Internal characteristics of domains in $\mathbb{C}^n$

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Abstract

This paper is devoted to internal capacity characteristics of a domain $D \subset \mathbb{C}^n$, relative to a point $a \in D$, which have their origin in the notion of the conformal radius of a simply connected plane domain relative to a point. Our main goal is studying internal Chebyshev constants and transfinite diameters for a domain $D \subset \mathbb{C}^n$ and its boundary ∂D relative to a point $a \in D$ in the spirit of the author's article (Math. USSR Sbornik **25** (1975), 350-364), where similar characteristics have been investigated for compact sets in $\mathbb{C}^n$. The central notion of directional Chebyshev constants is based on the asymptotic behavior of extremal monic "polynomials" and "copolynomials" in directions determined by the arithmetics of the indices set $\mathbb{Z}^n$. Some results are closely related to results on s-th Reiffen pseudometrics and internal directional analytic capacities of higher order (Jarnicki-Pflug, Nivoche) studying the asymptotic behavior of extremal "copolynomials" in varied directions as approaching to the point a.

1 Introduction

A well-known classical result of the geometric function theory (Fekete [8], Szegö [25]) is the coincidence of three characteristics of a compact set K in $\mathbb{C}$, which are defined in quite different ways:

$$d(K) = \tau(K) = c(K),$$

where $d(K)$ is the *transfinite diameter* (a geometric characterization), $\tau(K)$ is the *Chebyshev constant* (an approximation theory approach), *and* $c(K)$ is the *capacity* (a potential theory point of view). Multidimensional analogs of these characteristics were studied intensively in last decades, beginning with Leja's definition of the multivariate transfinite diameter [13] and author's article [26], where a multidimensional analog of Fekete's equality $d(K) = \tau(K)$ has been obtained. In [30], Section 3, one can find a survey of results concerned with relations among various capacity characteristics of compact sets in $\mathbb{C}^n$.

Our main goal is the study of *internal Chebyshev constants and transfinite diameters* for a domain D and its boundary ∂D in $\mathbb{C}^n$ relative to a point $a \in D$ in the spirit of [26], applying the general approach, developed in Section 4 of [30]. Namely, our considerations are based on two systems: the system of monomials

$$e_{i,a}(z) := (z-a)^{k(i)}, \ i \in \mathbb{N}, \tag{1}$$

where $i \to k(i) = (k_1(i), \ldots, k_n(i))$ is a standard enumeration of the set $\mathbb{Z}_+^n$ (see Section 2 below) and its biorthogonal system of analytic functionals $\{e'_{i,a}\}_{i\in\mathbb{N}}$ defined via

$$e'_{i,a}(f) = \frac{f^{(k(i))}(a)}{k(i)!}, \quad i \in \mathbb{N}, \quad f \in A(\{a\}), \tag{2}$$

here $A(\{a\})$ is the space of analytic germs at the point a. We investigate the asymptotic behavior of the *least deviation* (in proper norms related to the domain D) *from zero* δ_i of either (i) *"monic polynomials" with respect to the system* (2) $e'_{i,a} + \sum_{j<i} c_j \ e'_{j,a}$ $(\tau(a, \partial D)$ (in order to "measure the size of ∂D viewed from a") or (ii) *"monic copolynomials"*, that is, functions whose Taylor expansion at the point a is of the form: $e_{i,a} + \sum_{j>i} c_j \ e_{j,a}$ $(\tau(a, D)$ (for "measuring the size of D relative to a").

By analogy with [30], for an arbitrary domain $D \subset \mathbb{C}^n$, we introduce in Section 4 the directional Chebyshev constant $\tau(a, D; \theta)$, which describes the asymptotic behavior of extremal monic copolynomials by the system (1) in the direction θ, study properties of the characteristics $\tau(a, D; \theta)$ as a function of θ, and define the principal Chebyshev constant $\tau(a, D)$ as the geometric mean of directional ones. In Section 5 we consider, dual in a sense, directional Chebyshev constants $\tau(a, \partial D; \theta)$ and the principal Chebyshev constant $\tau(a, \partial D)$ that describe the asymptotic behavior of extremal monic polynomials by (2) and "measure the size of ∂D viewed from a point $a \in D$". It is shown that, in the case of a strictly pluriregular domain D, these characteristics are reciprocal one to another and remain the same, when normed spaces, used in their definition, vary in a wide range. Applying Theorem on Hilbert scales of analytic functions (see, e.g., [28, 29]) we show in Section 6 that the asymptotics of leading coefficients of orthonormal bases, obtained by Gram-Schmidt process from the systems (1) and (2) in proper Hilbert spaces, are expressed through Chebyshev constants. The transfinite diameter $d\,(a, \partial D)$ of the boundary ∂D viewed from a point $a \in D$ is introduced in Section 7 by means of extremal Vandermondians for the sequence (2). The equality $d\,(a, \partial D) = \tau\,(a, \partial D) = \tau\,(a, D)^{-1}$ is proved, which can be considered as an internal multivariate analog of the Fekete equality.

In Section 3 we consider the one-dimensional case, that displays a direct connection of the above internal characteristics with the logarithmic capacity of an appropriate compact set and, if D is simply connected, with its conformal radius related to a point.

Section 8 deals with internal analytic capacities of a domain relative to a point and is closely related to Jarnicki-Pflug's and Nivoche's results ([9, 10, 11, 16, 19]. Applying the latter, we give an expression of the Robin function in terms of internal orthonormal bases (which can be considered as an internal analog of Zeriahi's result [31], Theorem 2) and consider analogs of Szegö's equality by introducing some natural Chebyshev constants, though they are different from the considered above. The problem on analogs of Szegö's equality, concerned with the Chebyshev constants studied in Sections 4-6 (similar to Rumely's result for compact sets in $\mathbb{C}^n$ [22]), remains open, see Section 9, where some other conclusions and generalizations are discussed.

2 Preliminaries and notation

Given an open set $D \subset \mathbb{C}^n$ we denote by $A\,(D)$ the space of all analytic functions in D with usual locally convex topology of locally uniform convergence in D. If $K \subset \mathbb{C}^n$ is a compact set then the space $A\,(K)$ is the locally convex space of all germs of analytic functions on K, endowed with the standard inductive topology.

Definition 1 *A Stein manifold Ω is called pluriregular (strongly pseudoconvex, $\mathbb{C}^n$-pluriregular, P-pluriregular, hyperconvex) if there exists a negative function $u \in \operatorname{Psh}(\Omega)$ such that $u\,(z_j) \to 0$ for every sequence $\{z_j\}$ without limit points in Ω (shortly, if $z \to \partial\Omega$). We say that a domain D in a Stein manifold Ω is strictly pluriregular if there is a pseudoconvex do-*

main $\Delta : D \Subset \Delta \subset \Omega$ *and a continuous function* $u \in \mathrm{Psh}\,(\Delta)$ *such that* $D = \{z \in \Delta : u\,(z) < 0\}$*. If* $\dim \Omega = 1$*, we say that* D *is strictly regular.*

Notice that "strict pluriregularity" is somewhat weaker than "strict hyperconvexity" considered in [17, 18].

Definition 2 *The (generalized) pluripotential Green function of a Stein manifold* D *with a logarithmic singularity at a point* $a \in D$ *is defined via:*

$$g_D\,(a, z) := \limsup_{\zeta \to z} \sup \{u\,(\zeta) : u \in \mathcal{G}\,(a, D)\}\,, \tag{3}$$

where $\mathcal{G}\,(a, D)$ *consists of all negative functions* $u \in \mathrm{Psh}\,(D)$ *such that* $u\,(z) - \ln |\varphi\,(z)|$ *is bounded from above near* a*, where the mapping* $\varphi \in A\,(D)^n$ *represents local coordinates at* a *so that* $\varphi\,(a) = 0$ *(this definition does not depend on the choice of local coordinates; if* $D \subset \mathbb{C}^n$ *we take* $\varphi\,(z) = z - a$*).*

The following assertion will be needed (see, e.g., [30], Lemma 2.1)

Lemma 3 *Suppose* X*,* Y *is a pair of locally convex spaces and* $J : X \to Y$ *is an injective continuous linear operator with the dense image. Then the adjoint operator* $J^* : Y^* \to X^*$ *is also injective and, if* X *is reflexive, the image* $J^*\,(Y^*)$ *is dense in* X^**.*

Remark 4 *In what follows, we always treat the operator* J *as an identical embedding, identifying* x *with* Jx *and using the notation* $X \hookrightarrow Y$ *for a linear continuous embedding. In particular, we write also* $Y^* \hookrightarrow X^*$ *in the conditions of Lemma 3.*

We use the notation $|f|_E := \sup \{|f\,(z)| : z \in E\}$ for a function $f : E \to \mathbb{C}$. Denote by $\mathbb{Z}_+^n$ the set of all integer-valued vectors $k = (k_1, \ldots, k_\nu, \ldots, k_n)$ with non-negative coordinates. Let $|k| := k_1 + \ldots + k_\nu + \ldots + k_n$ be the degree of the multiindex k. Introduce an enumeration $\{k\,(i)\}_{i \in \mathbb{N}}$ of the set $\mathbb{Z}_+^n$ via conditions: the sequence $s\,(i) := |k\,(i)|$ is non-decreasing and on each set $\mathcal{K}_s := \{|k\,(i)| = s\}$ the enumeration coincides with the lexicographic order relative to $k_1, \ldots, k_n$. Denote by $i\,(k)$ the number assigned to k under this ordering. Notice, that the number of multiindices of degree not larger than s is $m_s := \binom{s+n}{s}$ and the number of ones of degree s is $N_s := m_s - m_{s-1} = \binom{s+n-1}{s}$, $s \geq 1$; $N_0 = 1$. Set

$$l_s := \sum_{q=0}^{s} q N_q \tag{4}$$

for $s = 0, 1, \ldots$.

We consider the standard $(n-1)$-simplex

$$\Sigma := \left\{ \theta = (\theta_\nu) \in \mathbb{R}^n : \theta_\nu \geq 0,\ \nu = 1, \ldots, n;\ \sum_{\nu=1}^{n} \theta_\nu = 1 \right\} \qquad (5)$$

and its interior Σ° (in the relative topology on the hyperplane containing Σ). For $\theta \in \Sigma$ we denote by $\mathcal{L}_\theta$ the set of all infinite sequences $L \subset \mathbb{N}$ such that $\frac{k(i)}{s(i)} \xrightarrow{L} \theta$. We set $k! := k_1! \cdots k_\nu! \cdots k_n!$ for $k = (k_\nu) \in \mathbb{Z}_+^n$. We use also the notation $|z| := \left(\sum_{\nu=1}^n |z_\nu|^2 \right)^{1/2}$.

Given a pair of Hilbert spaces $H_1 \hookrightarrow H_0$ with dense embedding, we denote by

$$H^\alpha = (H_0)^{1-\alpha} (H_1)^\alpha,\ \alpha \in \mathbb{R}$$

the Hilbert scale generated by the pair H_0, H_1 (see, e.g., [12]).

It is denoted by $H^\infty(D)$ the space of all bounded functions $f \in A(D)$ with the uniform norm $\|f\|_{H^\infty(D)} := |f|_D$. If D is bounded we consider its subspace $AC\left(\overline{D}\right)$ that consists of functions extendible continuously onto $\overline{D}$ and the Bergman space $AL^2(D)$ of all analytic functions square-integrable by the Lebesgue measure on D. By $\mathbb{U}_r(a)$ we denote the equilateral polydisk of radius $r > 0$ centered at $a \in \mathbb{C}^n$.

3 One-dimensional case: internal capacity characteristics

The *conformal radius of a simply connected domain* $D \subset \overline{\mathbb{C}}$ *with respect to the point* a is the number $r(a, D) := \frac{1}{|\omega'(a)|}$,where $\omega : D \to \mathbb{U}$ is a biholomorphic mapping such that $\omega(a) = 0$; it is supposed here that $\omega'(\infty) := \frac{d\omega(1/\zeta)}{d\zeta}|_{\zeta=0}$ if $a = \infty$; the number $r(\infty, D)^{-1}$ is called also *a conformal radius of the compact set* $K := \overline{\mathbb{C}} \smallsetminus D$ (see, e.g., [21]).

The *capacity of* D *relative to a point* $a \in D$ is defined via $c(a, D) := \exp(-\rho(a, D))$,where $\rho(a, D) := \lim_{z \to a} (g_D(a, z) - \ln|z - a|)$ is the *Robin constant* of D *relative to* $a \in D$ and $g_D(a, z)$ is the generalized (subharmonic!) Green function of D with the normalized (negative) logarithmic singularity at a. If D is a simply connected domain in $\mathbb{C}$ and $a \in D$, then the conformal radius $r(a, D)$ coincides with the capacity $c(a, D)$.

The characteristic $c(a,D)$ was considered by many authors also under the name "*interior* (or *inner*) *radius of D relative to a*". A related capacity characteristic, named "*radius of ∂D viewed from a point $a \in D$*", was also in consideration: $c(a,\partial D) := \exp \rho(a,D) = \frac{1}{c(a,D)}$.

Using biholomorphic mapping, these capacities can be reduced to the logarithmic capacity of a related compact set. Namely, set $K_a := \left\{ \frac{1}{z-a} : z \in \overline{\mathbb{C}} \setminus D \right\}$ if $a \neq \infty$ and $K_\infty = \overline{\mathbb{C}} \setminus D$, then

$$c(a,D) = \frac{1}{c(K_a)}, \; c(a,\partial D) = c(K_a). \tag{6}$$

where $c(K)$ is the logarithmic capacity of a compact set K in $\mathbb{C}$, that coincides, by Fekete-Szegö result, with its transfinite diameter $d(K)$ and Chebyshev constant $\tau(K)$.

For a fixed $a \in \mathbb{C}$ we consider a system of functions: $e_{s,a}(z) := \frac{1}{(z-a)^s}$, $s \in \mathbb{N}$ if $a \neq \infty$, and $e_{s,\infty}(z) = z^s$, $s \in \mathbb{N}$, otherwise. Given a domain $D \neq \overline{\mathbb{C}}$ and $a \in D$ we introduce the *Chebyshev constant* of ∂D *viewed from the point* a via

$$\tau(a,\partial D) := \lim_{s\to\infty} \inf \left\{ \left(\left| e_{s,a} + \sum_{0 \le j < s} c_j \; e_{j,a} \right|_{\partial D} \right)^{1/s} : c_j \in \mathbb{C} \right\}. \tag{7}$$

and the *transfinite diameter of ∂D viewed from the point a* via

$$d(a,\partial D) := \lim_{s\to\infty} \left(\sup \left\{ \left| \det \left(e_{\mu,a}(\zeta_\nu)\right)_{\mu,\nu=0}^{s} \right| : (\zeta_\nu) \in (\mathbb{C} \setminus D)^s \right\} \right)^{2/s(s-1)}. \tag{8}$$

Changing variables $z = a + \frac{1}{w}$ we obtain that

$$\tau(a,\partial D) = d(a,\partial D) = \tau(K_a) = c(K_a) = c(a,\partial D). \tag{9}$$

The representations (7) and (8) give a motivation for the notions of multivariate internal Chebyshev constants and transfinite diameter of ∂D viewed from a point $a \in D$, which we consider in the next sections. Only, for $n \geq 2$, one has to deal (see Section 7) with appropriate analytic functionals instead of the functions $\frac{1}{(z-a)^k}$, $k \in \mathbb{Z}^n$, which are not defined on $D \setminus \{a\}$ as analytic functions. Since the evaluation at a point has no sense for analytic functionals, we need to apply, in the definition of the transfinite diameter, the general approach suggested in Section 4 of [30]. As an application, we obtain there an expression of the capacity $c(a,D)$ via extremal Wronskians at the point a (Section 7, Corollary 21).

4 Internal Chebyshev constants

Given a domain D in $\mathbb{C}^n$ and a point $a \in D$ we define a sequence

$$\delta_i = \delta_i(a, D) := \inf\{|f|_D : f \in \mathcal{N}_i\}, \tag{10}$$

where

$$\mathcal{N}_i = \mathcal{N}_i(a, D) : \left\{ f \in H^\infty(D) : e'_{j,a}(f) = 0,\ j < i;\ e'_{i,a}(f) = 1 \right\},$$

where the functionals $e'_{i,a}$ are defined in (2). Hereafter it is assumed that $\inf \varnothing = +\infty$ (it may happen, for instance, if $H^\infty(D)$ consists only of constants).

Definition 5 *The directional Chebyshev constant of D relative to a point $a \in D$ in a direction $\theta \in \Sigma$ is a constant*

$$\tau(a, D; \theta) := \limsup_{\frac{k(i)}{|k(i)|} \to \theta} (\delta_i)^{1/s(i)} := \sup_{L \in \mathcal{L}_\theta} \limsup_{i \in L} (\delta_i)^{1/s(i)} \tag{11}$$

with δ_i defined in (10).

Lemma 6 *The set $\Sigma(a, D) := \{\theta \in \Sigma : \tau(a, D; \theta) < \infty\}$ is convex and the function $\ln \tau(a, D; \theta)$ is convex on $\Sigma(a, D)$.*

Proof. Given $\theta, \theta' \in \Sigma(a, D)$ and $0 < \alpha < 1$, take natural-valued sequences i_q, j_q, $r_q < R_q$ so that $s(i_q) = s(j_q)$ and

$$\frac{k(i_q)}{s(i_q)} \to \theta,\ \frac{k(j_q)}{s(j_q)} \to \theta',\ \ \frac{r_q}{R_q} \to \alpha \text{ as } q \to \infty.$$

For arbitrary $\varepsilon > 0$ find functions $f_{\varepsilon,q} \in \mathcal{N}_{i_q}$ and $g_{\varepsilon,q} \in \mathcal{N}_{j_q}$ such that

$$|f_{\varepsilon,q}|_D < \delta_{i_q}(1+\varepsilon),\ |g_{\varepsilon,q}|_D < \delta_{j_q}(1+\varepsilon).$$

Then the function $F(z) = (f_{\varepsilon,q})^{r_q} (g_{\varepsilon,q})^{R_q - r_q}$ belongs to $\mathcal{N}_{l_q}$, where $l_q = i\left(k^{(q)}\right)$ is the number corresponding to the multiindex $k^{(q)} = r_q k(i_q) + (R_q - r_q) k(j_q)$ in the enumeration (see Preliminary). Therefore

$$\delta_{l_q} \le |F|_D \le \left(\delta_{i_q}(1+\varepsilon)\right)^{r_q} \left(\delta_{j_q}(1+\varepsilon)\right)^{R_q - r_q}.$$

Then we logarithmize and divide by $s(l_q) = R_q\ s(i_q)$. By the construction, $\frac{k^{(q)}}{s(l_q)} \to \alpha\theta + (1-\alpha)\theta'$, therefore, passing to the upper limit by q and taking into account that $\varepsilon > 0$ is arbitrary, we obtain that

$$\ln \tau(a, D; \alpha\theta + (1-\alpha)\theta') \le \alpha \ln \tau(a, D; \theta) + (1-\alpha) \ln \tau(a, D; \theta') < \infty.$$

Therefore $\alpha\theta + (1-\alpha)\theta \in \Sigma(a, D)$ for every $\alpha \in (0,1)$, hence $\Sigma(a, D)$ is convex and the function $\tau(a, D; \theta)$ is convex on this set. ■

Corollary 7 *The function* $\ln \tau(a, D; \theta)$ *is continuous on the set* $\Sigma(a, D)$.

Lemma 8 *Let r be the radius of an inscribed equilateral polydisc for D, centered at a, then $\tau(a, D; \theta) \ge r$ for all $\theta \in \Sigma$. If the domain D is bounded and R is the radius of a circumscribed equilateral polydisc for D, centered at a, then $\tau(a, D; \theta)$ is bounded uniformly by R from above.*

By Lemmas 6 and 8 the function $\tau(a, D; \theta)$ is measurable and bounded from below. Therefore the following definition makes sense.

Definition 9 *The principal Chebyshev constant of D relative to $a \in D$ is the number:*

$$\tau(a, D) := \exp\left(\int_{\Sigma} \ln \tau(a, D; \theta)\ d\sigma(\theta)\right), \tag{12}$$

where σ is the normalized Lebesgue measure on Σ.

In general, $\tau(a, D)$ may be equal to $+\infty$, but if the domain D is bounded then $\tau(a, D) \le R$, where R is defined in Lemma 8.

Lemma 10 *Let D be a bounded domain in $\mathbb{C}^n$. Then there exists a usual limit in (11) for every $\theta \in \Sigma^\circ$.*

Proof. Suppose that there exist two subsequences $\{i_q\}$ and $\{j_q\}$ such that

$$\lim_{q\to\infty} \frac{k(i_q)}{s(i_q)} = \lim_{q\to\infty} \frac{k(j_q)}{s(j_q)} = \theta \in \Sigma^\circ, \tag{13}$$

but

$$\lim \left(\delta_{i_q}\right)^{1/s(i_q)} =: \alpha < \beta := \lim \left(\delta_{j_q}\right)^{1/s(j_q)}.$$

Going, if necessary, to subsequences, we assume that

$$k_\nu(j_q) \geq k_\nu(i_q) > 0,\ \nu = 1, \ldots, n;\ \ \frac{s(j_q)}{s(i_q)} \nearrow \infty.$$

Setting

$$r(q) := \inf\left\{\left[\frac{k_\nu(j_q)}{k_\nu(i_q)}\right] : \nu = 1, \ldots, n\right\},\ l(q) := k(j_q) - r(q)\,k(i_q), \tag{14}$$

we have from (13), (14) that $l_q \in \mathbb{Z}_+^n$ and

$$r(q) \sim \frac{s(j_q)}{s(i_q)},\quad |l(q)| = o(s(j_q)) \text{ as } q \to \infty. \tag{15}$$

Given $\varepsilon > 0$ choose $f_{\varepsilon,q} \in \mathcal{N}_{i_q}$ so that

$$|f_{\varepsilon,q}|_D < \delta_{i_q} + \varepsilon.$$

Then the function $F(z) := (z-a)^{l(q)} \cdot (f_{\varepsilon,q}(z))^{r(q)}$ satisfies the inequality

$$|F|_D \leq C^{|l(q)|}\left(\delta_{i_q} + \varepsilon\right)^{r(q)}, \tag{16}$$

where $C = \max\left\{|z-a| : z \in \overline{D}\right\}$. On the other hand, by the feature of the enumeration $k(i)$, we have that $F \in \mathcal{N}_{j_q}$, hence

$$|F|_D \geq \delta_{j_q}. \tag{17}$$

Since $\varepsilon > 0$ is arbitrary, combining the relations (16), (17), (15), we obtain that $\beta \leq \alpha$, which contradicts to the assumption that $\alpha < \beta$. ■

5 Strictly pluriregular domains

If D is strictly pluriregular domain, then $H^\infty(D)$ is a normed space. We show here that, in Definition 5, one can change $H^\infty(D)$ for a Banach space from a wide range, so that the defined characteristic remains intact. This allows to introduce notions of Chebyshev constants of ∂D viewed from a point $a \in D$. On the other hand, this permits to apply Hilbert space methods.

Let Y be a Banach space complying with the dense embedding $Y \hookrightarrow A(\{a\})$, $a \in \mathbb{C}^n$. Then, by Lemma 3, $A(\{a\})^* \hookrightarrow Y^*$. Each germ $f \in Y$ is represented by its Taylor expansion at the point a:

$$f(z) = \sum_{i=1}^{\infty} e'_{i,a}(f)\ e_{i,a}(z),$$

that converges absolutely and uniformly on some neighborhood of a; the functionals $e'_{i,a} \in A(\{a\})^* \hookrightarrow Y^*$ are defined in (2). We introduce two directional Chebyshev constants characterizing approximative properties of this system of functionals with respect to the spaces Y or $X := Y^*$. The first one describes the asymptotic behavior of the least deviation of "monic polynomials by the system of analytic functionals (2)" in the space X :

$$\tau_Y^*(a,\theta) := \limsup_{\frac{k(i)}{|k(i)|}\to\theta} (\Delta_{i,X})^{1/s(i)} := \sup_{L\in\mathcal{L}_\theta} \limsup_{i\in L} (\Delta_{i,X})^{1/s(i)},\ \theta \in \Sigma, \tag{18}$$

where

$$\Delta_{i,X} = \Delta_{i,a,X} := \inf \left\{ \left\| e'_{i,a} + \sum_{j<i} c_j\ e'_{j,a} \right\|_X : (c_j) \in \mathbb{C}^{i-1} \right\},\ i \in \mathbb{N}, \tag{19}$$

and $\mathcal{L}_\theta$ is defined in Preliminaries. One can see here an analogy with the one-dimensional case (see (7)) regarding that the linear continuous functionals (2) can be expressed via

$$e'_{i,a}(f) = \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}_r(a)} \frac{f(\zeta)\ d\,\zeta}{(\zeta-a)^{k(i)+I}},\quad f \in A(\{a\}),\ i \in \mathbb{N}, \tag{20}$$

where $I = (1,\ldots,1)$, and $\mathbb{T}_r(a) := \{z = (z_\nu) \in \mathbb{C}^n : |z_\nu - a_\nu| = r\}$ with some sufficiently small $r = r(f) > 0$.

The characteristic (18) is dual, in a sense, to the second one, defined via:

$$\tau_Y(a,\theta) := \liminf_{\frac{k(i)}{|k(i)|}\to\theta} (\delta_{i,Y})^{1/s(i)} := \inf_{L\in\mathcal{L}_\theta} \liminf_{i\in L} (\delta_{i,Y})^{1/s(i)},\ \theta \in \Sigma, \tag{21}$$

where

$$\begin{aligned} \delta_{i,Y} &= \delta_{i,a,Y} := \inf\{\|f\|_Y : f \in \mathcal{N}_i\}, \\ \mathcal{N}_i &= \mathcal{N}_{i,a,Y} := \left\{f \in Y : e'_{j,a}(f) = 0,\ j < i;\ \ e'_{i,a}(f) = 1\right\} \end{aligned} \tag{22}$$

If the space Y is closely related with a given strictly pluriregular domain D, then the first characteristic describes the size of the boundary ∂D viewed from the point a, while the second coincides on Σ°, as it will be shown below, with the characteristic $\tau(a, D; \theta)$ introduced in the previous section. In the next definition we deal with the special space $Y = AC\left(\overline{D}\right)$, but it will be shown below that the space Y can be varied in a quite wide range leaving the defined characteristics unchanged.

Definition 11 *Let D be a strictly pluriregular domain in $\mathbb{C}^n$, $a \in D$, and $Y = AC\left(\overline{D}\right)$. Then the number $\tau(a, \partial D; \theta) := \tau_Y^*(a, \theta)$ is called a directional Chebyshev constant of ∂D viewed from the point a in the direction $\theta \in \Sigma$. The principal Chebyshev constant of ∂D viewed from the point a is defined by the formula*

$$\tau(a, \partial D) := \exp\left(\int_{\Sigma} \ln \tau(a, \partial D; \theta) \; d\sigma(\theta)\right). \tag{23}$$

That the integral (23) exists follows from the relation $\tau(a, \partial D, \theta) = \tau(a, D, \theta)^{-1}$, $\theta \in \Sigma^\circ$, which will be proved below (see, Theorem 12).

Given a domain $D \subset \mathbb{C}^n$ and $a \in D$, consider the sublevel sets of the pluripotential Green function ($\lambda < 0$):

$$D_\lambda := \{z \in D : g_D(a, z) < \lambda\}, \quad K_\lambda := \{z \in D : g_D(a, z) \leq \lambda\}. \tag{24}$$

Theorem 12 *Let $D \subset \mathbb{C}^n$ be a strictly pluriregular domain, $a \in D$, Y any Banach space such that the dense embeddings hold:*

$$A\left(\overline{D}\right) \hookrightarrow Y \hookrightarrow A(D) \tag{25}$$

and $X = Y^*$. Then for each $\theta \in \Sigma^\circ$ the usual limit exists in the relations (18), (21) and

$$\tau_Y(a, \theta) = \tau(a, D; \theta) = \tau_Y^*(a, \theta)^{-1} = \tau(a, \partial D; \theta)^{-1} \quad \text{if } \theta \in \Sigma^\circ.$$

Moreover

$$\lim_{s \to \infty} \left| \prod_{i=m_{s-1}+1}^{m_s} \delta_{i,Y} \right|^{1/sN_s} = \lim_{s \to \infty} \left| \prod_{i=m_{s-1}+1}^{m_s} \frac{1}{\Delta_{i,X}} \right|^{1/sN_s} = \tau(a, D). \tag{26}$$

Therewith

$$\tau(a, D_\lambda; \theta) = \tau(a, D; \theta) \exp \lambda, \;\; \tau(a, \partial D_\lambda; \theta) = \tau(a, \partial D; \theta) \exp(-\lambda). \tag{27}$$

This theorem will be proved in the next section after some preliminary considerations.

6 Asymptotics of leading coefficients of internal orthonormal bases

Lemma 13 *Let $a \in \mathbb{C}^n$ and H be any Hilbert space with dense embedding $H \hookrightarrow A(\{a\})$, $a \in \mathbb{C}^n$. Let $\varphi_i' = \sum_{j=1}^{i} a_{ji}\ e_j'$ be the orthonormal system in the dual space $H^* \hookleftarrow A(\{a\})^*$, obtained by the Gram-Schmidt procedure applied to the system $\{e_i'\}$, defined by (2); $\{\varphi_i\} \subset H$ be the biorthogonal system to the system $\{\varphi_i'\}$. Then*

$$\delta_{i,H} = \frac{1}{\Delta_{i,H^*}} = |a_{ii}|$$

and for each $\theta \in \Sigma$, we have

$$\tau_H^*(a;\theta) = \limsup_{\frac{k(i)}{|k(i)|}\to\theta} \frac{1}{|a_{ii}|^{1/s(i)}},\ \ \tau_H(a,\theta) = \liminf_{\frac{k(i)}{|k(i)|}\to\theta} |a_{ii}|^{1/s(i)}.$$

Let H_λ be the Hilbert space of all $x = \sum_{i=1}^{\infty} \xi_i\ \varphi_i \in A(\{a\})$ with

$$\|x\|_{H_\lambda} := \left(\sum_{i=1}^{\infty} |\xi_i|^2 \exp(2\lambda s(i))\right)^{1/2} < \infty, \quad \lambda \le 0. \tag{28}$$

Then

$$\tau_{H_\lambda}(a,\theta) = \tau_H(a;\theta)\cdot\exp\lambda, \quad \tau_{H_\lambda}^*(a,\theta) = \tau_H^*(a;\theta)\cdot\exp(-\lambda),\ \ \lambda<0. \tag{29}$$

Proof. Consider also the dual Hilbert scale

$$G_\lambda := \left\{ x' = \sum_{i=1}^{\infty} \xi_i'\ \varphi_i' \in G : \|x'\|_{G_\lambda} := \left(\sum_{i=1}^{\infty} |\xi_i'|^2 \exp(-2\lambda s(i))\right)^{1/2} < \infty \right\},$$

with $\lambda \le 0$, $G_0 = G = H^*$. The system $\{\varphi_i\}$ is an orthogonal basis in each space H_λ and has an expansion

$$\varphi_i(z) = \sum_{j\ge i} b_{j,i}\ e_j(z), \tag{30}$$

converging in some neighborhood of the point a, while the system $\{\varphi_i'\}$ is an orthogonal basis in any G_λ. Therewith

$$b_{i,i}=\frac{1}{a_{i,i}},\ \left\|\varphi_i\right\|_{H_\lambda}=\exp\lambda s\left(i\right),\quad \left\|\varphi_i'\right\|_{G_\lambda}=\exp\left(-\lambda s\left(i\right)\right),\ i\in\mathbb{N},\ \lambda\leq 0.$$

By the extremal property of orthogonal systems, we have

$$\delta_{i.H_\lambda}=\left|a_{ii}\right|\exp\lambda s\left(i\right),\quad \Delta_{i.H_\lambda}=\left|a_{ii}\right|^{-1}\exp\left(-\lambda s\left(i\right)\right). \tag{31}$$

Logarithmizing and passing to the lower (upper) limit along subsequences $L\in\mathcal{L}_\theta,\ \theta\in\Sigma$ the proof is done. ■

Proof of Theorem 12. Since $A\left(\overline{D}\right)$ is a nuclear locally convex space, then, by Pietsch [20], Section 4.4, there exists a Hilbert space complying with dense embeddings

$$A\left(\overline{D}\right)\hookrightarrow H\hookrightarrow Y,\ A\left(\overline{D}\right)\hookrightarrow H\hookrightarrow AC\left(\overline{D}\right)\hookrightarrow A\left(D\right). \tag{32}$$

It is known (see, e.g., [27, 31, 28, 29]) that, under these restrictions on H, the system $\{\varphi_i\}$ is a common basis in the spaces $A\left(D\right),\ A\left(\{a\}\right),\ A\left(D_\lambda\right),$ $A\left(K_\lambda\right),\ \lambda<0$, and the following embeddings take place:

$$A\left(K_\lambda\right)\hookrightarrow H_\lambda\hookrightarrow A\left(D_\lambda\right),\ \lambda<0, \tag{33}$$

where H_λ is the scale (28) and the sublevel sets K_λ, D_λ are defined in (24). Therefore $H\hookrightarrow Y\hookrightarrow A\left(D\right)\hookrightarrow H_\lambda$ for every $\lambda<0$. Due to (31) and (32), there are positive constants $C,\ \ c=c\left(\lambda\right)$ such that

$$c\ \delta_{i,H_\lambda}\ =c\ \delta_{i,H}\ \exp\lambda s\left(i\right)\leq\ \delta_{i,Y}\ \leq C\ \delta_{i,H},\ i\in\mathbb{N}. \tag{34}$$

On the other hand, since $H\hookrightarrow H^\infty\left(D\right)\hookrightarrow A\left(D\right)\hookrightarrow H_\lambda$, $\lambda<0$, we obtain, taking into account Lemma 10, that

$$\limsup_{i\in L}\left(c\ \delta_{i,H}\right)^{1/s(i)}\ \exp\lambda\leq\tau\left(a,D;\theta\right)\leq\liminf_{i\in L}\left(C\ \delta_{i,H}\right)^{1/s(i)} \tag{35}$$

for any $L\in\mathcal{L}_\theta,\ \theta\in\Sigma^\circ$ and $\lambda<0$. Hence, if $\theta\in\Sigma^\circ$, the usual limit exists in (21) with $Y=H$ and $\tau\left(a,D;\theta\right)=\tau_H\left(a,\theta\right)$. Applying now (34) with an arbitrary Y satisfying the conditions of the theorem, we conclude the same with $\tau_Y\left(a,\theta\right)$ instead of $\tau_H\left(a,\theta\right)$. Then, applying the embeddings, dual to (33), we obtain

$$G_\lambda\hookrightarrow A\left(D\right)^*\hookrightarrow Y^*\hookrightarrow G,\ \lambda<0.$$

In the same token, by Lemma 13, we conclude that

$$\tau_Y^*(a,\theta) = \tau(a,\partial D;\theta) = \frac{1}{\tau(a,D;\theta)}, \;\; \theta \in \Sigma^\circ, \tag{36}$$

and the usual limit exists in (18) for $\theta \in \Sigma^\circ$.

An examination of the proofs of Lemmas 5 and 6 in [26] shows that, since the function $\tau(a,D;\theta)$ is continuous on Σ° (see Corollary 7 above) and the usual limits exist in (18), (21), we can establish in the same way as in [26] , the following relations:

$$\begin{aligned}
\lim_{s\to\infty} \frac{1}{N_s} \sum_{i=m_{s-1}+1}^{m_s} \ln \tau_{i,Y} &= \int_\Sigma \ln \tau(a,D;\theta)\; d\sigma(\theta) = \ln \tau(a,D), \\
\lim_{s\to\infty} \frac{1}{N_s} \sum_{i=m_{s-1}+1}^{m_s} \ln \tau_{i,Y}^* &= \int_\Sigma \ln \tau(a,\partial D;\theta)\; d\sigma(\theta) = \ln \tau(a,\partial D)
\end{aligned} \tag{37}$$

where σ is the normalized Lebesgue measure on Σ. Thus (26) is proved.

Applying (33) once more, we obtain

$$H_{\lambda+\varepsilon} \hookrightarrow H^\infty\left(\overline{D}_\lambda\right) \hookrightarrow H_{\lambda-\varepsilon}, \quad \lambda < 0,\; 0 < \varepsilon < -\lambda.$$

Therefore there exist constants $C = C(\lambda,\varepsilon)$ and $c = c(\lambda,\varepsilon)$ such that

$$\begin{aligned}
c\delta_{i,H_{\lambda-\varepsilon}} &= c\delta_{i,H} \exp(\lambda-\varepsilon) \le \delta_i(a,D_\lambda) \\
&\le C\;\delta_{i,H_{\lambda+\varepsilon}} = C\;\delta_{i,H} \exp(\lambda+\varepsilon), \quad i \in \mathbb{N},
\end{aligned}$$

here $\delta_i(a,D_\lambda)$ is defined in (10) with D_λ instead D. Passing to the limit along any sequence $L \in \mathcal{L}_\theta$, $\theta \in \Sigma^\circ$ and taking into account (29), we obtain that

$$\tau(a,D;\theta)\exp(\lambda-\varepsilon) \le \tau(a,D_\lambda;\theta) \le \tau(a,D;\theta)\exp(\lambda+\varepsilon), \, \theta \in \Sigma^\circ$$

The first relation in (27) follows by tending $\varepsilon \to 0$. The remained statements of the theorem can be derived easily from the proved ones by applying Lemma 13. ■

Summarizing the above considerations we obtain the main result of this section.

Theorem 14 *Let $Y = H$ be a Hilbert space satisfying the conditions of Theorem 12 and*

$$\varphi_i' = \sum_{j=1}^{i} a_{j,i}\ e_j', \quad \varphi_i = \sum_{j \geq i} b_{j,i}\ e_j$$

be the orthonormal systems constructed for the spaces H^* and H as in Lemma 13. Then

$$\lim_{i \in L} |b_{i,i}|^{1/s(i)} = \frac{1}{\tau\left(a, D; \theta\right)},\ \ L \in \mathcal{L}_\theta,\ \ \theta \in \Sigma^\circ.$$

The geometric mean of leading coefficients $a_{i,i} = \frac{1}{b_{i,i}}$ of degree s satisfies the asymptotic relation, determined by the principal Chebyshev constants:

$$\lim_{s \to \infty} \left(\left(\prod_{|k(i)|=s} |a_{i,i}| \right)^{1/N_s} \right)^{1/s} = \tau\left(a, \partial D\right) = \frac{1}{\tau\left(a, D\right)}. \tag{38}$$

Indeed, by Lemma 13, $\delta_{i,H} = |a_{i,i}| = \frac{1}{|b_{i,i}|}$, so it suffices to apply (26).

Proposition 15 *Let D be a bounded complete logarithmically convex n-circular domain in $\mathbb{C}^n$ and*

$$h\left(\theta\right) = h_D\left(\theta\right) := \sup \left\{ \sum_{\nu=1}^{n} \theta_\nu \ln |z_\nu| : z = (z_\nu) \in D \right\},\ \ \theta = (\theta_\nu) \in \Sigma$$

its characteristic function. Then $\tau\left(0, D; \theta\right) = \tau\left(\overline{D}, \theta\right) = \exp h\left(\theta\right),\ \ \theta \in \Sigma$, and

$$\tau\left(0, D\right) = \tau\left(\overline{D}\right) = \exp \int_\Sigma h\left(\theta\right)\ d\sigma\left(\theta\right), \tag{39}$$

where σ is the normalized Lebesgue measure on Σ (here $\tau\left(\overline{D}, \theta\right)$ and $\tau\left(\overline{D}\right)$ are, respectively, the directional and principal Chebyshev constants of a compact set $K = \overline{D}$, see [26, 30]).

Proof. Take any Hilbert space H complying with the embeddings

$$A\left(\overline{D}\right) \hookrightarrow H \hookrightarrow A\left(D\right) \tag{40}$$

and such that the monomials $e_i = z^{k(i)}$ are pairwise orthogonal; for instance, one can take the Bergman space $AL^2\left(D\right)$ of all functions analytic and square integrable in D. Then the system $\dfrac{e_i}{\|e_i\|_H}$ is an orthonormal polynomial basis p_i with $a_{i,i} = \dfrac{1}{\|e_i\|_H}$ in the frame of Theorem 6.1 from [30]; on the other hand, it is an orthonormal basis φ_i with $b_{i,i} = \dfrac{1}{\|e_i\|_H}$ in the context of Theorem 14. Therefore, by Theorem 14 above and Theorem 6.1 from [30],

$$\tau\left(\overline{D},\theta\right) = \lim_{\frac{k(i)}{s(i)}\to\theta} \left(\|e_i\|_H\right)^{1/s(i)} = \tau\left(0,D;\theta\right),\ \ \theta\in\Sigma^{\circ}, \tag{41}$$

where $\tau\left(\overline{D},\theta\right)$ is the directional Chebyshev constant of the compact set $K = \overline{D}$ in the direction θ (see [30]).

By (40), given $\varepsilon > 0$ there exist positive constants $c = c\left(\varepsilon\right)$ and $C = C\left(\varepsilon\right)$ such that

$$\begin{aligned} c\exp\left(\left(1-\varepsilon\right)h_D\left(\theta\left(i\right)\right)s\left(i\right)\right) &= c\left|e_i\right|_{(1-\varepsilon)D} \le \|e_i\|_H \le C\left|e_i\right|_{(1+\varepsilon)D} \\ &= C\exp\left(\left(1+\varepsilon\right)h_D\left(\theta\left(i\right)\right)s\left(i\right)\right), \end{aligned}$$

where $\theta\left(i\right) = \frac{k(i)}{s(i)}$, $i\in\mathbb{N}$. Hence, since the function h_D is continuous and $\varepsilon > 0$ is arbitrary, we obtain that

$$\lim_{\frac{k(i)}{s(i)}\to\theta} \left(\|e_i\|_H\right)^{1/s(i)} = \exp h_D\left(\theta\right),\ \ \theta\in\Sigma.$$

Combining this with (41), we obtain

$$\tau\left(\overline{D},\theta\right) = \tau\left(0,D;\theta\right) = \exp h_D\left(\theta\right),\ \ \theta\in\Sigma^{\circ}$$

and then, by integration, (39). ■

Corollary 16 *Let Y be any Banach space complying with the dense embeddings*

$$A\left(\overline{\mathbb{U}_r\left(a\right)}\right) \hookrightarrow Y \hookrightarrow A\left(\mathbb{U}_r\left(a\right)\right)$$

and $X = Y^*$. Then $\lim_{i\to\infty}\left(\Delta_{i,X}\right)^{1/s(i)} = \frac{1}{r}$, where $\Delta_{i,X}$ is defined in (19).

Problem 17 *Characterize all domains $D \Subset \mathbb{C}^n$ with $0\in D$ such that $\tau\left(\overline{D}\right) = \tau\left(0,D\right)$.*

7 Internal transfinite diameters

Let D be a domain in $\mathbb{C}^n$, $a \in D$, and $e_i' = e_{i,a}' \in A(D)^*$, $i \in \mathbb{N}$, be the system of analytic functionals determined by (2). Since, contrary to the one-dimensional case, an evaluation at a point has no sense for analytic functionals, there is no direct analog of Leja's Vandermondians. The general considerations of Section 4 in [30] turned to be useful for an alternative equivalent definition of the transfinite diameter for compact sets (see, [30], Theorem 5.1). This approach provides a way out in the present situation as well.

Definition 18 *The transfinite diameter of the boundary ∂D viewed from the point a is the number*

$$d(a, \partial D) := \limsup_{i \to \infty} \left(\tilde{\mathcal{V}}_i \right)^{1/l_{s(i)}} \tag{42}$$

where $s(i) = |k(i)|$ (see Preliminaries), l_s is defined in (4), and

$$\tilde{\mathcal{V}}_i = \sup \left\{ \left| \det \left(e_{i,\alpha}' (f_\beta) \right)_{\alpha,\beta=1}^{i} \right| : f_\beta \in \mathbb{B}_{H^\infty(D)},\ \beta = 1, \ldots, i \right\}. \tag{43}$$

is the sequence of extremal Vandermondians. The internal transfinite diameter of D with respect to the point a is defined via

$$d(a; D) := \frac{1}{d(a; \partial D)}.$$

Let D be a strictly pluriregular domain in $\mathbb{C}^n$, Y be a Banach space complying with the dense embeddings (25), and $X := Y^*$. Then, by Lemma 3

$$e_{i,a}' \in A(\{a\})^* \hookrightarrow A(D)^* \hookrightarrow X,\ i \in \mathbb{N}.$$

Set

$$\begin{aligned} \tilde{\mathcal{V}}_i^Y &:= \sup \left\{ \left| \det \left(e_\mu' (f_\nu) \right)_{\mu,\nu=1}^{i} \right| : f_\nu \in \mathbb{B}_Y,\ \nu = 1, \ldots, i \right\}, \\ d^Y &:= \limsup_{i \to \infty} \left(\tilde{\mathcal{V}}_i^Y \right)^{1/l_{s(i)}}. \end{aligned} \tag{44}$$

Theorem 19 *Under the above assumptions the usual limit exists in (44), which does not depend on the choice of the space* Y, $d^Y = d\left(a;\partial D\right)$, *and the equality holds*

$$d\left(a,D\right)=\frac{1}{d\left(a;\partial D\right)}=\tau\left(a,D\right)=\exp\left(\int\limits_{\Sigma}\ln\tau\left(a,D;\theta\right)\ d\sigma\left(\theta\right)\right), \qquad (45)$$

where σ *is the normalized Lebesgue measure on* Σ.

Proof. By Lemma 4.2 in [30], we have the estimates

$$\Delta_{i,X}\le\frac{\tilde{\mathcal{V}}_i^Y}{\tilde{\mathcal{V}}_{i-1}^Y}\le i\Delta_{i,X},\quad i\in\mathbb{N},$$

where $\Delta_{i,X}$ is defined in (19). Therefore

$$\prod_{i=m_{s-1}+1}^{m_s}\Delta_{i,X}\le\frac{\tilde{\mathcal{V}}_{m_s}^Y}{\tilde{\mathcal{V}}_{m_{s-1}}^Y}\le\left(m_s\right)^{N_s}\prod_{i=m_{s-1}+1}^{m_s}\Delta_{i,X}\ .$$

Since $\frac{\ln m_s}{s}\to 0$, then, due to (26), the asymptotic formula is true

$$\ln\tilde{\mathcal{V}}_{m_s}^Y-\ln\tilde{\mathcal{V}}_{m_{s-1}}^Y\sim sN_s\ln\tau\left(a,\partial D\right),\ s\to\infty. \qquad (46)$$

By summation from 1 to s (see, e.g., [5]), we derive the asymptotic formula

$$\ln\tilde{\mathcal{V}}_{m_s}^Y\sim\ln\tilde{\mathcal{V}}_{m_s}^Y-\ln\tilde{\mathcal{V}}_{m_0}^Y\sim\sum_{q=1}^{s}qN_q\cdot\ln\tau\left(a,\partial D\right)=l_s\ \ln\tau\left(a,\partial D\right),\ s\to\infty.$$

Let $m_{s-1}<i\le m_s$, that is $s\left(i\right)=s$. Take positive numbers r, R so that $\mathbb{U}_r\left(a\right)\Subset D\Subset\mathbb{U}_R\left(a\right)$, $2R>1>r/2$. Then, due to Corollary 16, there is i_0 such that

$$\left(\frac{1}{2R}\right)^{s(i)}\le\Delta_{i,X}\le\left(\frac{2}{r}\right)^{s(i)},\ i\ge i_0.$$

Therefore

$$\begin{aligned}\tilde{\mathcal{V}}_i^Y\ \left(\frac{1}{2R}\right)^{sN_s}\quad &\le\quad \tilde{\mathcal{V}}_i^Y\ \prod_{j=i+1}^{m_s}\Delta_{j,X}\le\tilde{\mathcal{V}}_{m_s}^Y\le\tilde{\mathcal{V}}_i^Y\ \left(m_s\right)^{N_s}\prod_{j=i+1}^{m_s}\Delta_{j,X}\\ &\le\quad \tilde{\mathcal{V}}_i^Y\ \left(m_s\right)^{N_s}\ \left(\frac{2}{r}\right)^{sN_s}\end{aligned}$$

for $i \geq i_0$ and $s = s(i)$. Since $\frac{sN_s}{l_s} \to 0$ as $s \to \infty$, we have, by (46), that

$$\ln \tilde{\mathcal{V}}_i^Y \sim l_{s(i)} \ln \tau(a, \partial D), \text{ as } i \to \infty,$$

which implies that the usual limit exists in the definition (44), that does not depend on Y: $d^Y = \tau(a, \partial D)$. Since $Y_1 := AC\left(\overline{D}\right) \hookrightarrow H^\infty(D) \hookrightarrow AL^2(D) =: Y_2$ and both spaces $Y_1,\ Y_2$ satisfy the conditions of Theorem, we obtain that $d^Y = d(a; \partial D)$. Then, by Theorem 12 and (12), we have

$$d(a; \partial D) = \tau(a, \partial D) = \exp\left(\int_\Sigma \ln \tau(a, \partial D; \theta)\ d\sigma(\theta)\right),$$

so (45) is proved. ■

Notice that

$$l_s \sim \lambda_s := \frac{s^{n+1}}{(n-1)!\,(n+1)}, \text{ as } s \to \infty. \tag{47}$$

The following statement can be proved similarly to Theorem 5.2 in [30].

Theorem 20 *Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then the Chebyshev constant $\tau(a; \partial D)$ is expressed by the formula:*

$$\tau(a; \partial D) = d(a, \partial D) = \left(\exp \sum_{\nu=1}^{n+1} \frac{1}{\nu}\right) \cdot \lim_{i \to \infty} \frac{(\mathcal{W}_{i,a})^{1/\lambda_{s(i)}}}{s(i)}, \tag{48}$$

where $\lambda_{s(i)}$ is defined in (47),

$$\mathcal{W}_{i,a} = \sup\left\{\left|\mathcal{W}_a\left((f_\nu)_{\nu=1}^i\right)\right| : |f_\nu|_D \leq 1,\ \nu = 1, \ldots, i\right\},$$

and

$$\mathcal{W}_a\left((f_\nu)_{\nu=1}^i\right) = \det\left(f_\nu^{(k(\mu))}(a)\right)_{\mu,\nu=1}^i$$

is the multivariate Wronskian of the system $\{f_\nu\}_{\nu=1}^i$, evaluated at the point a.

In particular, we get the following

Corollary 21 *Let D be a strictly regular domain in $\mathbb{C}$, $a \in D$. Then*

$$c(a, D) = \frac{1}{c(a, \partial D)} = \exp\left(-\frac{3}{2}\right) \lim_{s\to\infty} \frac{s}{(\mathcal{W}_{s,a})^{2/s^2}},$$

where

$$\mathcal{W}_{s,a} = \max\left\{\left|\det\left(f_\alpha^{(\beta)}(a)\right)_{a,\beta=0}^{s-1}\right| : |f_\alpha|_D \leq 1,\ \alpha = 0, \ldots, s-1\right\}.$$

In particular, if D is simply connected and $\omega : D \to \mathbb{B}$ is an analytic bijection such that $\omega(a) = 0$, then

$$|\omega'(a)| = \exp\frac{3}{2} \lim_{s\to\infty} \frac{(\mathcal{W}_{s,a})^{2/s^2}}{s}.$$

8 Internal Robin function and capacities in $\mathbb{C}^n$

For $n \geq 2$, the function $g_D(a, z) - \ln|z-a|$, in general, has infinitely many partial limits as $z \to a$. So, contrary to the case $n = 1$, there are many ways to define capacities of D related to the point a. By analogy with ([26, 27, 30]), one can define a natural capacity:

$$C(a, D) := \exp\left(-\limsup_{z\to a}\left(g_D(a, z) - \ln|z-a|\right)\right). \tag{49}$$

Similarly to the compact set case (for a survey of related results see [30]), in order to get an analog of Szegö equality, one can modify the definition of Chebyshev constants, by normalizing the leading homogeneous polynomial parts (relative to the variable $\zeta = z - a$), instead of normalizing the leading coefficients. Namely, let

$$\mathcal{M}_s := \left\{f \in A(D) : e'_{i,a}(f) = 0,\ s(i) < s\right\} \tag{50}$$

Given $f \in \mathcal{M}_s$, let

$$\hat{f}_s(z) = \sum_{s(i)=s} e'_{i,a}(f)\ (z-a)^{k(i)} = \lim_{|w|\to\infty} w^s f\left(a + \frac{z-a}{w}\right). \tag{51}$$

be its homogeneous part of degree s (it may be, in particular, the identical zero). Consider a Chebyshev-type characteristic (cf., [26, 27, 24]): $T(a,D) := \liminf_{s\to\infty} T_s(a,D)$, where

$$T_s(a,D) := \left(\inf\left\{|f|_D : f \in \mathcal{M}_s,\ \left|\hat{f}_s\right|_{\mathbb{B}^n} \geq 1\right\}\right)^{1/s}.$$

Theorem 22 *Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then $T(a,D) = C(a,D)$.*

This theorem will be proved below after some preliminary considerations. Without restrictions on D it may not be true, that is seen from

Example 23 *Let $D = \mathbb{B}_R \setminus K \subset \mathbb{C}$, where K is the standard Cantor set on the real line, $R > 1$, and $a \in D$. Then, since the set K is regular, but negligible for bounded analytic functions, we have $T(a,D) = T(a,\mathbb{B}_R) = C(a,\mathbb{B}_R) \neq C(a,D)$.*

The following notion was introduced in [2] (cf., [14, 1]).

Definition 24 *The Robin function of a Stein manifold D related to a point $a \in D$ is defined via*

$$\rho_D(a,\zeta) := \limsup_{|\lambda|\to 0}\left(g_D(a, a+\lambda\zeta) - \ln|\lambda|\right),\ \zeta \in \mathbb{C}^n.$$

Let D be a bounded domain in $\mathbb{C}^n$. Then the Robin function $\rho(\zeta) = \rho_D(a,\zeta)$ is continuous, plurisubharmonic in $\mathbb{C}^n$, and logarithmically homogeneous, that is

$$\rho(t\zeta) = \rho(\zeta) + \ln|t|,\ \zeta \in \mathbb{C}^n,\ t \in \mathbb{C}.$$

Therefore the open set

$$\breve{D} = \breve{D}_a := \{\zeta \in \mathbb{C}^n : \rho_D(a,\zeta) < 0\}.$$

is a complete circular domain, that is $\lambda z \in \breve{D}$ if $z \in \breve{D}$ and $|\lambda| \leq 1$. It is clear that

$$g_{\breve{D}_a}(0,\zeta) = \rho_D(a,\zeta),\quad \rho_{\breve{D}_a}(0,\zeta) \equiv \rho_D(a,\zeta).$$

Adapting Siciak's considerations ([24], 2.7) to our situation, we introduce a domain in $\overline{\mathbb{C}}\times\mathbb{C}^n$ via

$$\Delta := \left\{(w,\zeta) \in \overline{\mathbb{C}}\times\mathbb{C}^n : w \in \overline{\mathbb{C}} \setminus \{0\}\,,\ a + \frac{\zeta}{w} \in D\right\}.$$

Denote by $\mathcal{H}(\Delta)$ the set of all functions $v \in \operatorname{Psh}(\Delta)$, which are logarithmically homogeneous, that is $v(tw, t\zeta) = v(w,\zeta) + \ln|t|\,,\ t \in \mathbb{C} \setminus \{0\}$ (we set $t \cdot \infty = \infty$) and such that $v(1,\zeta) \leq C + \ln|\zeta|$ near the point $\zeta = 0$. Given a function $u \in \mathcal{G}(a, D)$ (see, Definition 2) we define a function

$$U(w,\zeta) := \begin{cases} u\left(a + \frac{\zeta}{w}\right) + \ln|w| & \text{if} \quad w \in \mathbb{C} \setminus \{0\}\,,\ \zeta \in wD - a, \\ \limsup\limits_{|w|\to\infty} u\left(a + \frac{\zeta}{w}\right) + \ln|w| & \text{if} \quad w = \infty,\ \zeta \in \mathbb{C}^n. \end{cases} \tag{52}$$

Then the mapping $S : \mathcal{G}(a, D) \to \mathcal{H}(\Delta)$ defined by $u(z) \to U(w,\zeta)$ is a bijection, its inverse is defined by $u(z) = U(1, z - a)$ (cf., [24], 2.7). Define the logarithmically homogeneous Green function:

$$h_\Delta(w,\zeta) := \limsup_{(\omega,\xi)\to(w,\zeta)} \sup\left\{v(\omega,\xi) : v \in \mathcal{H}(\Delta)\,;\ v|_{1\times(D-a)} \leq 0 \ \right\}.$$

This function is logarithmically homogeneous, plurisubharmonic in Δ. It is clear that

$$h_\Delta = S(g_D)\,;\quad g_D(a,z) = h_\Delta(1, z-a)\,,\ z \in D. \tag{53}$$

On the other hand, by the definition of ρ_D, we have

$$\rho_D(a,\zeta) = \limsup_{|w|\to\infty} \left(g_D\left(a, a + \frac{\zeta}{w}\right) + \ln|w|\right) \tag{54}$$

So, taking into account (53) and (52), we obtain the following

Proposition 25 *The Robin function is expressed via*

$$\rho_D(a,\zeta) = \limsup_{|w|\to\infty} h_\Delta(w,\zeta) = h_\Delta(\infty,\zeta)\,,\ \zeta \in \mathbb{C}^n.$$

Definition 26 *(cf., Jarnicki-Pflug [9, 10, 11], Nivoche [16, 17, 18])The ζ-directional analytic capacity of order s for a domain $D \subset \mathbb{C}^n$ relative to a* *point a is the number:*

$$\begin{aligned}\gamma_s(a,D;\zeta) \quad : \quad &= \left(\sup\left\{\frac{1}{s!}\left|d_\zeta^{(s)} f(a)\right| : f \in \mathcal{M}_s,\ |f|_D \le 1\right\}\right)^{-1/s} \qquad (55)\\ &= \left(\sup\left\{\left|\hat{f}_s(a+\zeta)\right| : f \in \mathcal{M}_s,\ |f|_D \le 1\right\}\right)^{-1/s},\ \zeta \in \mathbb{C}^n,\end{aligned}$$

$d_\zeta^{(s)}$ stays here for the derivative of order s in the direction of the tangent vector $\zeta \in \mathbb{C}^n$ at the point a and $\hat{f}$ is defined in (51). One can consider the reciprocal number:

$$\gamma_s(a,\partial D;\zeta) := (\gamma_s(a,D;\zeta))^{-1}$$

as an analytic capacity of order s of ∂D viewed from the point a in the direction ζ.

For every $\zeta \in \mathbb{C}^n$ the limit exists (see, e.g., [18]):

$$\gamma_\infty(a,D;\zeta) := \lim_{s\to\infty} \gamma_s(a,D;\zeta) = \inf\{\gamma_s(a,D;\zeta) : s \in \mathbb{N}\}. \qquad (56)$$

This characteristic can be considered as an *analytic capacity of infinite order* (*transfinite analytic capacity*) of D relative to the point a in a direction ζ.

Proposition 27 *(Nivoche [18])Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then*

$$-\ln\gamma_\infty(a,D;\zeta) \le \rho_D(a,\zeta),\ \zeta \in \mathbb{S}. \qquad (57)$$

The equality in (57) takes place quasi-everywhere on $\mathbb{S}$ (i.e., except a set $A \subset \mathbb{S}$ with $[A] = \left\{[z] \in \mathbb{CP}^{n-1} : z \in A\right\}$ polar in $\mathbb{CP}^{n-1}$).

We introduce related directional Chebyshev constants:

$$\begin{aligned}T_s(a,D;\zeta) \quad : \quad &= \inf\left\{|f|_D : f \in \mathcal{M}_s, \left|\ \hat{f}_s(a+\zeta)\right| \ge 1\right\}^{1/s},\ s \in \mathbb{N},\\ T(a,D;\zeta) \quad : \quad &= \liminf_{s\to\infty} T_s(a,D;\zeta) \qquad (58)\end{aligned}$$

with $\zeta \in \mathbb{S}$. It is easily seen that they coincide with their capacity counterparts (55) and (56): $T_s(a,D;\zeta) = \gamma_s(a,D;\zeta)$, $T(a,D;\zeta) = \gamma_\infty(a,D;\zeta)$, $\zeta \in \mathbb{S}$.

Proof of Theorem 22. It is obvious that

$$T_s(a,D)^{-1} = \left(\sup\left\{\left|\hat{f}_s\right|_{\mathbb{B}^n} : f \in \mathcal{M}_s,\ |f|_D \le 1\right\}\right)^{1/s}$$

So, taking into account (55), we have

$$T_s(a,D)^{-1} = \sup_{\zeta\in\mathbb{S}}\left\{\frac{1}{\gamma_s(D,a;\zeta)} : \zeta \in \mathbb{S}\right\}$$

It follows from the definition (49) that $C(a,D) = \exp(-\lambda(a,D))$ with

$$\lambda(a,D) := \max\{\rho_D(a,\zeta) : \zeta \in \mathbb{S}\}. \tag{59}$$

Then, by (57), we obtain that

$$-\ln T(a,D) = \limsup_{s\to\infty}(-\ln T_s(a,D)) \le \lambda(a,D).$$

In order to prove the converse estimate, suppose the contrary $-\ln T(a,D) < r < \lambda(a,D)$. Then, by (56), $-\ln\gamma_\infty(D,\zeta) \le r$, $\zeta \in \mathbb{S}$, hence, since the equality in (57) holds quasi-everywhere on $\mathbb{S}$, we would obtain that $\rho_D(a,\zeta) = \limsup_{\xi\to\zeta}(-\ln\gamma_\infty(D,\xi)) \le r < \lambda(a,D)$ for every $\zeta \in \mathbb{S}$. This contradicts to (59) and hence yields $C(a,D) = T(a,D)$. ■

Let D be a strictly pluriregular domain in $\mathbb{C}^n$ and Δ and u be as in Definition 1. Then there is $\varepsilon_0 > 0$ such that for every $\varepsilon : 0 < \varepsilon < \varepsilon_0$ the connected component Δ_ε of the set $\{z \in \Delta : u(z) < \varepsilon\}$ containing $\overline{D}$ is relatively compact in Δ. The following stability properties can be found, e.g., in [2, 7, 18].

Lemma 28 *Let D be a strictly pluriregular domain in $\mathbb{C}^n$. Then*

$$\begin{aligned} g_D(a,z) &= \lim_{\lambda\nearrow 0} g_{D_\lambda}(a,z), \quad g_D(a,z) = \lim_{\varepsilon\searrow 0} g_{\Delta_\varepsilon}(a,z),\ z \in D \setminus \{a\}, \\ \rho_D(a,\zeta) &= \lim_{\lambda\nearrow 0} \rho_{D_\lambda}(a,\zeta),\ \rho_D(a,\zeta) = \lim_{\varepsilon\searrow 0} \rho_{\Delta_\varepsilon}(a,\zeta),\ \zeta \in \mathbb{C}^n \setminus \{0\}, \end{aligned}$$

where D_λ are the sublevel domains defined in (24) and Δ_ε are defined above. Therewith the convergence is locally uniform in the all relations.

The following statement shows how the Robin function can be expressed in terms of orthonormal bases (cf., [31], Theorem 2).

Theorem 29 *Let D be a strictly pluriregular domain in $\mathbb{C}^n$, H any Hilbert space complying with dense embeddings $A\left(\overline{D}\right) \hookrightarrow H \hookrightarrow A(D)$, $\{\varphi_i\}$ the orthonormal basis from Lemma 13, and $g_i(a+\zeta) := \sum_{s(j)=s(i)} e'_{j,a}(\varphi_i)\,\zeta^{k(j)}$ be the homogeneous part of φ_i of degree $s(i)$, $i \in \mathbb{N}$. Then*

$$\rho_D(a,\zeta) = \limsup_{\xi\to\zeta} \limsup_{i\to\infty} \frac{\ln|g_i(a+\zeta)|}{s(i)}, \quad \zeta \in \mathbb{C}^n. \tag{60}$$

Proof. Take $\lambda < 0$. Then there exists a positive constant $c = c(\lambda)$ such that

$$c\,|f|_{D_\lambda} \le \|f\|_H, \quad f \in H, \tag{61}$$

where D_λ is defined in (24). Set

$$V(\xi) := \limsup_{i\to\infty} \frac{\ln|g_i(a+\xi)|}{s(i)} = \limsup_{s\to\infty} V_s(\xi),$$

where $V_s(\xi) = \sup\left\{\dfrac{\ln|g_i(a+\xi)|}{s} : s(i) = s\right\}$. Since, by (61),

$$\{c\varphi_i : s(i) = s\} \subset \mathcal{M}_s \cap \mathbb{B}_{H^\infty(D_\lambda)},$$

then

$$V_s(\xi) + \frac{\ln c}{s} \le -\ln\gamma_s(a, D_\lambda; \xi), \ \xi \in \mathbb{C}^n$$

and hence, by Proposition 27,

$$\rho_{D_\lambda}(a,\zeta) = \limsup_{\xi\to\zeta}(-\ln\gamma(a, D_\lambda; \xi)) \ge \limsup_{\xi\to\zeta} V(\xi), \ \zeta \in \mathbb{C}^n. \tag{62}$$

Let Δ, u be as in Definition 1. Then there exists ε_0 such that for every $\varepsilon : 0 < \varepsilon < \varepsilon_0$ the connected component Δ_ε of the set $\{z \in \Delta : u(z) < \varepsilon\}$ containing $\overline{D}$ is relatively compact in Δ. Consider $f \in \mathcal{M}_s \cap \mathbb{B}_{H^\infty\left(\overline{\Delta_\varepsilon}\right)}$. Then there exists $C = C(\varepsilon)$ such that $\|f\|_H \le C\,|f|_{\Delta_\varepsilon}$. Since

$$\hat{f}_s(a+\xi) = \sum_{s(i)=s} c_i\ g_i(a+\xi),$$

where $c_i = (f, \varphi_i)_H$ we have $|c_i| \le \|f\|_H \le C$. Therefore

$$\left|\hat{f}_s(a+\xi)\right| \le \sum_{s(i)=s} |c_i|\,|g_i(a+\xi)| \le CN_s \sup_{s(i)=s} |\ g_i(a+\xi)|.$$

Hence,

$$-\ln\gamma_s\left(a,\Delta_\varepsilon;\xi\right)\le V_s\left(\xi\right)+\frac{\ln CN_s}{s},\ \xi\in\mathbb{C}^n$$

and, after passing to the limit, by Proposition 27, we obtain

$$-\ln\gamma\left(a,\Delta_\varepsilon;\xi\right)\le V\left(\xi\right),\quad \xi\in\mathbb{C}^n$$

Thus

$$\rho_{\Delta_\varepsilon}\left(a,\zeta\right)=\limsup_{\xi\to\zeta}\left(-\ln\gamma\left(a,\Delta_\varepsilon;\xi\right)\right)\le\limsup_{\xi\to\zeta}V\left(\xi\right),\ \zeta\in\mathbb{C}^n. \tag{63}$$

Combining (62) and (63) and applying Lemma 28, we finalize the proof. ■

Introduce an average capacity:

$$\mathfrak{C}\left(a,D\right):=\exp\left(-\int_{\mathbb{S}}\ln\rho_D\left(a,\zeta\right)\ d\omega\left(\zeta\right)\right).$$

Corollary 30 *In the conditions of Theorem 29 we have*

$$\lim_{s\to\infty}\left(\exp\left(\frac{1}{s}\sup_{s(i)=s}\left\{\int_{\mathbb{S}}\ln\left|\hat{f}_i\left(a+\zeta\right)\right|\ d\omega\left(\zeta\right)\right\}\right)\right)=\frac{1}{\mathfrak{C}\left(a,D\right)}.$$

9 Conclusion

10.1 Theorem 29 and Proposition 25 could be useful in order to confirm the following conjecture (cf., [4], Theorem 2).

Conjecture 31 *It is likely that, for strictly pluriregular domains,* $\tau\left(0,\breve{D}_a;\theta\right)=\tau\left(a,D;\theta\right),\ \theta\in\Sigma$, *so that the directional Chebyshev constants and, hence, the transfinite diameter* $d\left(a,D\right)$ *would be determined uniquely by the Robin function* $\rho\left(a,D;\zeta\right)$*.*

The estimate $\tau\left(0,\breve{D}_a;\theta\right)\le\tau\left(a,D;\theta\right),\ \theta\in\Sigma$, can be easily proved similarly to [4]. In order to get the converse estimate $\tau\left(a,D;\theta\right)\le\tau\left(0,\breve{D}_a;\theta\right),\ \theta\in\Sigma$, *one needs to prove an internal analogue of Bloom's Theorem* ([3], Theorem 3.2).

10.2 Rumely [22] (see also [6]) discovered a formula expressing transfinite diameter of a compact set K via its Robin function. It remains open the following

Problem 32 *Let D be a strictly pluriregular domain in $\mathbb{C}^n$, $a \in D$. Write an analogue of Rumely formula for the expression of the internal transfinite diameter $d(a, \partial D)$ via the Robin function $\rho_D(a, z)$.*

10.3 Analytic characteristics (5), (9) and (43), (42) can be extended to Stein manifolds (they will depend on a choice of local coordinates at a!). Let D be a Stein manifold, $a \in D$ and the analytic mapping $\varphi = (\varphi_\nu) : D \to \mathbb{C}^n$ forms local coordinates at a with $\varphi(a) = 0$. For example, the directional Chebyshev constant $\tau_\varphi(a, D; \theta)$ can be defined as in Definition 5 only with the functionals (2) expressed in terms of the chosen local coordinates at a:

$$e'_i(f) = e'_{i,a}(\varphi; f) := \frac{1}{k(i)!} \frac{\partial^{|k(i)|} f(\varphi^{-1}(\zeta))}{\partial \zeta^{k(i)}} |_{\zeta=0} . \tag{64}$$

For concrete Stein manifolds one can use some preferable local coordinates at the point a. If, for example, D is an unbranched Riemann domain over $\mathbb{C}^n$, $\pi : D \to \mathbb{C}^n$ a projection, and $a \in D$, then it is natural to define Chebyshev constants applying the local coordinates $\varphi(z) = \pi(z) - \pi(a)$.

10.4 There is a different way to define the transfinite diameter and Chebyshev constants for an arbitrary Stein manifold D and given local coordinates φ at $a \in D$. Consider a continuous plurisubharmonic function u in D such that $\{u(z) < s\}$ is relatively compact in D for every $s \in \mathbb{N}$ and $u(a) < 1$. Let G_s be a connected component of $\{u(z) < s\}$ which contains a.

Definition 33 *The directional Chebyshev constants of the domain D relative to the point a and local coordinates φ are defined via*

$$\tilde{\tau}_\varphi(a, D; \theta) := \lim_{s \to +\infty} \tau_\varphi(a, G_s; \theta) = \sup_{s \in \mathbb{N}} \tau_\varphi(a, G_s; \theta), \tag{65}$$

and the corresponding principal Chebyshev constant and transfinite diameter are defined via

$$\tilde{\tau}_\varphi(a, D) := \lim_{s \to +\infty} \tau_\varphi(a, G_s), \ \tilde{d}_\varphi(a, D) := \lim_{s \to +\infty} d_\varphi(a, G_s). \tag{66}$$

If $D \subset \mathbb{C}^n$ and $\varphi(z) = z - a$, we use the notation $\tilde{\tau}(a, D; \theta)$, $\tilde{\tau}(a, D)$, $\tilde{d}(a, D)$, respectively.

For strictly pluriregular domains on Stein manifolds, these new characteristics coincide with $\tau_\varphi(a, D; \theta)$, $\tau_\varphi(a, D)$, $d_\varphi(a, D)$, respectively, but, in general, they do not so (see, e.g., Example 23). It is easily seen that the equality (45) holds with $\tilde{\tau}_\varphi$, $\tilde{d}_\varphi$ instead of τ, d.

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