İSTANBUL ANALYSIS SEMINARS

ON ENTROPY AND DIAMETERS

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Abstract: The ε-*entropy* of a compact set A in a metric space X is defined by the formula: $\mathcal{H}_\varepsilon(A) = \mathcal{H}_\varepsilon(A, X) := \ln N_\varepsilon(A, X)$, where $N_\varepsilon(A, X)$ is the smallest integer N such that A can be covered by N sets of diameter not greater than 2ε.

For a set A in a Banach space X the *Kolmogorov diameters (*or *widths) of* A *with respect to the unit ball* $\mathbb{B}_X$ of the space X are the numbers:

$$d_k(A) = d_k(A, \mathbb{B}_X) := \inf_{L \in \mathcal{L}_k} \sup_{x \in A} \inf_{y \in L} \|x - y\|_X, \quad k = 0, 1, \ldots, \tag{1}$$

where $\mathcal{L}_k$ is the set of all subspaces of X of dimension $\leq k$.

It was stated in [LT] that the asymptotics

$$-\ln d_k(A, \mathbb{B}_X) \sim \sigma\, k, \quad k \to \infty, \tag{2}$$

where A is an absolutely convex compact set in a Banach space X, implies the asymptotics

$$\mathcal{H}_\varepsilon(A, X) \sim \tau \left(\ln \frac{1}{\varepsilon}\right)^2, \ \tau = \frac{1}{\sigma}. \tag{3}$$

Analysing thoroughly the proof in [LT], one can see that a slightly weaker result has been proved there, namely, that the asymptotics

$$d_k(A, \mathbb{B}_X) \asymp e^{-\sigma k}, \quad k \to \infty,$$

which is harder than (2), implies (3). Our main goal is the following assertion generalizing and strengthening this result.

Theorem 1. *Let A be an absolutely convex compact set in a complex Banach space X and $\alpha > 0$. Then the asymptotics*

$$-\ln d_k\left(A, \mathbb{B}_X\right) \sim \sigma\ k^{1/\alpha}, \quad k \to \infty, \tag{4}$$

holds if and only if the asymptotics

$$\mathcal{H}_\varepsilon\left(A, X\right) \sim \tau\ \left(\ln \frac{1}{\varepsilon}\right)^{\alpha+1}$$

takes place with the constant $\tau = \frac{2}{(\alpha+1)\sigma^\alpha}$.

The proof is based on the following lemma (for a given positive sequence $a = (a_k)$ we use the notation $m_a(t) := \sharp\left\{k : a_k \leq t\right\},\ t > 0$).

Lemma 2. *Let A be a compact absolutely convex set in a complex Banach space X. Then*

$$2\int_0^{\frac{1}{2\varepsilon}} \frac{m_c(t)}{t}\, dt \leq \mathcal{H}_\varepsilon\left(A, X\right) \lesssim 2\int_0^{\frac{M}{\varepsilon}} \frac{m_b(t)}{t}\, dt,\ \ \varepsilon \searrow 0, \tag{5}$$

with some constant $M > 0$; here $b = \left(1/d_{k-1}\left(A, \mathbb{B}_X\right)\right)$ and $c = \left(k/d_{k-1}\left(A, \mathbb{B}_X\right)\right)$.

The left estimate in (5) is an easy adaptation of Mityagin's result ([M], Theorem 4, the right inequality) to the case of complex Banach spaces, while the right asymptotic inequality is an essential strengthening of the left inequality in that theorem; its proof is based on a quite refined technique from [LT].

Let K be a compact subset of an open set D on a Stein manifold Ω of dimension n, $H^\infty(D)$ the Banach space of all bounded and analytic in D functions endowed with the uniform norm, and A_K^D be a compact subset in the space of continuous functions $C(K)$ consisted of all restrictions of functions from the unit ball $\mathbb{B}_{H^\infty(D)}$. There are two old problems attributed to Kolmogorov: *for which pairs (K, D)*

(K1) *the asymptotics $H_\varepsilon\left(A_K^D\right) \sim \tau\ \left(\ln \frac{1}{\varepsilon}\right)^{n+1}$, $\varepsilon \to 0$, holds with some constant τ;*

(K2) *the asymptotics $-\ln d_k\left(A_K^D\right) \sim \sigma\ k^{1/n}$, $\ k \to \infty$, holds with some constant σ?*

In [Z] one can find some recent results as well as a survey of previous results, related to the problem **(K2)**. Now, due to Theorem 1, every result about **(K2)** can be translated to a statement about the problem **(K1)** and vice versa.

References

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