{n} - \phi_n(x)\right)^2$$

$$\leq \frac{c}{n} \left\|\phi_n'(x)\right\| \left\|F_n'' \varphi^2 w\right\| \leq \frac{c}{n} \left\|F_n'' \varphi^2 w\right\|$$

On the other hand by lemma 9

$$E_2(x) \leq C \left\|F_n'' \varphi^2 w\right\| D_n(x) \leq \frac{C}{n} \left\|F_n'' \varphi^2 w\right\|.$$

□

# Proof of Theorem 1

## Lemma

*If $f \in W^2_\omega$ then for $P_1 := P_1\left(f \circ \phi_n^{-1}\right)$ and $P_2 := P_2\left(f \circ \phi_n^{-1}\right)$ and we have*

$$\left\| w\left[f - P_1 \circ \phi_n\right]\right\|_{[0,\tau_{2n}]} \leq \frac{C}{n}\left\|f''\varphi^2 w\right\|.$$

$$\left\| w\left[f - P_2 \circ \phi_n\right]\right\|_{[\tau_{n-2,n},0]} \leq \frac{C}{n}\left\|f''\varphi^2 w\right\|.$$

## Proof.

By Lemma 8 we have $f \circ \phi_n^{-1} \in W_\omega^2$, then the proof of Lemma 3 in [10] gives

$$\max_{t\in\left[0,\frac{2}{n}\right]} \left|w\left(\phi_n^{-1}\right)\left|f\left(\phi_n^{-1}(t)\right) - P_1\left(f\circ\phi_n^{-1}, t\right)\right|\right| \le \frac{C}{n}\left\|\left(f\circ\phi_n^{-1}\right)''\varphi^2 w\right\|_{\left[0,\frac{2}{n}\right]}$$

and

$$\max_{t\in\left[1-\frac{2}{n},1\right]} \left|w\left(\phi_n^{-1}\right)\left|f\left(\phi_n^{-1}(t)\right) - P_2\left(f\circ\phi_n^{-1}, t\right)\right|\right| \le \frac{C}{n}\left\|\left(f\circ\phi_n^{-1}\right)''\varphi^2 w\right\|_{[1}$$

Now the proof of Lemma 8 gives the desired result. □

# Proof of Theorem 1

## Lemma

*Let* $F_n := F_n\left(f \circ \phi_n^{-1}\right)$. *If* $f \in W_\omega^2$ *then we have*

$$\left\| F_n'' \varphi^2 w \right\| \leq C \left\| f'' \varphi^2 w \right\|.$$

## Proof.

Apply the proofs of ([10], Lemma 4) and of Lemma 8 □

## Proof.

$(\textit{Theorem}1\ b)$ We know that for $\phi_n(x) \in \left[\frac{2}{n}, 1 - \frac{2}{n}\right]$

$$F_n \circ \phi_n(x) = F_n\left(f \circ \phi_n^{-1}, \phi_n(x)\right) = f \circ \phi_n^{-1}\left(\phi_n(x)\right) = f(x).$$

Then by Lemma 11 we deduce □

## Proof.

$$\begin{aligned}\left\| w\left[f - V_n^*(f)\right]\right\| &\le \left\| w\left[f - F_n\circ\phi_n\right] + w\left[F_n\circ\phi_n - V_n^*(f)\right]\right\| \\ &\le \frac{C}{n}\left\|F_n''\varphi^2 w\right\| + w\left[f - F_n\circ\phi_n\right] \\ &= \frac{C}{n}\left\|F_n''\varphi^2 w\right\| \\ &+ \max\left(\left\|w\left[f - F_n\circ\phi_n\right]\right\|_{\phi_n^{-1}\left[0,\frac{2}{n}\right]}, \left\|w\left[f - F_n\circ\phi_n\right]\right\|_{\phi_n^{-1}\left[1-\right.}\right.\end{aligned}$$

Now

$$\max_{x\in\ \phi_n^{-1}\left[0,\frac{2}{n}\right]} w(x)\left|f(x) - F_n\left(f\circ\phi_n^{-1}, \phi_n(x)\right)\right| = \max_{t\in\left[0,\frac{2}{n}\right]} w\left(\phi_n^{-1}(t)\right)\left|f\left(\phi_n^{-1}(\right.\right.$$

and Lemmas 11, 12 finish the proof. □

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