ON THE PARABOLIC INVERSE PROBLEM WITH AN UNKNOWN TIME DEPENDENT HEAT SOURCE

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INTRODUCTION

An inverse problem is the task that often occurs in many branches of science and mathematics where the values of some model parameter(s) must be obtained from the observed data. The transformation from data to model parameters is a result of the interaction of a physical system. Inverse problems arise for example in geophysics, medical imaging, remote sensing, ocean acoustic tomography, nondestructive testing, and astronomy. In inverse problems, the optimal overdetermination conditions are analyzed in some classical boundary conditions or and similar conditions given at a point. A literature review is given in

- J.R. Cannon, Hong-Ming Yin, Numerical solutions of some parabolic inverse problems, *Numerical Methods for Partial Differential Equations*, 2, 177-191, 1990.

Some widely used numerical methods (linearization, variational regularization of the Cauchy problem, relaxation methods, layer-stripping and discrete methods) are summarized in

- V.Isakov *Inverse problems for partial differential equations*, Applied Mathematical Sciences, Vol.127, Springer, 1998.

Some inverse boundary value problems are given in

- Yu. Ya. Belov *Inverse problems for partial differential equations*, Inverse and Ill-posed Problems Series, VSP, 2002.

and the generalized overdetermination conditions such as nonlocal, integral, and final overdetermination conditions are used by

- A.I. Prilepko, A. B. Kostin, "Some inverse problems for parabolic equations with final and integral observation," *Mat. Sb.*, 183, No. 4, 49-68 1992.
- J.R. Cannon,Yanping Lin, and Shingmin Wang, "Determination of a control parameter in a parabolic differential equation," *J. Austral. Math. Soc., Ser. B.*, 33, 149-163, 1991.

Inverse problems are typically ill posed, as opposed to the well-posed problems more typical when modeling physical situations where the model parameters or material properties are known. Of the three conditions for a well-posed problem suggested by Jacques Hadamard (existence, uniqueness, stability of the solution or solutions) the condition of stability is most often violated. While inverse problems are often formulated in infinite dimensional spaces, limitations to a finite number of measurements, and the practical consideration of recovering only a finite number of unknown parameters, may lead to the problems being recast in discrete form. In this case the inverse problem will typically be ill-conditioned. In these cases, regularization may be used to introduce mild assumptions on the solution and prevent overfitting. Many instances of regularized inverse problems can be interpreted as special cases of Bayesian inference. Cannon,et all. give approaches for the existence and uniqueness of a global solution pair (u, p) under some certain assumptions. Existence and uniqueness of a solution under some restrictions on the initial data are established in Ivanchov's article about the reconstruction of a free term in the heat equation. The well-posedness of a problem of determining the parameter is studied by Ashyralyev. The generic well-posedness of a linear inverse problem is studied for values of a diffusion parameter and generic local well-posedness of an inverse problem are proved by Choulli and Yamamoto where the unknown control function is in space variable.

- J.R. Cannon,Yanping Lin, and Shingmin Wang, "Determination of a control parameter in a parabolic differential equation," *J. Austral. Math. Soc., Ser. B.*, 33, 149-163, 1991.
- N.I. Ivanchov, "On the determination of unknown source in the heat equation with nonlocal boundary conditions", *Ukranian Mathematical Journal*, Vol. 47, No.10, 1995.
- A. Ashyralyev, On a problem of determining the parameter of a parabolic equation, *Ukranian Mathematical Journal,* 9, 1-11, 2010.
- M. Choulli, M. Yamamoto, Generic well-posedness of a linear inverse parabolic problem with diffusion parameter, *J.Inv. Ill-Posed Problems*, 7, 3, 241-254, 1999.
- M. Choulli, M. Yamamoto, Generic well-posedness of an inverse parabolic problem-the Hölder-space approach, *Inverse problems*, 12, 195-205, 1996.

The inverse problems are characterized, first of all, by the lack of coefficients and/or conditions. In

- A.A. Samarskii, P.N. Vabishchevich, *Numerical methods for solving inverse problems of mathematical physics,* Inverse and Ill-posed Problems Series, Walter de Gruyter,Berlin,Newyork, 2007.

coefficient inverse problems in which equation coefficients and/or the right hand side are unknown are distinguished. As a typical example of unknown coefficient inverse problems, the following parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial x}\left(k\left(x\right)\frac{\partial u(t,x)}{\partial x}\right) + p\left(t\right)q\left(x\right),\ 0 < x < l,\ 0 < t \leq T, \tag{1}$$

with the conditions

$$u\left(t,0\right) = u\left(t,l\right) = 0,\ 0 \leq t \leq T,$$
$$u\left(0,x\right) = \varphi\left(x\right), 0 \leq x \leq l,$$
$$u\left(t,x^{*}\right) = \rho\left(t\right), 0 < x^{*} < l, 0 < t \leq T$$

where $(u(t,x), k\left(x\right))$ is the solution pair is given. Three classes of inverse coefficient problems arising in engineering mechanics and computational material science are considered in

- A. Hasanov (Hasanoglu), Some new classes of inverse coeffcient problems in nonlinear mechanics and computational material science, Int. J. Non-Linear Mechanics, 46(5), 667-684, 2011.

Under same conditions, problem (1) where $(u(t,x), p\left(t\right))$ is the solution pair is known as right hand side identification problem.

Also in

- O. Demirdag, The boundary value problems for parabolic equations with a parameter, Master Thesis, The Graduate School of Science and Engineering, Fatih University.

problem (1) where $(u(t,x), q(x))$ is the solution pair with the conditions

$$u(t,0) = u(t,l) = 0,\ 0 \le t \le T,$$
$$u(0,x) = \varphi(x), 0 \le x \le l,$$
$$u(t^*,x) = \rho(x), 0 \le x \le l, 0 < t^* < T$$

is considered. Another type is boundary value inverse problems where missing boundary conditions can be identified, for instance, from measurements performed inside the domain. As an example, the parabolic equation (1) with the following conditions can be given.

$$u(t,0) = 0,\ 0 \le t \le T,$$
$$u(0,x) = u_0(x), 0 \le x \le l,$$
$$u(t,x^*) = \varphi(t), 0 < x^* < l, 0 < t \le T.$$

In this case, the unknown pair is $\left(u(t,x), k(x)\frac{\partial u(t,l)}{\partial x}\right)$. Evolutionary inverse problems in which initial conditions need to be identified are as the parabolic equation (1) with the following conditions

$$u(t,0) = u(t,l) = 0,\ 0 \le t \le T,$$
$$u(T,x) = u_T(x), 0 < x < l,$$
$$u(t,x^*) = \varphi(t), 0 < x^* < l, 0 < t \le T.$$

An important class of inverse problem is the determination of unknown right-hand sides of equations. In such problems, additional information about the solution is provided either throughout the calculation or over some part of the domain. The numerical algorithm for solving inverse problem of reconstructing a distributed right-hand side of a parabolic equation with local boundary conditions is studied

- V.T. Borukhov, P.N. Vabishchevich, Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation, *Computer Physics Communications,* 126, 32–36, 2000.
- A.A. Samarskii, P.N. Vabishchevich, *Numerical methods for solving inverse problems of mathematical physics,* Inverse and Ill-posed Problems Series, Walter de Gruyter,Berlin,Newyork, 2007.

In these articles, the numerical solution of the identification problem and well-posedness of the algorithm is presented. For reconstructing the right hand side function $f(t,x) = p(t)\, q(x)$ where $p(t)$ is the unknown function, the solution is observed in the form of $u(t,x) = \eta(t)\, q(x) + w(t,x)$ where $\eta(t) = \int_0^t p(s)\, ds$. Then, an approximation is given for $w(t,x)$ via fully implicit difference scheme. The solution of system constructed by the difference scheme is searched in the form

$$w_i^{n+1} = y_i + w_k^{n+1} z_i, i = 0, 1, \cdots, M,$$

where k is an interior grid point and the well-posedness of the algorithm is given by a apriori estimate

$$\max_{0 \le i \le M} |z_i| \le \tau \max_{0 < i < M} \left| \frac{1}{\psi_k} (a \psi_{\overline{x}})_{x,i} \right|$$

which is based on maximum principle. Thus, in the first article, $|z_i| < 1$ at small enough $\tau = O(1)$, i.e. it is necessary to use a sufficiently small time step.

In this talk, we investigate the well-posedness of the inverse problem of reconstructing the right side of a parabolic equation with nonlocal conditions

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u\left(t,x\right) + p\left(t\right)q\left(x\right) + f\left(t,x\right), \\ 0 < x < l,\ 0 < t \leq T, \\ u\left(t,0\right) = u\left(t,l\right), u_x\left(t,0\right) = u_x\left(t,l\right), 0 \leq t \leq T, \\ u\left(0,x\right) = \varphi\left(x\right),\ 0 \leq x \leq l, \\ u\left(t,x^*\right) = \rho\left(t\right),\ 0 \leq x^* \leq l, 0 \leq t \leq T, \end{cases} \tag{2}$$

where $u\left(t,x\right)$ and $p\left(t\right)$ are unknown functions, $a\left(x\right) \geq \delta > 0$ and $\sigma > 0$ is a sufficiently large number with assuming that
a) $q\left(x\right)$ is a sufficiently smooth function,
b) $q\left(x\right)$ and $q'\left(x\right)$ are periodic with length l,
c) $q\left(x^*\right) \neq 0$.

In contrast to Yamamoto's work, the unknown function in our problem depends on the time variable. Comparing to Vabischevich's work, we give the well-posedness in differential case and for the solution of nonlocal problem.

In the present work, the well-posedness of the right-hand side identification problem for one dimensional parabolic equation with nonlocal boundary conditions and multidimensional parabolic equation is considered. The difference schemes of the first and second orders of accuracy of these problems are presented. Under the applicability conditions, well-posedness of these difference schemes are investigated.

Let us briefly describe the contents of the various sections. It consists of fifth sections.

- **First section** is the introduction.
- **Second section** is about the well-posedness of the right-hand side identification problem for a parabolic equation.
- **Third section** includes the approximate solution of the right-hand side identification problem with nonlocal conditions. The first and second orders of difference schemes and their numerical analysis are given.
- **Fourth section** is the conclusion.

We consider the inverse problem of reconstructing the right side of a parabolic equation with nonlocal conditions

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u\left(t,x\right) + p\left(t\right) q\left(x\right) + f\left(t,x\right), \\ 0 < x < l,\ 0 < t \leq T, \\ \\ u\left(t,0\right) = u\left(t,l\right), u_x\left(t,0\right) = u_x\left(t,l\right), 0 \leq t \leq T, \\ \\ u\left(0,x\right) = \varphi\left(x\right),\ 0 \leq x \leq l, \\ \\ u\left(t,x^*\right) = \rho\left(t\right),\ 0 \leq x^* \leq l, 0 \leq t \leq T, \end{cases} \tag{3}$$

where $u\left(t,x\right)$ and $p\left(t\right)$ are unknown functions, $a\left(x\right) \geq \delta > 0$ and $\sigma > 0$ is a sufficiently large number. Assume that
a) $q\left(x\right)$ is a sufficiently smooth function,
b) $q\left(x\right)$ and $q'\left(x\right)$ are periodic with length l,
c) $q\left(x^*\right) \neq 0$.

In this talk, positive constants, which can be differ in time will be indicated with an M. On the other hand $M(\alpha,\beta,\cdots)$ is used to focus on the fact that the constant depends only on $\alpha,\beta,\cdots$. With the help of A we introduce the fractional spaces $E_\alpha, 0<\alpha<1$, consisting of all $v\in E$ for which the following norm is finite:

$$\|v\|_{E_\alpha}=\sup_{\tau>0}\tau^{1-\alpha}\|A\exp\{-\tau A\}v\|_E . \tag{4}$$

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}^{\alpha}[0,l]$, $\alpha\in(0,1)$, of all continuous functions $\phi(x)$ defined on $[0,l]$ with $\phi(0)=\phi(l),\phi'(0)=\phi'(l)$ satisfying a Hölder condition for which the following norm is finite

$$\|\phi\|_{\overset{\circ}{C}{}^{\alpha}[0,l]}=\|\phi\|_{C[0,l]}+\sup_{0<x<x+h<l}\frac{|\phi(x+h)-\phi(x)|}{h^\alpha},$$

$$\|\phi\|_{C[0,l]}=\max_{0\le x\le l}|\phi(x)| .$$

Then, the following theorem on well-posedness of problem (3) is established.

Theorem 2.1. Let $\varphi(x) \in \overset{\circ}{C}^{2\alpha+2}[0,l]$, $\rho'(t) \in C[0,T]$ and $f(t,x) \in C\left([0,T], \overset{\circ}{C}^{2\alpha}[0,l]\right)$. Then for the solution of problem (3), the following coercive stability estimates

$$\|u_t\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha+2}[0,l]\right)} \leq M(x^*,q)\|\rho'\|_{C[0,T]}$$

$$+M(a,\delta,\sigma,\alpha,x^*,q,T)\left(\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,l]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|\rho\|_{C[0,T]}\right), \quad (5)$$

$$\|p\|_{C[0,T]} \leq M(x^*,q)\|\rho'\|_{C[0,T]}$$

$$+M(a,\delta,\sigma,\alpha,x^*,q,T)\left[\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,l]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|\rho\|_{C[0,T]}\right] \quad (6)$$

hold.

Proof. Let us search for the solution of the inverse problem in the following form

$$u(t,x) = \eta(t)\, q(x) + w(t,x), \tag{7}$$

where

$$\eta(t) = \int_0^t p(s)\, ds. \tag{8}$$

Taking derivatives from (7), we get

$$\frac{\partial u(t,x)}{\partial t} = p(t)\, q(x) + \frac{\partial w(t,x)}{\partial t} \tag{9}$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \eta(t)\, \frac{d^2 q(x)}{dx^2} + \frac{\partial^2 w(t,x)}{\partial x^2}.$$

Moreover if we substitute $x = x^*$ in equation (7), we get

$$u(t,x^*) = \eta(t)\, q(x^*) + w(t,x^*) = \rho(t)$$

and

$$\eta(t) = \frac{\rho(t) - w(t,x^*)}{q(x^*)}. \tag{10}$$

Taking derivative of both sides, we get

$$p(t) = \frac{\rho'(t) - w_t(t, x^*)}{q(x^*)}. \tag{11}$$

Using triangle inequality, from it follows that

$$\begin{aligned} |p(t)| &= \left| \frac{\rho'(t) - w_t(t, x^*)}{q(x^*)} \right| \leq M(x^*, q)\left(\left|\rho'(t)\right| + \left|w_t(t, x^*)\right|\right) \\ &\leq M(x^*, q)\left(\max_{0 \leq t \leq T}\left|\rho'(t)\right| + \max_{0 \leq t \leq T}\max_{0 \leq x^* \leq l}\left|w_t(t, x^*)\right|\right) \\ &\leq M(x^*, q)\left(\max_{0 \leq t \leq T}\left|\rho'(t)\right| + \max_{0 \leq t \leq T}\left\|w_t(t)\right\|_{\overset{\circ}{C}^{2\alpha}[0,l]}\right) \end{aligned} \tag{12}$$

for any $t, t \in [0, T]$.

Here, using equations (7) and (10) and under the assumptions on $q(x)$, one can show that $w(t,x)$ is the solution of the following problem

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} = a(x)\frac{\partial^2 w(t,x)}{\partial x^2} + a(x)\frac{\rho(t)-w(t,x^*)}{q(x^*)}\frac{d^2 q(x)}{dx^2} \\ -\sigma\frac{\rho(t)-w(t,x^*)}{q(x^*)}q(x) - \sigma w(t,x) + f(t,x), 0 < x < l,\ 0 < t \leq T, \\ w(t,0) = w(t,l), w_x(t,0) = w_x(t,l), 0 \leq t \leq T, \\ w(0,x) = \varphi(x),\ 0 \leq x \leq l. \end{cases} \tag{13}$$

So, the end of proof of Theorem 2.1 is based on the inequality (12) and the following theorem.

Theorem 2.2. For the solution of problem (13), the following coercive stability estimate

$$\|w_t\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} \leq M\left(a,\delta,\sigma,\alpha,x^*,q,T\right)\left(\|\varphi\|_{\overset{\circ}{C}^{2\alpha}[0,l]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|\rho\|_{C[0,T]}\right) \tag{14}$$

holds.

Proof. We can rewrite the problem (13) in the abstract form

$$\begin{cases} w_t + Aw = \left(aq'' - \sigma q\right)\frac{\rho(t)-w(t,x^*)}{q(x^*)} + f\left(t\right), 0 < t \leq T, \\ w\left(0\right) = \varphi \end{cases} \tag{15}$$

in a Banach space $E = \overset{\circ}{C}[0,l]$ with the positive operator A defined by

$$Au = -a(x)\frac{\partial^2 u\left(t,x\right)}{\partial x^2} + \sigma u$$

with

$$D\left(A\right) = \left\{u\left(x\right) : u, u', u'' \in C[0,l], u\left(0\right) = u\left(l\right), u_x\left(0\right) = u_x\left(l\right)\right\}.$$

Here, $f\left(t\right) = f\left(t,x\right)$ and $w\left(t\right) = w\left(t,x\right)$ are known and unknown abstract functions defined on $[0,T]$ with values in $E = \overset{\circ}{C}[0,l]$, $w\left(t,x^*\right)$ is unknown scalar function defined on $[0,T]$, $q = q\left(x\right)$, $q'' = q''\left(x\right)$, $\varphi = \varphi\left(x\right)$ and $a = a\left(x\right)$ are elements of $E = \overset{\circ}{C}[0,l]$ and $q\left(x^*\right)$ is a number.

By the Cauchy formula, the solution can be written as

$$w(t) = e^{-tA}\varphi - \int_0^t e^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} w(s, x^*)\, ds + \int_0^t e^{-(t-s)A} \frac{\rho(s)(aq'' - \sigma q)}{q^*} ds + \int_0^t e^{-(t-s)A} f(s)\, ds. \tag{16}$$

Taking the derivative of both sides, we obtain that

$$w_t(t) = -Ae^{-tA}\varphi + \int_0^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} w(s, x^*)\, ds - \frac{aq'' - \sigma q}{q^*} w(t, x^*) - \int_0^t Ae^{-(t-s)A} \frac{\rho(s)(aq'' - \sigma q)}{q^*} ds + \frac{(aq'' - \sigma q)}{q^*}\rho(t) - \int_0^t Ae^{-(t-s)A} f(s)\, ds + f(t).$$

Applying the formula

$$\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w\left(s,x^*\right)ds=\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}\int_0^s w_z\left(z,x^*\right)dzds$$

$$+\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}\varphi\left(x^*\right)ds$$

and changing the order of integration, we obtain that

$$\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w\left(s,x^*\right)ds=\int_0^t\int_z^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w_z\left(z,x^*\right)dsdz$$

$$+\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}\varphi\left(x^*\right)ds.$$

Then, the following presentation of the solution of (13)

$$w_t\left(t\right)=Ae^{-tA}\varphi+\int_0^t\int_z^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w_z\left(z,x^*\right)dsdz$$

$$+\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}\varphi\left(x^*\right)ds-\int_0^t Ae^{-(t-s)A}\frac{\rho\left(s\right)\left(aq''-\sigma q\right)}{q^*}ds$$

$$-\int_0^t Ae^{-(t-s)A}f\left(s\right)ds+\frac{\left(aq''-\sigma q\right)}{q^*}\left(\rho\left(t\right)-w\left(t,x^*\right)\right)+f\left(t\right)=\sum_{k=1}^{6}G_k\left(t\right)$$

Here,

$$G_1(t) = Ae^{-tA}\varphi,$$

$$G_2(t) = \int_0^t \int_z^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} w_z(z, x^*)\, ds dz,$$

$$G_3(t) = \int_0^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \varphi(x^*)\, ds,$$

$$G_4(t) = -\int_0^t Ae^{-(t-s)A} \frac{\rho(s)(aq'' - \sigma q)}{q^*} ds,$$

$$G_5(t) = -\int_0^t Ae^{-(t-s)A} f(s)\, ds,$$

$$G_6(t) = \frac{(aq'' - \sigma q)}{q^*} (\rho(t) - w(t, x^*)) + f(t).$$

It is very well known that, from the fact that the operators $R, \exp\{-\lambda A\}$ and A commute, it follows that (Ashyralyev, Sobolevskii)

$$\|R\|_{E_\alpha \to E_\alpha} \leq \|R\|_{E \to E} . \tag{17}$$

Now, let us estimate $G_k(t)$ for any $k = 1, 2, 3, 4, 5, 6$ separately. Applying the definition of norm of the spaces E_α and (17), we get

$$\begin{aligned} \|G_1(t)\|_{E_\alpha} &= \left\|Ae^{-tA}\varphi\right\|_{E_\alpha} \leq \left\|e^{-tA}\right\|_{E_\alpha \to E_\alpha} \|A\varphi\|_{E_\alpha} \\ &\leq \left\|e^{-tA}\right\|_{E \to E} \|A\varphi\|_{E_\alpha} . \end{aligned}$$

Using estimate (A. Ashyralyev, P.E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations,* Birkhäuser Verlag, Basel, Boston, Berlin, 1994.)

$$\|\exp\{-tA\}\|_{E \to E} \leq Me^{-\delta t}, \tag{18}$$

we get

$$\|G_1(t)\|_{E_\alpha} \leq M_1 \|A\varphi\|_{E_\alpha} \tag{19}$$

for any t, $t \in [0, T]$.

Let us estimate $G_2(t)$

$$\|G_2(t)\|_{E_\alpha} = \left\| \int_0^t \int_z^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} w_z(z, x^*)\, dsdz \right\|_{E_\alpha}$$

$$\leq \int_0^t \int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds \, |w_z(z, x^*)| \, dz.$$

equation (4), we have that

$$\int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds = \int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds$$

$$+ \sup_{\lambda > 0} \int_z^t \left\| \lambda^{1-\alpha} Ae^{-\lambda A} Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds.$$

By the definition of norm of the spaces E_α , we get

$$\int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds = \int_z^t (t-s)^{\alpha-1} \left\| (t-s)^{1-\alpha} Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds$$

$$\leq \int_z^t (t-s)^{\alpha-1} ds \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} = \left(-\frac{(t-s)^\alpha}{\alpha} \right]_z^t \Bigg) \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha}$$

$$\leq \frac{T^\alpha}{\alpha} \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} = M_2 (a, \sigma, \alpha, x^*, q, T) .$$

Using estimate (Ashyralyev, Sobolevskii, 1994)

$$\left\| A^{\alpha} \exp\left\{-tA\right\} \right\|_{E\to E} \leq Me^{-\delta t} t^{-\alpha}, \tag{20}$$

we can obtain that

$$\int_{z}^{t} \left\| \lambda^{1-\alpha} Ae^{-\lambda A} Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^{*}} \right\|_{E} ds$$

$$\leq \int_{z}^{t} \frac{2^{2-\alpha}\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds \left\| \frac{\lambda+t-s}{2} Ae^{-\frac{\lambda+t-s}{2}A} \right\|_{E\to E}$$

$$\times \left\| \left(\frac{\lambda+t-s}{2} \right)^{1-\alpha} Ae^{-\frac{\lambda+t-s}{2}A} \frac{aq''-\sigma q}{q^{*}} \right\|_{E} ds$$

$$\leq M_{3}(\alpha) \left\| \frac{aq''-\sigma q}{q^{*}} \right\|_{E_{\alpha}} \int_{z}^{t} \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds$$

$$= M_{3}(\alpha) \left\| \frac{aq''-\sigma q}{q^{*}} \right\|_{E_{\alpha}} \left(\left. \frac{\lambda^{1-\alpha}}{(\alpha-1)(\lambda+t-s)^{1-\alpha}} \right]_{z}^{t} \right)$$

$$\leq M_{3}(\alpha) \left\| \frac{aq''-\sigma q}{q^{*}} \right\|_{E_{\alpha}} \left(\frac{\lambda^{1-\alpha}}{(1-\alpha)(\lambda+t-z)^{1-\alpha}} \right] \Bigg)$$

for any $\lambda > 0$.

Then,

$$\sup_{\lambda>0}\int_{z}^{t}\left\|\lambda^{1-\alpha}Ae^{-\lambda A}Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^{*}}\right\|_{E}ds$$

$$\leq M_{3}(\alpha)\left\|\frac{aq''-\sigma q}{q^{*}}\right\|_{E_{\alpha}}\frac{1}{(1-\alpha)}=M_{4}(a,\sigma,\alpha,x^{*},q).$$

Then, we get

$$\int_{z}^{t}\left\|Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^{*}}\right\|_{E_{\alpha}}ds\leq M_{5}(a,\sigma,\alpha,x^{*},q,T) \tag{21}$$

for any s, $0\leq z\leq s\leq t$ and

$$\|G_{2}(t)\|_{E_{\alpha}}\leq M_{6}(a,\sigma,\alpha,x^{*},q,T)\int_{0}^{t}|w_{z}(z,x^{*})|\,dz$$

$$\leq M_{6}(a,\sigma,\alpha,x^{*},q,T)\int_{0}^{t}\max_{0\leq x\leq l}|w_{z}(z,x)|\,dz\leq M_{6}(a,\sigma,\alpha,x^{*},q,T)\int_{0}^{t}\|w_{z}\|_{E_{\alpha}}dz. \tag{22}$$

$G_3(t)$ is estimated as follows

$$\|G_3(t)\|_{E_\alpha} = \left\| \int_0^t A e^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \varphi(x^*)\, ds \right\|_{E_\alpha}$$

$$\leq \left\| \int_0^t A e^{-(t-s)A} \frac{aq'' - \sigma q}{q^*}\, ds \right\|_{E_\alpha} |\varphi(x^*)| .$$

Since

$$|\varphi(x^*)| \leq \max_{0 \leq x \leq l} |\varphi(x)| = \|\varphi\|_E \leq \|\varphi\|_{E_\alpha} \leq \left\|A^{-1}\right\|_{E_\alpha \to E_\alpha} \|A\varphi\|_{E_\alpha} \leq M \|A\varphi\|_{E_\alpha} \qquad (23)$$

and using the estimate (21) and choosing $z = 0$, we obtain

$$\|G_3(t)\|_{E_\alpha} \leq M_7(a, \sigma, \alpha, x^*, q, T) \|A\varphi\|_{E_\alpha} \qquad (24)$$

for any $t \in [0, T]$.

By the estimate (21), the estimation of $G_4(t)$ is as follows

$$\|G_4(t)\|_{E_\alpha} = \left\| \int_0^t Ae^{-(t-s)A} \rho(s) \frac{aq'' - \sigma q}{q^*} ds \right\|_{E_\alpha} \leq M_8(a, \sigma, \alpha, x^*, q, T) \|\rho\|_{C[0,T]}. \tag{25}$$

Now, let us estimate $G_5(t)$. By the definition of the norm of the spaces E_α, we get

$$\|G_5(t)\|_{E_\alpha} = \left\| \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_{E_\alpha}$$

$$= \left\| \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_E + \sup_{\lambda>0} \lambda^{1-\alpha} \left\| Ae^{-\lambda A} \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_E .$$

Using equation (4), we have that

$$\left\|\int_0^t Ae^{-(t-s)A}f(s)\,ds\right\|_E \leq \int_0^t (t-s)^{\alpha-1}\left\|(t-s)^{1-\alpha}Ae^{-(t-s)A}f(s)\right\|_E ds$$

$$\leq \int_0^t (t-s)^{\alpha-1}ds\,\|f\|_{C(E_\alpha)} = \|f\|_{C(E_\alpha)}\left(-\frac{(t-s)^\alpha}{\alpha}\right]_0^t\Bigg)$$

$$= \frac{t^\alpha}{\alpha}\|f\|_{C(E_\alpha)} \leq M_9(\alpha,T)\|f\|_{C(E_\alpha)}. \tag{26}$$

Now, we consider the second term. Using the equation (4), we get

$$\lambda^{1-\alpha}\left\| Ae^{-\lambda A}\int_0^t Ae^{-(t-s)A}f(s)\,ds\right\|_E \le \lambda^{1-\alpha}\int_0^t \left\| Ae^{-\frac{t-s+\lambda}{2}A}\right\|_{E\to E}$$

$$\times\left\| Ae^{-\frac{t-s+\lambda}{2}A}f(s)\right\|_E ds = \lambda^{1-\alpha}\int_0^t \left(\frac{t-s+\lambda}{2}\right)^{\alpha-1}\left(\frac{t-s+\lambda}{2}\right)^{-1}$$

$$\times\left\|\frac{t-s+\lambda}{2}Ae^{-\frac{t-s+\lambda}{2}A}\right\|_{E\to E}\left\|\left(\frac{t-s+\lambda}{2}\right)^{1-\alpha}Ae^{-\frac{t-s+\lambda}{2}A}f(s)\right\|_E ds$$

$$\le M_{10}\lambda^{1-\alpha}\int_0^t\left(\frac{t-s+\lambda}{2}\right)^{\alpha-2}\|f\|_{E_\alpha}ds$$

$$\le M_{10}\lambda^{1-\alpha}\int_0^t\left(\frac{t-s+\lambda}{2}\right)^{\alpha-2}ds\,\|f\|_{C(E_\alpha)} = M_{10}\|f\|_{C(E_\alpha)}\left(\left.\frac{\left(\frac{t-s+\lambda}{2}\right)^{\alpha-1}}{1-\alpha}\right]_0^t\right)$$

for any $\lambda > 0$.

Then,

$$\sup_{\lambda>0}\lambda^{1-\alpha}\left\|Ae^{-\lambda A}\int_0^t Ae^{-(t-s)A}f(s)\,ds\right\|_E$$

$$\leq M_{10}\|f\|_{C(E_\alpha)}\left(\frac{2^{1-\alpha}}{1-\alpha}\right)\leq M_{11}(\alpha)\|f\|_{C(E_\alpha)}. \tag{27}$$

By the estimates (26) and (27), we get

$$\|G_5(t)\|_{E_\alpha}\leq M_{12}(\alpha,T)\|f\|_{C(E_\alpha)}. \tag{28}$$

Let estimate $G_6(t)$.

$$\|G_6(t)\|_{E_\alpha} = \left\| \frac{(aq'' - \sigma q)}{q^*} (\rho(t) - w(t, x^*)) + f(t) \right\|_{E_\alpha}.$$

Applying the formula

$$w(t, x^*) = w(0, x^*) + \int_0^t w_z(z, x^*)\, dz = \varphi(x^*) + \int_0^t w_z(z, x^*)\, dz,$$

the definition of the norm of the spaces E_α and the estimate (23), we get

$$\|G_6(t)\|_{E_\alpha} \leq \left\| \frac{(aq'' - \sigma q)}{q^*} \right\|_{E_\alpha} \left(\|\rho\|_{C[0,T]} + M_{13} \|A\varphi\|_{E_\alpha} + \int_0^t \|w_z\|_{E_\alpha} dz \right) + \|f\|_{C(E_\alpha)}$$

$$= M_{14}(a, \delta, \sigma, x^*, q) \left(\|\rho\|_{C[0,T]} + \|A\varphi\|_{E_\alpha} + \int_0^t \|w_z\|_{E_\alpha} dz \right) + \|f\|_{C(E_\alpha)}. \qquad (29)$$

Combining the estimates (19) , (22) , (24) , (25) , (28) and (29) , we get

$$\|w_t\|_{E_\alpha} \leq M_1 \|A\varphi\|_{E_\alpha} + M_6(a,\sigma,\alpha,x^*,q,T) \int_0^t \|w_z\|_{E_\alpha} dz$$

$$+M_7(a,\sigma,\alpha,x^*,q,T)\|A\varphi\|_{E_\alpha} + M_8(a,\sigma,\alpha,x^*,q,T)\|\rho\|_{C[0,T]} + M_{12}(\alpha,T)\|f\|_{C(E_\alpha)}$$

$$+M_{14}(a,\delta,\sigma,x^*,q)\left(\|\rho\|_{C[0,T]} + \|A\varphi\|_{E_\alpha} + \int_0^t \|w_z\|_{E_\alpha} dz\right) + \|f\|_{C(E_\alpha)}. \quad (30)$$

From (30), using Gronwall's integral inequality, we get

$$\|w_t\|_{E_\alpha} \leq e^{M_{15}(a,\sigma,\alpha,x^*,q,T)} \Big[M_1 \|A\varphi\|_{E_\alpha}$$
$$+ M_7(a,\sigma,\alpha,x^*,q,T) \|A\varphi\|_{E_\alpha} + M_8(a,\sigma,\alpha,x^*,q,T) \|\rho\|_{C[0,T]}$$
$$M_{14}(a,\delta,\sigma,x^*,q) \left(\|\rho\|_{C[0,T]} + \|A\varphi\|_{E_\alpha} \right) + (M_{12}(\alpha,T)+1) \|f\|_{C(E_\alpha)} \Big]. \qquad (31)$$

The following theorem finishes the proof of Theorem 2.2.
Theorem 2.3. (Ashyralyev, Sobolevskii) For $0 < \alpha < \frac{1}{2}$ the norms of the spaces $E_\alpha(C[0,l], A)$ and $C^{2\alpha}[0,l]$ are equivalent.

Note that, in similar manner one can obtain the well-posedness of the inverse problem of reconstructing the right side of a multidimensional parabolic equation under restrictions for $q(x)$. Let us consider boundary value problem for the multidimensional parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \sum\limits_{|r|=2m} a_r(x) \dfrac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} - \sigma u(t,x) + p(t) q(x) \\ +f(t,x), \ x \in \mathbb{R}^n, 0 < t < T, |r| = r_1 + r_2 + \cdots + r_n, \\ u(0,x) = \varphi(x), x \in \mathbb{R}^n, \\ u(t,x^*) = \rho(t), 0 \le t \le T, \ x^* \in \Omega \subset \mathbb{R}^n, \end{cases} \tag{32}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a_r(x) \ge \delta > 0$ is sufficiently smooth function and $\sigma > 0$ is a sufficiently large number. Here, we assume that $q(x)$ is sufficiently smooth and bounded function and $q(x^*) \neq 0$.

It is assumed that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)\,(i\xi_1)^{r_1}\cdots(i\xi_n)^{r_n}\,,\xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x)\,\frac{\partial^{|r|}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1\,|\xi|^{2m} \leq (-1)^m\,B^x(\xi) \leq M_2\,|\xi|^{2m} < \infty$$

for $\xi \neq 0$.

Then, the following theorem on well-posedness of problem (32) exists.

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}^{\alpha}(\mathbb{R}^n)$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $\mathbb{R}^n$ satisfying a Hölder condition with the norm

$$\|\phi\|_{\overset{\circ}{C}{}^{\alpha}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |\phi(x)| + \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha}}.$$

Theorem 2.4. Let $\varphi(x) \in \overset{\circ}{C}{}^{2\alpha+2}(\mathbb{R}^n)$, $\rho'(t) \in C[0,T]$ and $f(t,x) \in C\left([0,T], \overset{\circ}{C}{}^{2\alpha}(\mathbb{R}^n)\right)$.
Then for the solution of problem (32), the following coercive stability estimates

$$\|u_t\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}(\mathbb{R}^n)\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha+2m}(\mathbb{R}^n)\right)} \leq M(x^*,q)\|\rho'\|_{C[0,T]}$$

$$+M(a,\delta,\sigma,\alpha,x^*,q,T)\left(\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2m}(\mathbb{R}^n)} + \|f\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}(\mathbb{R}^n)\right)} + \|\rho\|_{C[0,T]}\right),$$

$$\|p\|_{C[0,T]} \leq M(x^*,q)\|\rho'\|_{C[0,T]}$$

$$+M(a,\delta,\sigma,\alpha,x^*,q,T)\left[\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2m}(\mathbb{R}^n)} + \|f\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}(\mathbb{R}^n)\right)} + \|\rho\|_{C[0,T]}\right]$$

hold.
The proof of Theorem 2.4 follows the scheme of the proof of Theorem 2.1 and it is based on the following theorems.

Theorem 2.5. (Ashyralyev, Sobolevskii) For $0 < \alpha < \frac{1}{2m}$ and the indicator $\mu \in (0,1)$, the norms of the spaces $E_\alpha\left(C^\mu\left(\mathbb{R}^n\right), A\right)$ and $C^{\mu+2m\alpha}\left(\mathbb{R}^n\right)$ are equivalent.
Theorem 2.6. (Ashyralyev, Sobolevskii) The solution of elliptic problem

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} u(x)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} - \sigma u(x) = f(x), x \in \mathbb{R}^n$$

obey the coercivity inequality

$$\sum_{|r|=2m} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \right\|_{C^\mu(\mathbb{R}^n)} \leq M(\mu) \|f\|_{C^\mu(\mathbb{R}^n)} .$$

For the approximate solution of the problem (3), the Rothe difference scheme

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} = a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} - \sigma u_n^k + p^k q_n + f(t_k, x_n), \\ p^k = p(t_k), q_n = q(x_n), x_n = nh, \ t_k = k\tau, \\ 1 \le k \le N, 1 \le n \le M-1, Mh = l, N\tau = T, \\ \\ u_0^k = u_M^k, -3u_0^k + 4u_1^k - u_2^k = u_{M-2}^k - 4u_{M-1}^k + 3u_M^k, 0 \le k \le N, \\ \\ u_n^0 = \varphi(x_n), \ 0 \le n \le M, \\ \\ u_{\left\lfloor \frac{x^*}{h} \right\rfloor}^k = u_s^k = \rho(t_k), 0 \le k \le N, 0 \le s \le M \end{cases} \tag{33}$$

is constructed. Here, $q_s \neq 0, q_0 = q_M$ and $-3q_0 + 4q_1 - q_2 = q_{M-2} - 4q_{M-1} + 3q_M$ are assumed.

Let A be a strongly positive operator. With the help of A we introduce the fractional spaces $E'_{\alpha}(E,A)\,, 0<\alpha<1$, consisting of all $v\in E$ for which the following norms are finite:

$$\|v\|_{E'_{\alpha}}=\sup_{\lambda>0}\left\|\lambda^{\alpha}A\left(\lambda+A\right)^{-1}v\right\|_{E}. \quad (34)$$

To formulate our results, we introduce the Banach space $\overset{\circ}{C_h}{}^{\alpha}=\overset{\circ}{C}{}^{\alpha}\,[0,l]_h\,,\ \alpha\in(0,1)$, of all grid functions $\phi^h=\{\phi_n\}_{n=1}^{M-1}$ defined on $[0,l]_h=\{x_n=nh, 0\le n\le M, Mh=l\}$ with $\phi_0=\phi_M$, $-3\phi_0^k+4\phi_1^k-\phi_2^k=\phi_{M-2}^k-4\phi_{M-1}^k+3\phi_M^k$ equipped with the norm

$$\|\phi_h\|_{\overset{\circ}{C_h}{}^{\alpha}}=\|\phi_h\|_{C_h}+\sup_{1\le n<n+r\le M}\left|\phi_{n+r}-\phi_n\right|(rh)^{-\alpha},$$

$$\|\phi_h\|_{C_h}=\max_{1\le n\le M}|\phi_n|.$$

Moreover, $C_{\tau}\left(E\right)=C\left([0,T]_{\tau}\,,E\right)$ is the Banach space of all grid functions $\phi^{\tau}=\{\phi\left(t_k\right)\}_{k=1}^{N-1}$ defined on $[0,T]_{\tau}\,\{t_k=k\tau, 0\le k\le N, Nh=T\}$ with values in E equipped with the norm

$$\|\phi^{\tau}\|_{C_{\tau}(E)}=\max_{1\le k\le N}\|\phi\left(t_k\right)\|_{E}.$$

Then, the following theorem on well-posedness of problem (33) is established.

Theorem 2.7. For the solution of problem (33), the following coercive stability estimates

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} + \left\| \left\{ D_h^2 u_k^h \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} \leq M(q,s) \left\| \left\{ \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau}$$

$$+M(\widetilde{a},\phi,\alpha,T)\left(\left\| D_h^2 \varphi^h \right\|_{\overset{\circ}{C}_h^{2\alpha}} + \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right),$$

$$\left\| p^\tau \right\|_{C[0,T]_\tau} \leq M(q,s) \left\| \left\{ \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau}$$

$$+M(\widetilde{a},\phi,\alpha,T)\left[\left\| D_h^2 \varphi^h \right\|_{\overset{\circ}{C}_h^{2\alpha}} + \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right]$$

hold. Here, $f^h(t_k) = \{f(t_k, x_n)\}_{n=1}^{M-1}$, $\varphi^h = \{\varphi(x_n)\}_{n=1}^{M-1}$, $\rho^\tau = \{\rho(t_k)\}_{k=0}^N$, $D_h^2 u^h = \left\{ \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} \right\}_{n=1}^{M-1}$ and $\widetilde{a} = \frac{1}{q_s}\left(a D_h^2 q^h - \sigma q^h\right)$.

Proof. We search the solution of (33) in the following form

$$u_n^k = \eta^k q_n + w_n^k, \tag{35}$$

where

$$\eta^k = \sum_{i=1}^{k} p^i \tau, 1 \leq k \leq N, \eta^0 = 0. \tag{36}$$

Taking difference derivatives from (35) , we get

$$\frac{u_n^k - u_n^{k-1}}{\tau} = \frac{\eta^k - \eta^{k-1}}{\tau} q_n + \frac{w_n^k - w_n^{k-1}}{\tau} = p^k q_n + \frac{w_n^k - w_n^{k-1}}{\tau}$$

and

$$\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = \eta^k \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} + \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2}$$

for any $n, 1 \leq n \leq M-1$. Moreover for the interior grid point u_s^k, we have that

$$u_s^k = \eta^k q_s + w_s^k = \rho(t_k)$$

and

$$\eta^k = \frac{\rho(t_k) - w_s^k}{q_s}. \tag{37}$$

From the last equality, taking the difference derivative it follows that

$$p^k = \frac{1}{q_s}\left(\frac{\rho(t_k) - \rho(t_{k-1})}{\tau} - \frac{w_s^k - w_s^{k-1}}{\tau}\right). \tag{38}$$

Using the triangle inequality, we get

$$\left|p^k\right| \leq M(q,s)\left(\left|\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right| + \left|\frac{w_s^k - w_s^{k-1}}{\tau}\right|\right)$$

$$\leq M(q,s)\left(\max_{1\leq k\leq N}\left|\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right| + \max_{1\leq k\leq N}\max_{0\leq s\leq M}\left|\frac{w_s^k - w_s^{k-1}}{\tau}\right|\right)$$

$$\leq M(q,s)\left(\max_{1\leq k\leq N}\left|\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right| + \max_{1\leq k\leq N}\left\|\frac{w_k^h - w_{k-1}^h}{\tau}\right\|_{\overset{\circ}{C}_h^{2\alpha}}\right) \quad (39)$$

for any $k, 1 \leq k \leq N$.

Here, $\left\{w_k^h\right\}_{k=0}^N$ is the solution of the following difference scheme

$$\begin{cases} \frac{w_n^k - w_n^{k-1}}{\tau} = a(x_n)\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + a(x_n)\frac{\rho(t_k)-w_s^k}{q_s}\frac{q_{n+1}-2q_n+q_{n-1}}{h^2} \\ -\sigma\frac{\rho(t_k)-w_s^k}{q_s}q_n - \sigma w_n^k + f(t_k, x_n), x_n = nh,\ t_k = k\tau, \\ 1 \leq k \leq N, 1 \leq n \leq M-1, Mh = l, N\tau = T, \\ w_0^k = w_M^k, -3w_0^k + 4w_1^k - w_2^k = w_{M-2}^k - 4w_{M-1}^k + 3w_M^k, 0 \leq k \leq N, \\ w_n^0 = \varphi(x_n),\ 0 \leq n \leq M. \end{cases} \quad (40)$$

Therefore, the end of proof of Theorem 2.7 is based on the inequality (39) and the following theorem.

Theorem 2.8. For the solution of problem (40), the following coercive stability estimate

$$\left\| \left\{ \frac{w_k^h - w_{k-1}^h}{\tau} \right\}_{k=1}^{N} \right\|_{C_\tau\left(\overset{\circ}{C}_h{}^{2\alpha}\right)} \leq M\left(\widetilde{a}, \phi, \alpha, T\right) \left(\left\| \varphi^h \right\|_{\overset{\circ}{C}_h{}^{2\alpha}} + \left\| \left\{ f^h\left(t_k\right) \right\}_{k=1}^{N} \right\|_{C_\tau\left(\overset{\circ}{C}_h{}^{2\alpha}\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right)$$

holds.

Proof. We can rewrite the difference scheme (40) in the abstract form

$$\left\{ \begin{array}{l} \frac{w_k^h - w_{k-1}^h}{\tau} + A_h^x w_k^h = \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \frac{\rho(t_k) - w_s^k}{q_s} \\ + f^h\left(t_k\right), t_k = k\tau, 1 \leq k \leq N, N\tau = T, \\ w_0^h = \varphi^h \end{array} \right. \tag{41}$$

in a Banach space $E = \overset{\circ}{C}\left[0, l\right]_h$ with the positive operator A_h^x defined by

$$A_h^x u^h = \left\{ -a(x_n) \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \sigma u \right\}_{n=1}^{M-1} \tag{42}$$

acting on grid functions u^h such that satisfies the condition

$$u_0 = u_M, -3u_0 + 4u_1 - u_2 = u_{M-2} - 4u_{M-1} + 3u_M.$$

Let denote $R=(I+\tau A_h^x)^{-1}$. In (41), we have that

$$w_k^h = R w_{k-1}^h + R\tau\left(\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)\frac{\rho(t_k)-w_s^k}{q_s}+f^h(t_k)\right),$$

$\forall k, 1\le k\le N$. By recurrence relations, we get

$$w_k^h = R^k\varphi^h + \sum_{m=1}^{k} R^{k-m+1}\frac{\tau}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)\rho(t_m)$$
$$-\sum_{m=1}^{k} R^{k-m+1}\frac{\tau}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)w_s^m + \sum_{m=1}^{k} R^{k-m+1}\tau f^h(t_m).$$

Taking the difference derivative of both sides, we obtain that

$$\frac{w_k^h-w_{k-1}^h}{\tau} = \frac{R^k-R^{k-1}}{\tau}\varphi^h + \frac{1}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)\rho(t_k)$$
$$+\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)\rho(t_m)$$
$$-\frac{1}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)w_s^k$$
$$-\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)w_s^m$$
$$+f^h(t_k)+\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)f^h(t_m).$$

Applying the formula

$$\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)w_s^m=\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\varphi\left(x_s\right)$$
$$+\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\sum_{j=1}^{m}\frac{w_s^j-w_s^{j-1}}{\tau}\tau$$

and changing the order of summation, we obtain that

$$\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)w_s^m=\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\varphi\left(x_s\right)$$
$$+\sum_{j=1}^{k}\sum_{m=j}^{k}\left(R^{k-m+1}-R^{k-m}\right)\frac{w_s^j-w_s^{j-1}}{\tau}\tau \qquad (43)$$

Then, the following presentation of the solution of (40)

$$\frac{w_k^h - w_{k-1}^h}{\tau} = \frac{R^k - R^{k-1}}{\tau}\varphi^h + \frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)\rho\left(t_k\right)$$

$$+\sum_{m=1}^{k}\left(R^{k-m+1} - R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)\rho\left(t_m\right)$$

$$-\frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)w_s^k$$

$$-\sum_{m=1}^{k}\left(R^{k-m+1} - R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)\varphi\left(x_s\right)$$

$$-\sum_{j=1}^{k}\sum_{m=j}^{k}\left(R^{k-m+1} - R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)\frac{w_s^j - w_s^{j-1}}{\tau}\ \tau$$

$$+f^h\left(t_k\right) + \sum_{m=1}^{k}\left(R^{k-m+1} - R^{k-m}\right)f^h\left(t_m\right) = \sum_{k=1}^{6} J_k$$

is obtained.

Here,

$$J_1^k = \frac{R^k - R^{k-1}}{\tau} \varphi^h,$$

$$J_2^k = \sum_{m=1}^{k} \left(R^{k-m+1} - R^{k-m} \right) \frac{1}{q_s} \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \rho \left(t_m \right),$$

$$J_3^k = -\sum_{m=1}^{k} \left(R^{k-m+1} - R^{k-m} \right) \frac{1}{q_s} \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \varphi \left(x_s \right),$$

$$J_4^k = -\sum_{j=1}^{k} \sum_{m=j}^{k} \left(R^{k-j+1} - R^{k-j} \right) \frac{\tau}{q_s} \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \frac{w_s^m - w_s^{m-1}}{\tau},$$

$$J_5^k = \sum_{m=1}^{k} \left(R^{k-m+1} - R^{k-m} \right) f^h \left(t_m \right),$$

$$J_6^k = \frac{1}{q_s} \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \left(\rho \left(t_k \right) - w_s^k \right) + f^h \left(t_k \right).$$

Now, let us estimate J_r^k for any $r = 1, 2, 3, 4, 5, 6$ separately. We start with J_1^k.

$$J_1^k = \frac{R^k - R^{k-1}}{\tau} \varphi^h = R^k \frac{I - R^{-1}}{\tau} \varphi^h = R^k A_h^x \varphi^h. \tag{44}$$

Then, applying the definition of norm of the spaces $E_\alpha^{'}$ and (44), we get

$$\begin{aligned} \left\| J_1^k \right\|_{E_\alpha^{'}} &= \left\| R^k A_h^x \varphi^h \right\|_{E_\alpha^{'}} \leq \left\| R^k \right\|_{E_\alpha^{'} \to E_\alpha^{'}} \left\| A_h^x \varphi^h \right\|_{E_\alpha^{'}} \\ &\leq \left\| R^k \right\|_{E \to E} \left\| A_h^x \varphi^h \right\|_{E_\alpha^{'}} . \end{aligned}$$

Using estimate

$$\left\| R^k \right\|_{E \to E} \leq M, \tag{45}$$

we get

$$\left\| J_1^k \right\|_{E_\alpha^{'}} \leq M_1 \left\| A_h^x \varphi^h \right\|_{E_\alpha^{'}} \tag{46}$$

for any k, $1 \leq k \leq N$.

Let us estimate J_2^k

$$\left\|J_2^k\right\|_{E'_\alpha} = \left\|\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)\rho\left(t_m\right)\right\|_{E'_\alpha}$$

$$= \left\|\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\widetilde{a}\rho\left(t_m\right)\right\|_{E'_\alpha} \leq \max_{1\leq m\leq N}\rho\left(t_m\right)\sum_{m=1}^{k}\left\|\left(R^{k-m+1}-R^{k-m}\right)\widetilde{a}\right\|_{E'_\alpha}.$$

Using

$$R^{k-m+1}-R^{k-m} = R^{k-m+1}\left(I-R^{-1}\right) = -R^{k-m+1}A_h^x\tau \tag{47}$$

and the definition of norm of the spaces E'_α, we get

$$\left\|\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)\widetilde{a}\right\|_{E'_\alpha} \leq \sum_{m=1}^{k}\left\|R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E$$

$$+\sup_{\lambda>0}\sum_{m=1}^{k}\left\|\lambda^\alpha A_h^x\left(\lambda+A_h^x\right)^{-1}R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E.$$

Let estimate each term seperately. We divide first term into two parts.

$$\sum_{m=1}^{k}\left\|R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E=\sum_{m=1}^{k-1}\left\|R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E+\|RA_h^x\tau\widetilde{a}\|_E .$$

In the first part, by the definition of norm of the spaces $E_\alpha^{'}$ and equation (Ashyralyev, Sobolevskii, 1994)

$$(I+\tau A)^{-k}=\frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-t}\exp\{-\tau tA\}\,dt \text{ (for all k} \geq 2), \tag{48}$$

we deduce that

$$\sum_{m=1}^{k-1}\left\|R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E\leq\sum_{m=1}^{k-1}\frac{\tau}{(k-m)!}\int_0^\infty\frac{t^{k-m}}{(\tau t)^{1-\alpha}}e^{-t}\left\|(\tau t)^{1-\alpha}A_h^x e^{-\tau tA}\widetilde{a}\right\|_E dt.$$

$$\leq\|\widetilde{a}\|_{E_\alpha'}\sum_{m=1}^{k-1}\frac{\tau}{(k-m)!}\int_0^\infty\frac{t^{k-m}}{(\tau t)^{1-\alpha}}e^{-t}dt=\|\widetilde{a}\|_{E_\alpha'}\sum_{m=1}^{k-1}\frac{\tau^\alpha}{(k-m)!}\int_0^\infty t^{k-m-1+\alpha}e^{-t}dt$$

$$=\|\widetilde{a}\|_{E_\alpha'}\sum_{m=1}^{k-1}\frac{\tau^\alpha}{(k-m)!}\int_0^\infty t^{(k-m-1)\alpha+\alpha}e^{-\alpha t}t^{(k-m-1)(1-\alpha)}e^{-(1-\alpha)t}dt.$$

Using the Hölder inequality with $p = \frac{1}{\alpha}, q = \frac{1}{1-\alpha}$ and the definition of the gamma function, we deduce that

$$\sum_{m=1}^{k-1} \left\| R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq \|\widetilde{a}\|_{E_\alpha'} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \left(\int_0^\infty \left(t^{(k-m-1)\alpha+\alpha} e^{-\alpha t} \right)^{\frac{1}{\alpha}} dt \right)^\alpha$$

$$\times \left(\int_0^\infty \left(t^{(k-m-1)(1-\alpha)} e^{-(1-\alpha)t} \right)^{\frac{1}{1-\alpha}} dt \right)^{1-\alpha}$$

$$= \|\widetilde{a}\|_{E_\alpha'} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \left(\int_0^\infty t^{k-m} e^{-t} dt \right)^\alpha \left(\int_0^\infty t^{k-m-1} e^{-t} dt \right)^{1-\alpha}$$

$$= \|\widetilde{a}\|_{E_\alpha'} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \left(\Gamma(k-m+1) \right)^\alpha \left(\Gamma(k-m) \right)^{1-\alpha}.$$

Using $\Gamma(n) = (n-1)!$ and $\Gamma(n) = (n-1)\Gamma(n-1)$, we get

$$\sum_{m=1}^{k-1} \left\| R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq \|\widetilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} (k-m)^\alpha \Gamma(k-m)$$

$$= \|\widetilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)^{1-\alpha}} = \|\widetilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau}{((k-m)\tau)^{1-\alpha}}$$

$$\leq M_2 \|\widetilde{a}\|_{E'_\alpha} \int_0^{k\tau} \frac{1}{(k\tau - s)^{1-\alpha}} ds = M_2 \|\widetilde{a}\|_{E'_\alpha} \left(-\frac{(k\tau - s)^\alpha}{\alpha} \right]_0^{k\tau} \right).$$

So, we have that

$$\sum_{m=1}^{k-1} \left\| R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq M_2 \left\| \widetilde{a} \right\|_{E'_\alpha} \frac{(k\tau)^\alpha}{\alpha} \leq M_3 (\alpha, T) \left\| \widetilde{a} \right\|_{E'_\alpha}. \tag{49}$$

In the second part, we have that

$$\left\| R A_h^x \tau \widetilde{a} \right\|_E \leq \left\| R A_h^x \tau \right\|_{E \to E} \left\| \widetilde{a} \right\|_E \leq M_4 \left\| \widetilde{a} \right\|_{E'_\alpha}. \tag{50}$$

Combining the estimates $(49), (50)$, we get

$$\sum_{m=1}^{k} \left\| R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq M_5 (\alpha, T) \left\| \widetilde{a} \right\|_{E'_\alpha}. \tag{51}$$

Let estimate the second term. Using the Cauchy-Riesz formula (Ashyralyev, Sobolevskii, 1994)

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) (z - A)^{-1} dz, \tag{52}$$

we get

$$\sum_{m=1}^{k} \lambda^\alpha A_h^x \left(\lambda + A_h^x\right)^{-1} R^{k-m+1} A_h^x \tau \widetilde{a} = \frac{1}{2\pi i} \int_{S_1 \cup S_2} \sum_{m=1}^{k} \frac{z}{(1+z)^{k-m+1}} \frac{\lambda^\alpha}{\lambda + z\tau^{-1}} A_h^x \left(z - \tau A_h^x\right)^{-1} \widetilde{a} dz$$

$$= \frac{1}{2\pi i} \int_{S_1 \cup S_2} \sum_{m=1}^{k} \frac{\left(\frac{z}{\tau}\right)^{-\alpha}}{(1+z)^{k-m+1}} \frac{\lambda^\alpha}{\lambda\tau + z} \left(\frac{z}{\tau}\right)^\alpha A_h^x \left(\frac{z}{\tau} - A_h^x\right)^{-1} \widetilde{a} dz.$$

Since $z=\rho e^{\pm i\phi}$, with $|\phi|\leq\frac{\pi}{2}$, the estimate

$$\left\|(\lambda-A)^{-1}\right\|_{E\to E}\leq\frac{M(\phi)}{1+|\lambda|} \tag{53}$$

yields

$$\left\|\left(\frac{z}{\tau}\right)^{\alpha}A_h^x\left(\frac{z}{\tau}-A_h^x\right)^{-1}\widetilde{a}\right\|_E\leq M_6\left\|\left(\frac{\rho}{\tau}\right)^{\alpha}A_h^x\left(\frac{\rho}{\tau}+A_h^x\right)^{-1}\widetilde{a}\right\|_E,\ \frac{1}{|\lambda\tau+z|}\leq\frac{M_6}{\lambda\tau+\rho}.$$

Hence

$$\left\|\sum_{m=1}^{k}\lambda^{\alpha}A_h^x\left(\lambda+A_h^x\right)^{-1}R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E\leq M_6\int_0^{\infty}\sum_{m=1}^{k}\frac{\rho^{1-\alpha}}{\left[1+2\rho\cos\phi+\rho^2\right]^{\frac{k-m+1}{2}}}\frac{(\lambda\tau)^{\alpha}\,d\rho}{\lambda\tau+\rho}\left\|\widetilde{a}\right\|_{E_\alpha'}.$$

Summing the geometric progression, we get

$$\left\|\sum_{m=1}^{k}\lambda^{\alpha}A_h^x\left(\lambda+A_h^x\right)^{-1}R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E\leq M_6\int_0^{\infty}\sum_{m=1}^{k}\frac{\rho^{1-\alpha}}{\left[1+2\rho\cos\phi+\rho^2\right]^{\frac{1}{2}}}$$

$$\times\left(1-\frac{1}{\left[1+2\rho\cos\phi+\rho^2\right]^{\frac{1}{2}}}\right)^{-1}\frac{(\lambda\tau)^{\alpha}\,d\rho}{\lambda\tau+\rho}\left\|\widetilde{a}\right\|_{E_\alpha'}$$

$$\leq M_6\int_0^{\infty}\frac{(\lambda\tau)^{\alpha}\varkappa(\rho)\,d\rho}{(\lambda\tau+\rho)\,\rho^{\alpha}}\left\|\widetilde{a}\right\|_{E_\alpha'}.$$

Since the function

$$\varkappa(\rho) = \frac{\rho}{\left[1 + 2\rho\cos\phi + \rho^2\right]^{\frac{1}{2}} - 1} = \frac{1 + \left[1 + 2\rho\cos\phi + \rho^2\right]^{\frac{1}{2}}}{2\cos\phi + \rho}$$

does not increase for $\rho \geq 0$, we have $\varkappa(0) = \frac{1}{\cos\phi} \geq \varkappa(\rho)$ for all $\rho > 0$. Consequently,

$$\left\| \sum_{m=1}^{k} \lambda^{\alpha} A_h^x \left(\lambda + A_h^x\right)^{-1} R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq \frac{M_6}{\cos\phi} \int_0^{\infty} \frac{(\lambda\tau)^{\alpha}\, d\rho}{(\lambda\tau + \rho)\, \rho^{\alpha}} \left\|\widetilde{a}\right\|_{E'_\alpha}$$

for any $\lambda > 0$. So,

$$\sup_{\lambda>0} \left\| \sum_{m=1}^{k} \lambda^{\alpha} A_h^x \left(\lambda + A_h^x\right)^{-1} R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq M_7(\phi, \alpha) \left\|\widetilde{a}\right\|_{E'_\alpha}. \tag{54}$$

Then using (51) and (54), we get

$$\left\| \sum_{m=1}^{k} \left(R^{k-m+1} - R^{k-m} \right) \widetilde{a} \right\|_{E'_\alpha} \leq M_8 \left(\phi, \alpha, T \right) \| \widetilde{a} \|_{E'_\alpha} \tag{55}$$

and

$$\left\| J_2^k \right\|_{E'_\alpha} \leq \max_{1 \leq m \leq N} \rho \left(t_m \right) M_8 \left(\phi, \alpha, T \right) \| \widetilde{a} \|_{E'_\alpha} . \tag{56}$$

Now let us estimate J_3^k.

$$\left\| J_3^k \right\|_{E'_\alpha} = \left\| \sum_{m=1}^{k} \left(R^{k-m+1} - R^{k-m} \right) \frac{1}{q_s} \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \varphi \left(x_s \right) \right\|_{E'_\alpha} .$$

$$\leq \sum_{m=1}^{k} \left\| \left(R^{k-m+1} - R^{k-m} \right) \frac{1}{q_s} \left(a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \right\|_{E'_\alpha} \left| \varphi \left(x_s \right) \right| .$$

Since

$$|\varphi(x_s)| \leq \max_{0\leq s\leq M} |\varphi(x_s)| = \left\|\varphi^h\right\|_E \leq \left\|\varphi^h\right\|_{E'_\alpha}$$

$$\leq \left\|\left(A_h^x\right)^{-1}\right\|_{E'_\alpha \to E'_\alpha} \left\|A_h^x \varphi^h\right\|_{E'_\alpha} \leq M \left\|A_h^x \varphi^h\right\|_{E'_\alpha} \tag{57}$$

and using (55), we get

$$\|J_3\|_{E'_\alpha} \leq M_9(\phi, \alpha, T) \|\widetilde{a}\|_{E'_\alpha} \left\|A_h^x \varphi^h\right\|_{E'_\alpha}. \tag{58}$$

J_4^k can be estimated as follows:

$$\left\|J_4^k\right\|_{E'_\alpha} = \left\|\sum_{j=1}^{k}\sum_{m=j}^{k}\left(R^{k-m+1} - R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)\frac{w_s^j - w_s^{j-1}}{\tau}\ \tau\right\|_{E'_\alpha}$$

$$\leq \sum_{j=1}^{k}\sum_{m=j}^{k}\left\|\left(R^{k-m+1} - R^{k-m}\right)\frac{1}{q_s}\left(a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q\right)\right\|_{E'_\alpha}\left|\frac{w_s^j - w_s^{j-1}}{\tau}\right|\ \tau.$$

Using equation (55), we get

$$\left\|J_4^k\right\|_{E'_\alpha} \leq M_8\left(\phi,\alpha,T\right)\|\widetilde{a}\|_{E'_\alpha}\sum_{j=1}^{k}\left|\frac{w_s^j-w_s^{j-1}}{\tau}\right|\tau \leq M_8\left(\phi,\alpha,T\right)\|\widetilde{a}\|_{E'_\alpha}\sum_{j=1}^{k}\max_{0\leq s\leq M}\left|\frac{w_s^j-w_s^{j-1}}{\tau}\right|\tau$$

for any j, $1\leq j\leq m\leq k$.
Then, we have that

$$\left\|J_4^k\right\|_{E'_\alpha} \leq M_8\left(\phi,\alpha,T\right)\|\widetilde{a}\|_{E'_\alpha}\sum_{j=1}^{k}\left\|\frac{w_j^h-w_{j-1}^h}{\tau}\right\|_{E'_\alpha}\tau. \tag{59}$$

Now, let estimate J_5^k. By the definition of the norm of the spaces E'_α and equation (55), we get

$$\left\|J_5^k\right\|_{E'_\alpha} = \left\|\sum_{m=1}^{k}\left(R^{k-m+1}-R^{k-m}\right)f^h\left(t_m\right)\right\|_{E'_\alpha} \leq M_{10}\left(\phi,\alpha,T\right)\left\|\left\{f^h\left(t_k\right)\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)}. \tag{60}$$

Let estimate J_6^k.

$$\left\|J_6^k\right\|_{E'_\alpha} = \left\|\frac{1}{q_s}\left(a\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}-\sigma q\right)\left(\rho\left(t_k\right)-w_s^k\right)+f^h\left(t_k\right)\right\|_{E'_\alpha}.$$

Applying the formula

$$w_s^k = w_s^0+\sum_{j=1}^{k}\frac{w_s^j-w_s^{j-1}}{\tau}\tau = \varphi\left(x_s\right)+\sum_{j=1}^{k}\frac{w_s^j-w_s^{j-1}}{\tau}\tau, \tag{61}$$

we obtain

$$\left\|J_6^k\right\|_{E'_\alpha} \leq \|\widetilde{a}\|_{E'_\alpha}\left(\max_{1\leq k\leq N}\left|\rho\left(t_k\right)\right|+M_{11}\left\|A_h^x\varphi^h\right\|_{E'_\alpha}+\sum_{j=1}^{k}\left\|\frac{w_j^h-w_{j-1}^h}{\tau}\right\|_{E'_\alpha}\tau\right)+\left\|\left\{f^h\left(t_k\right)\right\}_{k=1}^N\right\|_{C_\tau\left(E'_\alpha\right)}. \tag{62}$$

Combining the estimates $(46), (56), (58), (59), (60)$ and (62), we get

$$\left\|\frac{w_k^h-w_{k-1}^h}{\tau}\right\|_{E'_\alpha} \leq M_1\left\|A_h^x\varphi^h\right\|_{E'_\alpha}+\max_{1\leq m\leq N}\rho\left(t_m\right)M_8\left(\phi,\alpha,T\right)\|\widetilde{a}\|_{E'_\alpha}$$

$$+M_9\left(\phi,\alpha,T\right)\|\widetilde{a}\|_{E'_\alpha}\left\|A_h^x\varphi\right\|_{E'_\alpha}+M_8\left(\phi,\alpha,T\right)\|\widetilde{a}\|_{E'_\alpha}\sum_{j=1}^{k}\left\|\frac{w_j^h-w_{j-1}^h}{\tau}\right\|_{E'_\alpha}\tau$$

$$+\left(M_{10}\left(\phi,\alpha,T\right)+1\right)\left\|\left\{f^h\left(t_k\right)\right\}_{k=1}^N\right\|_{C_\tau\left(E'_\alpha\right)}$$

$$+\|\widetilde{a}\|_{E'_\alpha}\left(\max_{1\leq k\leq N}\left|\rho\left(t_k\right)\right|+M_{11}\left\|A_h^x\varphi^h\right\|_{E'_\alpha}+\sum_{j=1}^{k}\left\|\frac{w_j^h-w_{j-1}^h}{\tau}\right\|_{E'_\alpha}\tau\right)$$

or

$$\left\| \frac{w_k^h - w_{k-1}^h}{\tau} \right\|_{E'_\alpha} \leq \left(1 - (1 + M_8(\phi, \alpha, T)) \|\widetilde{a}\|_{E'_\alpha} \tau\right)^{-1} \left[M_1 \left\| A_h^x \varphi^h \right\|_{E'_\alpha} \right.$$

$$+ \max_{1 \leq m \leq N} |\rho(t_m)| M_8(\phi, \alpha, T) \|\widetilde{a}\|_{E'_\alpha} + M_9(\phi, \alpha, T) \|\widetilde{a}\|_{E'_\alpha} \|A_h^x \varphi\|_{E'_\alpha}$$

$$+ (M_{10}(\phi, \alpha, T) + 1) \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau(E'_\alpha)} + \|\widetilde{a}\|_{E'_\alpha} \left(\max_{1 \leq k \leq N} |\rho(t_k)| + M_{11} \left\| A_h^x \varphi^h \right\|_{E'_\alpha} \right)$$

$$\left. + \|\widetilde{a}\|_{E'_\alpha} (1 + M_8(\phi, \alpha, T)) \sum_{j=1}^{k-1} \left\| \frac{w_j^h - w_{j-1}^h}{\tau} \right\|_{E'_\alpha} \tau \right].$$

Using the discrete analogue of Gronwall's inequality and the last inequality, we get

$$\left\| \frac{w_k^h - w_{k-1}^h}{\tau} \right\|_{E'_\alpha} \leq e^{M_{12}(\widetilde{a}, \phi, \alpha, T)} \left[M_{13}(\widetilde{a}, \phi, \alpha, T) \left\| A_h^x \varphi^h \right\|_{E'_\alpha} + M_{13}(\widetilde{a}, \phi, \alpha, T) \|\rho^\tau\|_{C[0,T]_\tau} \right.$$

$$\left. + M_{14}(\widetilde{a}, \phi, \alpha, T) \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau(E'_\alpha)} \right]$$

for every k, $1 \leq k \leq N$.

Then, we have that

$$\left\| \left\{ \frac{w_k^h - w_{k-1}^h}{\tau} \right\}_{k=1}^{N} \right\|_{C_\tau(E'_\alpha)} \leq M_{15}\left(\widetilde{a}, \phi, \alpha, T\right) \left(\left\| A_h^x \varphi^h \right\|_{E'_\alpha} + \left\| \left\{ f^h\left(t_k\right) \right\}_{k=1}^{N} \right\|_{C_\tau(E'_\alpha)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right).$$

The following theorem finishes the proof of Theorem 2.8.
Theorem 2.9. (Ashyralyev 2007) For $0 < \alpha < \frac{1}{2}$ the norms of the spaces $E'_\alpha\left(C[0,l]_h, A_h^x\right)$ and $C^{2\alpha}\left[0,l\right]_h$ are equivalent.
Note that, in similar manner one can study the well-posedness of difference schemes for the inverse problem of reconstructing the right side of a multidimensional parabolic equation (32).

- A. Ashyralyev, Fractional spaces generated by the positive differential and difference operators in a Banach space, *PISMME,* Springer, 13-22, 2007.

The discretization of problem (32) is carried out in two steps. In the first step, the grid space $\mathbb{R}_h^n \,(0 < h \leq h_0)$ is defined as the set of all points of the Euclidean space $\mathbb{R}^n$ whose coordinates are given by

$$x_n = s_n h,\ \ s_n = 0, \pm 1, \pm 2, \cdots, n = 1, \cdots, m.$$

The operator

$$B_h^x = h^{-2m} \sum_{2m \leq |s| \leq S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}}$$

acts on functions is defined on the entire space $\mathbb{R}_h^n$. Here $s \in \mathbb{R}^{2n}$ is a vector with nonnegative integer coordinates,

$$\Delta_{k\pm} f^h(x) = \pm \left(f^h(x \pm e_k h) - f^h(x) \right),$$

where e_k is the unit vector of the axis x_k.
The function $A^x(\xi h, h)$ is obtained by replacing the operator $\Delta_{k\pm}$ with the expression $\pm\left(\exp\{\pm i \xi_k h\} - 1\right)$, and is called the symbol of the difference operator B_h^x.

It will be assumed that for $|\xi_k h| \leq \pi$ and fixed x the symbol $A^x(\xi h, h)$ of the operator $B_h^x = A_h^x - \sigma I_h$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geq M|\xi|^{2m}, |\arg A^x(\xi h, h)| \leq \phi < \phi_0 \leq \frac{\pi}{2}$$

Suppose that the coefficient b_s^x of the operator $B_h^x = A_h^x - \sigma I_h$ is bounded and satisfies the inequality

$$\left| b_s^{x+e_k h} - b_s^x \right| \leq M h^{\varepsilon}, x \in \mathbb{R}_h^n,$$

where $\varepsilon \in (0, 1]$ is a fixed point.
In the second step, problem (32) is replaced by the differences schemes

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} = h^{-2m} \sum\limits_{2m \leq |s| \leq S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} u_k^h(x) - \sigma u_k^h(x) \\ +p^k q^h(x) + f^h(t_k, x), x \in \mathbb{R}_h^n, p^k = p(t_k), t_k = k\tau, 1 \leq k \leq N, N\tau = T, \\ u_n^0 = \varphi^h(x), \ x \in \mathbb{R}_h^n, \\ u^k(y) = \rho(t_k), 0 \leq k \leq N, y = \left[\left| \frac{x^*}{h} \right| \right] h \in \Omega \subset \mathbb{R}_h^n. \end{cases} \quad (63)$$

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}^{\alpha}\left(\mathbb{R}_h^n\right)$, $\alpha \in (0,1)$, of all bounded grid functions $\phi^h(x)$ defined on $\mathbb{R}_h^n$ equipped with the norm

$$\left\|\phi^h\right\|_{\overset{\circ}{C}{}^{\alpha}\left(\mathbb{R}_h^n\right)} = \sup_{x\in\mathbb{R}_h^n}\left|\phi^h(x)\right| + \sup_{\substack{x,y\in\mathbb{R}_h^n \\ x\neq y}} \frac{\left|\phi^h(y)-\phi^h(x)\right|}{\left|y-x\right|^{\alpha}}.$$

The following theorem on well-posedness of problem (63) exists.

Theorem 2.10. For the solution of problem (63), the following coercive stability estimates

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}{}^{2\alpha}\left(\mathbb{R}_h^n\right)\right)} + \left\| \left\{ h^{-2m} \sum_{2m \le |s| \le S} \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} u_k^h \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h{}^{2\alpha}\left(\mathbb{R}_h^n\right)\right)}$$

$$\le M(q,s) \left\| \left\{ \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau}$$

$$+ M(\widetilde{a}, \phi, \alpha, T) \left[\left\| h^{-2m} \sum_{2m \le |s| \le S} \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} \varphi^h \right\|_{\overset{\circ}{C}{}^{2\alpha}\left(\mathbb{R}_h^n\right)} \right.$$

$$\left. + \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h{}^{2\alpha}\left(\mathbb{R}_h^n\right)\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right],$$

$$
\begin{aligned}
\left\|p^{\tau}\right\|_{C[0,T]_{\tau}} \leq\ & M(q,s)\left\|\left\{\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_{\tau}} \\
& + M(\widetilde{a},\phi,\alpha,T)\left[\left\|h^{-2m}\sum_{2m\leq|s|\leq S}\Delta_{1-}^{s_1}\Delta_{1+}^{s_2}\cdots\Delta_{n-}^{s_{2n-1}}\Delta_{n+}^{s_{2n}}\varphi^{h}\right\|_{\overset{\circ}{C}^{2\alpha}\left(\mathbb{R}_h^n\right)}\right. \\
& \left. + \left\|\left\{f^{h}(t_k)\right\}_{k=1}^{N}\right\|_{C_{\tau}\left(\overset{\circ}{C}_h^{2\alpha}\left(\mathbb{R}_h^n\right)\right)} + \left\|\rho^{\tau}\right\|_{C[0,T]_{\tau}}\right]
\end{aligned}
$$

hold.

The proof of Theorem 2.10 follows the scheme of the proof of Theorem 2.7 and it is based on the following theorems.

Theorem 2.11. (Ashyralyev, Sobolevskii) For $0 < \alpha < \frac{1}{2m}$ and the indicator $\mu \in (0,1)$, the norms of the spaces $E_\alpha\left(C^\mu\left(\mathbb{R}_h^n\right), A\right)$ and $C^{\mu+2m\alpha}\left(\mathbb{R}_h^n\right)$ are equivalent.
Theorem 2.12. (Ashyralyev, Sobolevskii) The solution of elliptic problem

$$h^{-2m} \sum_{2m \le |s| \le S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} u^h(x) - \sigma u^h(x) = \varphi^h(x), x \in \mathbb{R}_h^n$$

obey the coercivity inequality

$$\sum_{2m \le |s| \le S} \left\| h^{-2m} \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} u^h \right\|_{C^\mu\left(\mathbb{R}_h^n\right)} \le M(\mu) \left\| \varphi^h \right\|_{C^\mu\left(\mathbb{R}_h^n\right)}.$$

For the approximate solution of the problem (3) , the Crank-Nicholson difference scheme

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau} = \frac{a(x_n)}{2}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\ -\sigma\frac{u_n^k+u_n^{k-1}}{2}+\frac{p^k+p^{k-1}}{2}q_n+f\left(t_k-\frac{\tau}{2},x_n\right), \\ p^k=p\left(t_k\right),q_n=q\left(x_n\right),x_n=nh,\ t_k=k\tau, \\ 1\le k\le N, 1\le n\le M-1, Mh=l, N\tau=T, \\ u_0^k=u_M^k, -3u_0^k+4u_1^k-u_2^k=u_{M-2}^k-4u_{M-1}^k+3u_M^k, 0\le k\le N, \\ u_n^0=\varphi\left(x_n\right),\ 0\le n\le M, \\ u_s^k+\frac{u_{s+1}^k-u_s^k}{h}\left(x^*-sh\right)=\rho\left(t_k\right), 0\le k\le N, 0\le s=\left[\left|\frac{x^*}{h}\right|\right]h\le M \end{cases} \tag{64}$$

is constructed. Here, $q_s, q_{s+1}\neq 0, q_0=q_M$ and $-3q_0+4q_1-q_2=q_{M-2}-4q_{M-1}+3q_M$ are assumed. Then, the following theorem on well-posedness of problem (64) is established.

Theorem 3.1. For the solution of problem (64), the following coercive stability estimates

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} + \left\| \left\{ D_h^2 u_k^h \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} \leq M(q,s) \left\| \left\{ \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau}$$

$$+M(\widetilde{a},\phi,\alpha,T)\left(\left\| D_h^2 \varphi^h \right\|_{\overset{\circ}{C}_h^{2\alpha}} + \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right),$$

$$\left\| p^\tau \right\|_{C[0,T]_\tau} \leq M(q,s) \left\| \left\{ \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau}$$

$$+M(\widetilde{a},\phi,\alpha,T)\left[\left\| D_h^2 \varphi^h \right\|_{\overset{\circ}{C}_h^{2\alpha}} + \left\| \left\{ f^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau\left(\overset{\circ}{C}_h^{2\alpha}\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right]$$

hold. Here, $f^h(t_k) = \left\{ f\left(t_k - \frac{\tau}{2}, x_n\right) \right\}_{n=1}^{M-1}$, $\varphi^h = \{\varphi(x_n)\}_{n=1}^{M-1}$ and $\rho^\tau = \{\rho(t_k)\}_{k=0}^N$.

We have not been able to obtain a sharp estimate for the constants figuring in the stability inequalities. So, we will provide the following results of numerical experiments of the following problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u\left(t,x\right) + p\left(t\right)\cos\left(2x\right) + f\left(t,x\right), \\ f\left(t,x\right) = \left(4e^{-t} - t^2 - 1\right)\cos\left(2x\right),\ x \in \left(0,\pi\right),\ t \in \left(0,1\right], \\ u(0,x) = \cos\left(2x\right),\ x \in [0,\pi], \\ u(t,0) = u(t,\pi),\ u_x(t,0) = u_x(t,\pi),\ t \in [0,1], \\ u(t,\frac{1}{4}) = \cos\left(\frac{1}{2}\right)e^{-t}. \end{cases} \tag{65}$$

The exact solution of the given problem is $u\left(t,x\right) = e^{-t}\cos\left(2x\right)$ and for the control parameter $p\left(t\right)$ is $1 + t^2$.

The first order of accuracy difference scheme
For the approximate solution of the problem (65), applying the Rothe difference scheme (33), we get

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} - u_n^k + p^k q_n + f\left(t_k, x_n\right), \\ f\left(t_k, x_n\right) = \left(4e^{-t_k} - t_k^2 - 1\right)\cos 2x_n, \\ p^k = p\left(t_k\right), q_n = \cos\left(2x_n\right), x_n = nh,\ \ t_k = k\tau, \\ 1 \le k \le N, 1 \le n \le M-1, Mh = \pi, N\tau = T, \\ \\ u_n^0 = \cos\left(2x_n\right),\ \ 0 \le n \le M, \\ \\ u_0^k = u_M^k, -3u_0^k + 4u_1^k - u_2^k = u_{M-2}^k - 4u_{M-1}^k + 3u_M^k, 0 \le k \le N, \\ \\ u_s^k = \rho\left(t_k\right), \rho\left(t_k\right) = \cos\left(\frac{1}{2}\right)e^{-t_k}, 0 \le k \le N, s = \left\lfloor \frac{1}{4h} \right\rfloor \end{cases} \tag{66}$$

is constructed.

The value of $p(t_k)$ at the grid points can be obtained from the equation (38)

$$p^k = \frac{1}{\cos(2x_s)} \left(\frac{\rho(t_k) - \rho(t_{k-1})}{\tau} - \frac{w_s^k - w_s^{k-1}}{\tau} \right), 1 \leq k \leq N, \quad (67)$$

where $w_s^r, r = k, k-1$ is the solution of the difference scheme

$$\begin{cases} \frac{w_n^k - w_n^{k-1}}{\tau} = \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + \frac{\rho(t_k) - w_s^k}{\cos(2x_s)} \frac{\cos(2x_{n+1}) - 2\cos(2x_n) + \cos(2x_{n-1})}{h^2} \\ -\frac{\rho(t_k) - w_s^k}{\cos(2x_s)} \cos(2x_n) - w_n^k + f(t_k, x_n), x_n = nh, \ t_k = k\tau, \\ 1 \leq k \leq N, 1 \leq n \leq M-1, Mh = l, N\tau = T, \\ w_0^k = w_M^k, -3w_0^k + 4w_1^k - w_2^k = 3w_{M-2}^k - 4w_{M-1}^k + w_M^k, 0 \leq k \leq N, \\ w_n^0 = \cos(2x_n), \ 0 \leq n \leq M. \end{cases} \quad (68)$$

The difference scheme (68) can be arranged as

$$\begin{cases} \left(-\frac{1}{h^2}\right) w_{n+1}^k + \left(\frac{1}{\tau} + \frac{2}{h^2} + 1\right) w_n^k + \left(-\frac{1}{h^2}\right) w_{n-1}^k + \left(-\frac{1}{\tau}\right) w_n^{k-1} \\ + \left(\frac{\cos(2x_{n+1}) - 2\cos(2x_n) + \cos(2x_{n-1})}{\cos(2x_s) h^2} - \frac{\cos(2x_n)}{\cos(2x_s)}\right) w_s^k \\ = \frac{\rho(t_k)}{\cos(2x_s)} \frac{\cos(2x_{n+1}) - 2\cos(2x_n) + \cos(2x_{n-1})}{h^2} - \frac{\rho(t_k)}{\cos(2x_s)} \cos(2x_n) + f(t_k, x_n), \\ \rho(t_k) = \cos(\frac{1}{2}) e^{-t_k}, f(t_k, x_n) = (4e^{-t_k} - t_k^2 - 1) \cos 2x_n, \\ x_n = nh, \; t_k = k\tau, 1 \le k \le N, 1 \le n \le M-1, Mh = l, N\tau = T, \\ w_0^k = w_M^k, -3w_0^k + 4w_1^k - w_2^k = w_{M-2}^k - 4w_{M-1}^k + 3w_M^k, 0 \le k \le N, \\ w_n^0 = \cos(x_n), \; 0 \le n \le M. \end{cases}$$

First, applying the first order of accuracy difference scheme (68), we obtain $(M+1) \times (M+1)$ system of linear equations and we write them in the matrix form

$$Aw^k + Bw^{k-1} = D\varphi^k, \quad 1 \le k \le N, \; w^0 = \{\cos 2x_n\}_{n=0}^M \tag{69}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & . & 0 & 0 & -1 \\ x & y & x & 0 & . & z_1 & . & 0 & 0 & 0 \\ 0 & x & y & x & . & z_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & z_{n-1} & . & x & y & x \\ -3 & 4 & -1 & 0 & . & 0 & . & -1 & 4 & -3 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & v & 0 & . & 0 & 0 & 0 \\ 0 & 0 & v & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & v & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}.$$

Here,

$$x = -\frac{1}{h^2}, \; y = \frac{1}{\tau} + \frac{2}{h^2} + 1, v = -\frac{1}{\tau},$$

$$z_c = \frac{\cos(2x_{c+1}) - 2\cos(2x_c) + \cos(2x_{c-1})}{\cos(2x_s)\, h^2} - \frac{\cos(2x_c)}{\cos(2x_s)}, \text{ in } (s+1)^{th} \text{ column},$$

$$w^r = \begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \qquad \textit{for } r = k+1, k,$$

$$\varphi^k = \begin{bmatrix} 0 \\ \phi_1^k \\ \vdots \\ \phi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\phi_n^k = \frac{\rho(t_k)}{\cos(2x_s)} \frac{\cos(2x_{n+1}) - 2\cos(2x_n) + \cos(2x_{n-1})}{h^2} - \frac{\rho(t_k)}{\cos(2x_s)} \cos(2x_n) + f(t_k, x_n),$$

and D is $(M+1) \times (M+1)$ identity matrix.

Using (69), we can obtain that

$$w^k = A^{-1}\left(D\varphi^k - Bw^{k-1}\right), \quad k = 1, 2, \cdots, N, \; w^0 = \{\cos 2x_n\}_{n=0}^{M}. \tag{70}$$

Then, we can reach to the solution of $w_n^k, 0 \leq k \leq N, \; 0 \leq n \leq M$. Applying the first order of accuracy difference scheme (66) and (67), we have again $(M+1) \times (M+1)$ system of linear equations and we write them in the matrix form where

$$A_2 u^k + B_2 u^{k-1} = D\varphi^k, \quad 1 \leq k \leq N, \; u^0 = \{\cos 2x_n\}_{n=0}^{M}$$

where

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & -1 \\ x & y & x & 0 & . & 0 & 0 & 0 \\ 0 & x & y & x & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & x & y & x \\ -3 & 4 & -1 & 0 & . & -1 & 4 & -3 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & v & 0 & . & 0 & 0 & 0 \\ 0 & 0 & v & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & v & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)} .$$

Here,

$$x = -\frac{1}{h^2}, \; y = \frac{1}{\tau} + \frac{2}{h^2} + 1, v = -\frac{1}{\tau},$$

$$u^r = \begin{bmatrix} u_0^r \\ \vdots \\ u_M^r \end{bmatrix}_{(M+1)\times 1} \quad \textit{for } r = k+1, k,$$

$$\varphi^k = \begin{bmatrix} 0 \\ \phi_1^k \\ \vdots \\ \phi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1} ,$$

$$\phi_n^k = p^k q_n + f\left(t_k, x_n\right).$$

To solve the resulting difference equations, we again apply the iterative method given in (70).

The second order of accuracy difference scheme

For the approximate solution of the problem (65), applying the Crank-Nicholson difference scheme (33), we get

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} - \frac{u_n^k + u_n^{k-1}}{2} + \frac{p^k + p^{k-1}}{2} q_n \\ + f\left(t_k - \frac{\tau}{2}, x_n\right), f\left(t_k, x_n\right) = \left(4e^{-t_k} - t_k^2 - 1\right)\cos 2x_n, \\ p^k = p\left(t_k\right), q_n = \cos\left(2x_n\right), x_n = nh, \ t_k = k\tau, \\ 1 \le k \le N, 1 \le n \le M-1, Mh = \pi, N\tau = T, \\ u_n^0 = \cos\left(2x_n\right), \ 0 \le n \le M, \\ u_0^k = u_M^k, -3u_0^k + 4u_1^k - u_2^k = u_{M-2}^k - 4u_{M-1}^k + 3u_M^k, 0 \le k \le N, \\ u_s^k + \frac{u_{s+1}^k - u_s^k}{h}\left(x^* - sh\right) = \rho\left(t_k\right), \rho\left(t_k\right) = \cos(\frac{1}{2})e^{-t_k}, 0 \le k \le N, s = \left\lfloor \frac{1}{4h} \right\rfloor \end{cases}$$

is constructed.

The value of $p(t_k)$ at the grid points can be obtained from the equation (38)

$$p^k = \frac{\frac{\rho(t_k)-\rho(t_{k-1})}{\tau} - (1-y)\frac{w_s^k - w_s^{k-1}}{\tau} - y\frac{w_{s+1}^k - w_{s+1}^{k-1}}{\tau}}{(1-y)\,q_s + yq_{s+1}}, 1 \le k \le N,$$

where $y = \frac{x^*-sh}{h}$ and $w_s^r, r = k, k-1$ is the solution of the difference scheme

$$\begin{cases} \frac{w_n^k - w_n^{k-1}}{\tau} = \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{2h^2} + \frac{w_{n+1}^{k-1} - 2w_n^{k-1} + w_{n-1}^{k-1}}{2h^2} \\ + \frac{q_{n+1}-2q_n+q_{n-1}}{2h^2}\left(\frac{\rho(t_k) - (1-y)\,w_s^k - yw_{s+1}^k}{(1-y)\,q_s + yq_{s+1}} + \frac{\rho(t_{k-1}) - (1-y)\,w_s^{k-1} - yw_{s+1}^{k-1}}{(1-y)\,q_s + yq_{s+1}}\right) \\ -\frac{q_n}{2}\left(\frac{\rho(t_k) - (1-y)\,w_s^k - yw_{s+1}^k}{(1-y)\,q_s + yq_{s+1}} + \frac{\rho(t_{k-1}) - (1-y)\,w_s^{k-1} - yw_{s+1}^{k-1}}{(1-y)\,q_s + yq_{s+1}}\right) - \frac{w_n^k + w_n^{k-1}}{2} \\ +f\left(t_k - \frac{\tau}{2}, x_n\right), x_n = nh,\ t_k = k\tau, 1 \le k \le N, 1 \le n \le M-1, Mh = l, N\tau = T, \\ w_0^k = w_M^k, -3w_0^k + 4w_1^k - w_2^k = 3w_{M-2}^k - 4w_{M-1}^k + w_M^k, 0 \le k \le N, \\ w_n^0 = \cos(2x_n),\ 0 \le n \le M. \end{cases} \quad (71)$$

The difference scheme (71) can be arranged as

$$\begin{cases} \left(-\frac{1}{2h^2}\right) w_{n+1}^{k-1} + \left(-\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2}\right) w_n^{k-1} + \left(-\frac{1}{2h^2}\right) w_{n-1}^{k-1} \\ + \left(-\frac{1}{2h^2}\right) w_{n+1}^{k} + \left(\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2}\right) w_n^{k} + \left(-\frac{1}{2h^2}\right) w_{n-1}^{k} \\ + \left(\frac{(q_{n+1}-2q_n+q_{n-1})(1-y)}{2h^2((1-y)q_s+yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s+yq_{s+1})}\right) w_s^k + \left(\frac{(q_{n+1}-2q_n+q_{n-1})y}{2h^2((1-y)q_s+yq_{s+1})} + \frac{q_n y}{2((1-y)q_s+yq_{s+1})}\right) w_{s+1}^k \\ + \left(\frac{(q_{n+1}-2q_n+q_{n-1})(1-y)}{2h^2((1-y)q_s+yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s+yq_{s+1})}\right) w_s^{k-1} + \left(\frac{(q_{n+1}-2q_n+q_{n-1})y}{2h^2((1-y)q_s+yq_{s+1})} + \frac{q_n y}{2((1-y)q_s+yq_{s+1})}\right) w_{s+1}^{k-1} \\ = \frac{\frac{(q_{n+1}-2q_n+q_{n-1})}{2h^2} - \frac{q_n}{2}}{((1-y)q_s+yq_{s+1})} \left(\rho\left(t_k\right) + \rho\left(t_{k-1}\right)\right) + f\left(t_k - \frac{\tau}{2}, x_n\right), \\ \rho\left(t_k\right) = \cos(\frac{1}{2})e^{-t_k}, f\left(t_k, x_n\right) = \left(4e^{-t_k} - t_k^2 - 1\right)\cos 2x_n, \\ x_n = nh,\ t_k = k\tau, 1 \le k \le N, 1 \le n \le M-1, Mh = l, N\tau = T, \\ w_0^k = w_M^k, -3w_0^k + 4w_1^k - w_2^k = w_{M-2}^k - 4w_{M-1}^k + 3w_M^k, 0 \le k \le N, \\ w_n^0 = \cos\left(x_n\right),\ 0 \le n \le M. \end{cases}$$

First, applying the first order of accuracy difference scheme (68), we obtain $(M+1)\times(M+1)$ system of linear equations and we write them in the matrix form

$$Aw^k + Bw^{k-1} = D\varphi^k, \quad 1 \le k \le N,\ w^0 = \{\cos 2x_n\}_{n=0}^{M}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & -1 \\ a & b & a & 0 & . & z_1 & j_1 & . & 0 & 0 & 0 \\ 0 & a & y & a & . & z_2 & j_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & z_{n-1} & j_{n-1} & . & a & b & a \\ -3 & 4 & -1 & 0 & . & 0 & 0 & . & -1 & 4 & -3 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 \\ a & c & a & . & z_1 & j_1 & . & 0 & 0 & 0 \\ 0 & a & c & . & z_2 & j_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & z_{n-1} & j_{n-1} & . & a & c & a \\ 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}.$$

Here,

$$a = -\frac{1}{2h^2},\ b = \frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2}, c = -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2},$$

$$z_n = \left(\frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} \right), \text{ in } (s+1)^{th} \text{ column},$$

$$j_n = \left(\frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} \right), \text{ in } (s+2)^{th} \text{ column},$$

$$w^r = \begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \qquad for\ r = k+1, k,$$

Using the iterative method given in (70), we can reach to the solution of $w_n^k, 0 \leq k \leq N,\ 0 \leq n \leq M$. Applying the second order of accuracy difference scheme (66) and (67), we have again $(M+1) \times (M+1)$ system of linear equations and we write them in the matrix form where

$$A_2 u^k + B_2 u^{k-1} = D\varphi^k, \quad 1 \leq k \leq N,\ u^0 = \{\cos 2x_n\}_{n=0}^{M}$$

where

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & -1 \\ a & b & a & 0 & . & 0 & 0 & 0 \\ 0 & a & b & a & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & a & b & a \\ -3 & 4 & -1 & 0 & . & -1 & 4 & -3 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 \\ a & c & a & . & 0 & 0 & 0 \\ 0 & a & c & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & a & c & a \\ 0 & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}.$$

Here,

$$a = -\frac{1}{2h^2},\ b = \frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2}, c = -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2},$$

$$u^r = \begin{bmatrix} u_0^r \\ \vdots \\ u_M^r \end{bmatrix}_{(M+1)\times 1} \qquad \textit{for } r = k+1, k,$$

Error analysis

Table 1 gives the error analysis between the exact solution of $p\left(t\right)$ and the solutions derived by the numerical process. The error is computed by the following formula.

$$E_p = \max_{1 \leq k \leq N} \left| p(t_k) - p_k \right| .$$

Table 1. Error analysis for $p\left(t\right)$.

	N=30	N=60	N=90
Max. Error	0.3201	0.1451	0.0810

Table 2 gives the error analysis between the exact solution and the solutions derived by difference schemes. Table 2 is constructed for $N = M = 30$, 60 and 90 respectively. For their comparison,

the errors are computed by

$$E = \max_{\substack{1 \leq k \leq N \\ 1 \leq n \leq M}} \left| u(t_k, x_n) - u_n^k \right| .$$

Table 2. Error analysis for the exact solution $u(t,x)$.

Method	N=M=30	N=M=60	N=M=90
1^{st} order of accuracy	0.0628	0.0345	0.0214
2^{nd} order of accuracy	1.3240×10^{-4}	4.2111×10^{-5}	1.2088×10^{-5}

For finding of the control parameter $p(t)$, we use $u(t,x^*)=\gamma(t)$ for $t=0$, values of its first and second derivatives and smoothness of $\gamma(t)$ in t. Therefore, we will consider the error between $\gamma(t_k)$ and $u^k_{\left\lfloor \frac{x^*}{h} \right\rfloor}$. Table 3 gives the maximum error for $h=\frac{1}{M}$ and $N=M=30$, 60 and 90 respectively.

Table 3. Error analysis between $\gamma(t_k)$ and $u^k_{\left\lfloor \frac{x^*}{h} \right\rfloor}$

Method	N=M=30	N=M=60	N=M=90
1^{st} order of accuracy	0.0354	0.0179	0.0116
2^{nd} order of accuracy	4.2321×10^{-5}	1.1632×10^{-5}	$3.1262.10^{-6}$

Thus, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

This work is devoted to the study of the well-posedness of the right-hand side identification problem for a parabolic equation. The following original results are obtained:

- The well-posedness in $C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)$ of the inverse problem of reconstructing the right side of a parabolic equation with nonlocal conditions

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) + p(t)q(x) + f(t,x), \\ 0 < x < l,\ 0 < t \le T, \\ u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), 0 \le t \le T, \\ u(0,x) = \varphi(x),\ 0 \le x \le l, \\ u(t,x^*) = \rho(t),\ 0 \le x^* \le l, 0 \le t \le T \end{cases} \tag{72}$$

  is established.
- Theorem on the well-posedness in $C\left([0,T], C^{2m\alpha}(\mathbb{R}^n)\right)$ of the inverse problem of reconstructing the right side of a multidimensional parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum\limits_{|r|=2m} a_r(x)\frac{\partial^{|r|}u(t,x)}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} - \sigma u(t,x) + p(t)q(x) \\ +f(t,x),\ x\in\mathbb{R}^n, 0<t<T, |r| = r_1+r_2+\cdots+r_n, \\ u(0,x) = \varphi(x), x\in\mathbb{R}^n, \\ u(t,x^*) = \rho(t), 0\le t\le T,\ x^*\in\Omega\subset\mathbb{R}^n \end{cases} \tag{73}$$

- For the approximate solution of the problem (72) the Rothe difference scheme

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} = a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} - \sigma u_n^k + p^k q_n + f(t_k, x_n), \\ p^k = p(t_k), q_n = q(x_n), x_n = nh, \ t_k = k\tau, \\ 1 \le k \le N, 1 \le n \le M-1, Mh = l, N\tau = T, \\ u_0^k = u_M^k, -3u_0^k + 4u_1^k - u_2^k = u_{M-2}^k - 4u_{M-1}^k + 3u_M^k, 0 \le k \le N, \\ u_n^0 = \varphi(x_n), \ 0 \le n \le M, \\ u_{\left[\left|\frac{x^*}{h}\right|\right]}^k = u_s^k = \rho(t_k), 0 \le k \le N, 0 \le s \le M \end{cases} \tag{74}$$

is presented. The coercive stability estimates for the problem (74) are established.

- The well-posedness of difference scheme

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} = h^{-2m} \sum\limits_{2m \le |s| \le S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} u_k^h(x) - \sigma u_k^h(x) \\ + p^k q^h(x) + f^h(t_k, x), x \in \mathbb{R}_h^n, p^k = p(t_k), t_k = k\tau, 1 \le k \le N, N\tau = T, \\ u_n^0 = \varphi^h(x), \ x \in \mathbb{R}_h^n, \\ u^k(y) = \rho(t_k), 0 \le k \le N, y = \left[\left|\frac{x^*}{h}\right|\right] h \in \Omega \subset \mathbb{R}_h^n \end{cases}$$

for the inverse problem of reconstructing the right side of a multidimensional parabolic equation (73) is presented.

- The Crank-Nicholson difference scheme

$$\left\{\begin{array}{l} \frac{u_n^k-u_n^{k-1}}{\tau}=\frac{a(x_n)}{2}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\ -\sigma\frac{u_n^k+u_n^{k-1}}{2}+\frac{p^k+p^{k-1}}{2}q_n+f\left(t_k-\frac{\tau}{2},x_n\right), \\ p^k=p\left(t_k\right),q_n=q\left(x_n\right),x_n=nh,\ t_k=k\tau, \\ 1\leq k\leq N,1\leq n\leq M-1,Mh=l,N\tau=T, \\ \\ u_0^k=u_M^k,-3u_0^k+4u_1^k-u_2^k=u_{M-2}^k-4u_{M-1}^k+3u_M^k,0\leq k\leq N, \\ \\ u_n^0=\varphi\left(x_n\right),\ 0\leq n\leq M, \\ \\ u_s^k+\frac{u_{s+1}^k-u_s^k}{h}\left(x^*-sh\right)=\rho\left(t_k\right),0\leq k\leq N,0\leq s=\left[\left|\frac{x^*}{h}\right|\right]\leq M \end{array}\right. \tag{75}$$

is constructed. For the solution of problem (75), coercive stability estimates are obtained.

- The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Thank you for your attention