On Operators of Strong type B

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An operator from E into X is called an operator of strong type B if T'' maps the band $B(E)$ generated by E into X. The space of operators of strong type B will be denoted by $W_{st}(E, X)$.
A subset A of E is called b-order bounded if it is order bounded in E''. For example every positive increasing norm bounded net in E is b-order bounded. An operator $T : E \rightarrow X$ is called b-weakly compact if $T(A)$ is relatively weakly compact for each b-order bounded subset A of E. b-weakly compact operators have been characterized as those which do not preserve a subspace (or a sublattice) isomorphic to c_0. The space of b-weakly compact operators will be denoted by $W_b(E, X)$.

Each operator of strong type B is b-weakly compact. One of the problems asked in [9] was the existence of a b-weakly compact operator which is not of strong type B. This question was settled in [4-5] where it was shown that there exists a b-weakly compact operator which is not of strong type B.
The space of order weakly compact operators will be denoted by $W_o(E, X)$.
Among the various classes of operators that are introduced we have ;

$$W(E, X) \subseteq W_{st}(E, X) \subseteq W_b(E, X) \subseteq W_o(E, X)$$

where inclusions may well be proper. For example the identity operator on c_0 is order weakly compact but not b-wealy compact.

$W_{st}(E, X)$ is a closed subspace of $L(E, X)$ in the norm topology. If F has order continuous norm then $W_{st}(E, X)$ is an order ideal in $L(E, F)$.

If $T : E \to F$ is of strong type B and $S : F \to X$ is bounded then $S \circ T$ is of strong type B. If $T : E \to F$ is regular and $S : F \to X$ is of strong type B, then $S \circ T$ is strong type B.

If X is a Banach space that does not contain c_0 then each bounded operator $T : E \to X$ factors over a KB-space F as $S \circ Q$ where $Q : E \to F$ is a lattice homomorphism and S is bounded. Consequently, each bounded operator $T : E \to X$ is of strong type B.Thus, when E or F is a KB-space, then every bounded operator is of strong type B.

The following theorem of Ghousoub and B.Johnson sheds more light on operators of strong type B. For a Banach lattice E, $I(E)$ will denote the closure of the order ideal generated by E in E''.

Theorem Let $T : E \to X$ be a bounded operator. Then, the following are equivalent:

1) T is of strong type B.

2) $T''|_{I(E)}$ does not preserve a copy of c_0.

3) There exist a Banach space Y and bounded linear operators $S : E \to Y$ and $R : Y \to X$ such that neither R or S preserve a copy of c_0, and $T = R \circ S$.

4) There exist a KB-space F, an interval preserving lattice homomorphism $R : I(E) \to F$, and an operator $S : F \to X$ such that $T''|_{I(E)} = S \circ R$.

It follows that an operator is of strong type B if and only if it factors over a KB-space.

Operators of strong type B are of substance only when they do not coincide with weakly compact or b-weakly compact operators. For example, when E has order continuous norm or when $E = C(S)$ for some compact Hausdorff space S then $W_b(E, X) = W_{st}(E, X)$. It may also be the case that there are no b-weakly compact or operators of strong type B, as in the case of E being an M-space. Recall that E has property V if every non-weakly compact operator preserves a copy of c_0. Similarly, if E has V_0 then every non-weakly compact operator preserves a copy of c_0.

Recall that E has the strict Dieudonne propert if every $T : E \to X$ is either weakly compact or preserve a copy of c_0. Thus when E has the strict Dieudonne property then $W(E, X) = W_{st}(E, X) = W_b(E, X)$ for every X. For example, this is the case when E is σ-Dedekind complete and E' has the Schur property or when E has the Grothendieck property.

In order to identify when we have $W(E,X) = W_{st}(E,X)$, we need a lemma.

Lemma Suppose E' does not have order continuous norm. Then there exist a disjoint sequence (e_n) of positive elements in E with $||e_n|| \leq 1$ for all n and $0 \leq \phi \in E'$ and $\epsilon > 0$ satisfying $\phi(e_n) > \epsilon$ for all n. Moreover, the components ϕ_n of ϕ in the carriers C_{e_n} form an order bounded disjoint sequence in E'_+ such that $\phi_n(e_n) = \phi(e_n)$ for all n and $\phi_n(e_m) = 0$ if $n \neq m$.

In this context let us recall a theorem of Grothendieck: Every operator from a Banach lattice not containing a copy of complemented copy of l^1 into a Banach space not containing a copy of c_0 is weakly compact.

Proposition 1 The following are equivalent:

1) $W(E,X) = W_{st}(E,X)$.

2) One of the following hold:

a) E' is a KB-space,

b) X is reflexive.

$(1 \Rightarrow 2)$. Suppose E' is not a KB-space and X is not reflexive. Then we construct an operator of strong type B which is not weakly compact. Since E' is not a KB-space, it follows from the preceding lemma that there exist a disjoint sequence (u_n) in E_+ with $||u_n|| \leq 1$ for all n and some $\phi \in E'_+$, $\epsilon > 0$ such that $\phi(u_n) > \epsilon$ for all n. The components ϕ_n of ϕ in the carriers C_{u_n} of u_n form an order bounded disjoint sequence in E'_+ such that $\phi(u_n) = \phi_n(u_n)$ and $\phi_m(u_n) = 0$ if $n \neq m$. Note that we have $0 \leq \phi_n \leq \phi$ for all n. Let us define $T_1 : E \to l^1$ by

$$T_1(x) = \left(\frac{\phi_n(x)}{\phi(u_n)} \right)$$

for each $x \in E$.

Since

$$\sum_{n=1}^{\infty}\left|\frac{\phi_n(x)}{\phi(u_n)}\right| \leq \frac{1}{\epsilon}\phi(|x|)$$

the operator T_1 is well-defined, positive and is of strong type B. On the other hand since X is not reflexive, the closed unit ball B_X of X is not weakly compact. Hence we can find a sequence (y_n) in B_X without any weakly convergent subsequences. Let us define the operator $T_2 : l^1 \rightarrow X$ by $T_2(\alpha_n) = \Sigma\alpha_n y_n$. Since $\Sigma||\alpha_n y_n|| \leq \Sigma|\alpha_n|$, T_2 is well-defined. We consider the operator $T = T_2 \circ T_1 : E \rightarrow l^1 \rightarrow X$ which is defined by

$$T(x) = \sum_n \frac{\phi_n(x)}{\phi(u_n)} y_n$$

for each $x \in E$. As T factors over the KB-space l^1, T is of strong type B. However as $T(u_n) = y_n$ and since (y_n) is chosen not to have any weakly convergent subsequences T is not weakly compact. □

With regards to this result, there is a Banach lattice E with E' is a KB-space and a non-weakly compact operator $T : E \longrightarrow c_0$ which is b-weakly compact [4].

Corollary The following are equivalent:

1) Each strong type B operator $T : E \longrightarrow F$ is weakly compact.

2) Each positive strong type B operator $T : E \longrightarrow F$ is weakly compact.

3) One of the following holds:

a) E' is a KB-space.

b) F is reflexive.

The next result yields a characterization of the order continuity of the dual norm.

Corollary The following are equivalent:

1) Each strong type B operator T on E is weakly compact.
2) Each positive strong type B operator on E is weakly compact.
3) E' is a KB-space.

Corollary Let F be an infinite dimensional AL-space. Then the following are equivalent:

1) Each strong type B operator $T : E \to F$ is weakly compact.
2) Each positive strong type B operator $T : E \to F$ is weakly compact.
3) E' is a KB-space.

Corollary

Let E be an infinite dimensional AL-space. Then the following are equivalent:
1) Each strong type B operator $T : E \to F$ is weakly compact.
2) F is reflexive.
Corollary Let X be a non-reflexive Banach space. Then the following are equivalent:
1) $W_o(E, X) = W(E, X)$.
2) One of the following holds:
a) E has the positive Grothendieck property.
b) E' is a KB-space and c_0 is not embeddable in X.

Corollary

Let F be a non-reflexive Banach space. Then the following are equivalent:
1) $W_o(E, F) = W(E, F)$.
2) One of the following holds:
a) E has the positive Grothendieck property.
b) E' and F are KB-spaces.
Proposition 2 If each positive operator of strong type B $T : E \to F$ is compact, then one of the following holds.
1) E' is a KB-space.
2) F is finite dimensional.

Proof

Suppose the two statements are not true. We construct a positive operator of strong type B from E into F which is not compact. As the norm of E' is not order continuous, E contains a sublattice isomorphic to l^1 and there exists a positive projection $P : E \to l^1$. Since F is infinite dimensional, there exists a disjoint norm bounded sequence (y_n) in F_+ which does not converge to zero in norm. Let $S : l^1 \to F$ be the operator defined by $S(\alpha_n) = \sum_n \alpha_n y_n$ for each (α_n) in l^1. Since $S(e_n) = y_n$ for each n, S is not compact. Consider the positive operator $T = S \circ P : E \to l^1 \to F$. Since T factors over the KB-space l^1, T is an operator of strong type B. T is not compact. Because if it were then the operator $R = T \circ i$ where i is the embedding of l^1 into E would be compact. As $T \circ i$ is the identity on l^1, this is not possible. □

It is a well-known theorem that E is a KB-space if and only if every operator $T : c_0 \to E$ is weakly compact.

Proposition 3 The following are equivalent:

1) For each Banach lattice E, each $T : E \to X$ is of strong type B.

2) Each $T : c_0 \to X$ is of strong type B.

3) c_0 is not embeddable in X.

The next result yields a characterization of KB-spaces in terms of operators of strong type B.

Corollary For a Banach lattice F, the following are equivalent:

1) For any E, each $T : E \longrightarrow F$ is of strong type B.

2) Each $T : c_0 \longrightarrow F$ is of strong type B.

3) Each $T : c_0 \longrightarrow F$ is compact.

4) Each positive $T : c_0 \longrightarrow F$ is of strong type B.

5) F is a KB-space.

Operators of strong type B does not satisfy the duality property. Indeed, the identity of l^1 is of strong type B but its adjoint, the identity on l^∞ is not of strong type B. On the other hand the identity of c_0 is not of strong type B, but its adjoint, the identity on l^1 is of strong type B.

Proposition 4 The following are equivalent:

1) Each $T : F' \to E'$ is of strong type B.

2) If $T : E \to F$ is of strong type B then so is T' .

3) One of the following holds:

a) E' is a KB-space.

b) F' is a KB-space.

Proof

$(2 \Rightarrow 3)$.Suppose neither E' nor F' is a KB-space. Then we wish to show that there is an operator $T : E \to F$ which is of strong type B but its adjoint T' is not. Since E' does not have order continuous norm, there exists an order bounded disjoint sequence (f_n) in E'_+ such that $||f_n|| = 1$ for all n. Let $f = \bigvee_{n=1}^{\infty} f_n$ in E'. Let us define $T_1 : E \to l^1$ by $T_1(x) = (f_n(x))$ for each $x \in E$. Since

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \sum_n f_n(|x|) \leq f(|x|)$$

for each $x \in E$, T_1 is well-defined and positive.

Since F' is not a KB-space, there exists an order bounded disjoint sequence (g_n) in F' such that $||g_n|| = 1$ for all n. Next we choose y_n in F_+ with $||y_n|| = 1$ and $g_n(y_n) \geq \frac{1}{2}$. Let $T_2 : l^1 \to F$ be defined as $T_2(\alpha_n) = \Sigma_n \alpha_n y_n$ for all (α_n) in l^1. The operator $T = T_2 \circ T_1 : E \to F$ is defined as $T(x) = \Sigma_n f_n(x) y_n$ for $x \in E$. T is of strong type B. Its adjoint is defined by $T'(h) = \Sigma_n h(y_n) f_n$ for $h \in F'$. Let us observe that (g_n) is an order bounded disjoint sequence in F' and

$$||T'(g_n)|| = ||\sum_n g_m(y_k) f_k|| \geq ||g_n(y_n) f_n|| \geq \frac{1}{2}$$

for each n.Thus T' is not of strong type B. □

It is natural to ask whether T is of strong type B when T' is of strong type B. This problem was studied for b-weakly compact operators in [2] under the assumption that E has order continuous norm. However when E has order continuous norm then $W_{st}(E,F) = W_b(E,F)$. Hence their result can be stated for operators of strong type B.

Proposition 5 Let E have order continuous norm. Then the following are equivalent:

1) Each $T : E \to F$ is of strong type B.

2) T is of strong type B whenever $T' : F' \to E'$ is of strong type B.

3) One of the following holds:

a) E is a KB-space.

b) F is a KB-space.

Let us observe that if E does not have order continuous norm then there exists a pair of Banach lattices E, F where $W_b(E, F) = W(E, F)$ without E or F being KB-spaces. Take for example $E = l^\infty$, $F = c_0$.

Proposition 6 The following are equivalent:

1) Each operator $T : E \to F$ is of strong type B.

2) T is of strong type B whenever T' is.

3) One of the following holds:

a) E' is a KB-space.

b) F is a KB-space.

Proof

$(2 \Rightarrow 3)$. Suppose that E' and F are not KB-spaces. Consider the order bounded disjoint sequence (ϕ_n) given by the lemma. Let $g_n = \frac{\phi_n}{\phi(u_n)}$. Then $g_n(u_n) = 1$ for all n and $g_n(u_m) = 0$ for $n \neq m$. (g_n) is an order bounded disjoint sequence in E'_+. Since the topology $|\sigma|(E', E)$ of E' is Lebesgue, $g_n \to 0$ for $|\sigma|(E', E)$. Therefore the operator $T_1 : E \to c_0$ defined by $T_1(x) = (g_n(x))$ for each $x \in E$ is well-defined and positive. Moreover $T_1(u_n) = e_n$ for each n.

Since F is not a KB-space, c_0 is lattice embeddable in F. Let $T_2 : c_0 \to F$ be an embedding and let $y_n = T_2(e_n)$. Let us also observe that the sequence (y_n) is bounded away from zero. That is, $||y_n|| \geq K$ for some K and all n. The operator $T = T_2 \circ T_1 : E \to c_0 \to F$ is not of strong type B whereas T' is of strong type B. □

We have the following characterization of operators of strong type B.

Proposition 7 An operator $T : E \to X$ is strong type B if and only if $(T''(x_n))$ is convergent for each norm bounded increasing sequence (x_n) in $I(E)$.

Recall that an operator $T : E \to F$ is called semi-compact if for each $\epsilon > 0$, there exists $u \in F_+$ such that $T(B_E) \subseteq [-u, u] + \epsilon B_F$.

A semi-compact operator need not be of strong type B. For example, the identity operator on l^∞ is semi-compact but not of strong type B. On the other hand, the identity on l^1 is of strong type B, but it is not semi-compact.

Proposition 8

Let F be σ-Dedekind complete. Then the following are equivalent:
1) Each semi-compact $T : E \longrightarrow F$ is of strong type B.
2) Each positive semi-compact $T : E \longrightarrow F$ is of strong type B.
3) One of the following is true:
a) E is a KB-space.
b) F has order continuous norm.

Proposition 9

The following are equivalent:
1) Each $0 \leq T : F' \to E'$ is of strong type B.
2) Adjoint of each $0 \leq T : E \to F$ is strong type B.
3) Each semi-compact $T : F' \to E'$ is of strong type B.
4) For each $0 \leq T : E \to F$ such that T' is semi-compact, T' is of strong type B.
5) Each semi-compact $0 \leq T : F' \to E'$ is strong type B.
6) One of the following holds:
a) E' is a KB-space.
b) F' is a KB-space.

Square of an operator of strong type B need not be weakly compact. Consider for example the identity operator on $L^1[0,1]$. To this end we have the following:

Proposition 10 The following are equivalent:

1) Let $0 \leq S, T : E \rightarrow E, 0 \leq S \leq T, T \in W_{st}(E,E)$, then S is weakly compact.

2) $0 \leq T : E \rightarrow E$ be of strong type B, then T is weakly compact.

3) If $0 \leq T : E \rightarrow E$ is of strong type B, then T^2 is weakly compact.

4) E' is a KB-space.

If T'' is an order weakly compact operator then by Theorem 3.5.8. in [8], there exists a KB-space F and an interval preserving lattice homomorphism $Q : E \rightarrow F$ and a bounded operator $S : F \rightarrow X$ such that $T = S \circ Q$. It follows that every operator $T : E \rightarrow X$ with T'' is order weakly compact is of strong type B. Thus, if T' is semi-compact then T'' is order weakly compact by Theorem 3.6.18 in [8] and $T : E \rightarrow X$ is of strong type B.

If F has order continuous norm, then each positive semi-compact operator $T : E \to F$ is weakly compact. It is also known that if each positive semi-compact operator on a σ-Dedekind complete Banach lattice is weakly compact then E has order continuous norm [3]. We now generalize this asking positive semi-compact operator to be order weakly compact.

Proposition 11 Let E be σ-Dedekind complete. If each positive semi-compact operator on E is order weakly compact, then E has order continuous norm.

Proof

Assume that the norm of E is not order continuous. Then E contains a copy of l^∞ and there exists a positive projection, say P onto the copy of l^∞. Let i be the embedding of l^∞ into E. Then the operator $i \circ P : E \to l^\infty \to E$ is semi-compact but not an order weakly compact operator. Because if it were, then $P \circ (i \circ P)$ would also be order weakly compact. This would imply that the restriction of $P \circ (i \circ P)$ to l^∞ is also order weakly compact. But this is not the case. □

Proposition 12

Let $T : E \longrightarrow F$ be an order bounded operator. Then
1) If E' is a KB-space and if T maps order intervals of E into norm precompact subsets of F, then the operator T' has the same property.
2) If F has order continuous norm, then T maps order intervals onto norm precompact subsets of F whenever its dual T' has the same property.

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