Hypercyclicity of Weighted Backward Shifts on Spaces of Real Analytic Functions

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27.11.2015
Outline

- Introduction
- Multipliers on the spaces of real analytic functions
- Density arguments
- Conditions on hypercyclicity of the weighted backward shifts
- Hypercyclicity of the usual backward shift
An operator $T$ on a topological vector space $X$ is called \textit{hypercyclic} if there is some $x \in X$ such that the set
\[
\{x, Tx, T^2x, \cdots, T^n x, \cdots\}
\]
is dense in $X$. 
An operator $T$ on a topological vector space $X$ is called hypercyclic if there is some $x \in X$ such that the set

$$\{x, Tx, T^2x, \cdots, T^n x, \cdots\}$$

is dense in $X$.

The set $\{x, Tx, T^2x, \cdots\}$ is called the orbit of $x$ under $T$. 
(Birkhoff transitivity theorem) An operator $T$ on a separable Fréchet space $X$ is hypercyclic, if and only if, it is topologically transitive, that is, for any pair of nonempty open subsets $U$, $V$ of $X$, there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$
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$$T^n(U) \cap V \neq \emptyset.$$ 

Remark: Any hypercyclic operator on a general topological vector space is topologically transitive, however the converse may not be true.

(Bonet 2000) A topologically transitive linear operator on an arbitrary locally convex space need not be hypercyclic.
Examples of hypercyclic operators:
(Birkhoff 1929) The *translation operators*

\[ T_a f(z) = f(z + a), \ a \neq 0 \]

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Examples of hypercyclic operators:

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(MacLane 1952) The *differentiation operator* \( D : f \mapsto f' \) on \( H(\mathbb{C}) \),

(Rolewicz 1962) The multiples of the *backward shift* 

\[ \lambda B(x_n)_{n \in \mathbb{N}} = (\lambda x_{n+1})_{n \in \mathbb{N}} \]

for any \( \lambda \) with \( |\lambda| > 1 \), on the sequence spaces \( l_p, 1 \leq p < \infty \), or \( c_0 \).
Let $X$ be a Fréchet sequence space with canonical unit sequences $e_n$. Then, for a sequence of nonzero scalars $\omega = (\omega_n)_{n \in \mathbb{N}}$, the operator $B_\omega : X \to X$ defined by

$$B_\omega e_n = \omega_n e_{n-1}, \quad n \geq 1, \quad e_0 = 0,$$

is called a weighted backward shift on $X$. Theorem (Grosse-Erdmann 2000) The operator $B_\omega : X \to X$, acting on a Fréchet sequence space $X$ in which $(e_n)_{n \in \mathbb{N}}$ is a basis, is hypercyclic, if and only if, there is an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that $(n_k \prod_{\nu=1}^{k} \omega_\nu)^{-1} e_{n_k} \to 0$ in $X$ as $k \to \infty$. 


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**Theorem (Grosse-Erdmann 2000)**

The operator $B_\omega : X \rightarrow X$, acting on a Fréchet sequence space $X$ in which $(e_n)_{n \in \mathbb{N}}$ is a basis, is hypercyclic, if and only if, there is an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$\left( \prod_{\nu=1}^{n_k} \omega_{\nu} \right)^{-1} e_{n_k} \rightarrow 0$$

in $X$ as $k \rightarrow \infty$. 
Let $A(\Omega)$ denote the space of all complex-valued real analytic functions on an open set $\Omega$ in $\mathbb{R}$, that is, every function in $A(\Omega)$ develops into a Taylor series at each point of $\Omega$. 

Topology on $A(\Omega)$

- Projective limit topology $A(\Omega) = \text{proj}_{N \in \mathbb{N}} H(\mathcal{K}_N) = \text{proj}_{N \in \mathbb{N}} \text{ind}_{n \in \mathbb{N}} H(\infty)(U_N, n)$, where $(\mathcal{K}_N)_{N \in \mathbb{N}}$ is a compact increasing exhaustion of $\Omega$, and $(U_N, n)_{n \in \mathbb{N}}$ are fundamental sequences of neighborhoods of $\mathcal{K}_N$ for each $N$.

- Inductive limit topology $A(\Omega) = \text{ind}_H(U)$, where the inductive limit is taken over all complex neighborhoods of $\Omega$. 

Introduction
The Spaces of Real Analytic Functions

Let \( A(\Omega) \) denote the space of all complex-valued real analytic functions on an open set \( \Omega \) in \( \mathbb{R} \), that is, every function in \( A(\Omega) \) develops into a Taylor series at each point of \( \Omega \).

### Topology on \( A(\Omega) \)

- **Projective limit topology**

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  A(\Omega) = \text{proj}_{N \in \mathbb{N}} H(K_N) = \text{proj}_{N \in \mathbb{N}} \text{ind}_{n \in \mathbb{N}} H^\infty(U_{N,n}),
  \]

  where \( (K_N)_{N \in \mathbb{N}} \) is a compact increasing exhaustion of \( \Omega \), and \( (U_{N,n})_{n \in \mathbb{N}} \) are fundamental sequences of neighborhoods of \( K_N \) for each \( N \).
Let $A(\Omega)$ denote the space of all complex-valued real analytic functions on an open set $\Omega$ in $\mathbb{R}$, that is, every function in $A(\Omega)$ develops into a Taylor series at each point of $\Omega$.

### Topology on $A(\Omega)$

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  where $(K_N)_{N \in \mathbb{N}}$ is a compact increasing exhaustion of $\Omega$, and $(U_N, n)_{n \in \mathbb{N}}$ are fundamental sequences of neighborhoods of $K_N$ for each $N$.

- **Inductive limit topology**
  
  $$A(\Omega) = \text{ind} H(U),$$

  where the inductive limit is taken over all complex neighborhoods of $\Omega$. 
(Martineau 1966) These topologies are equivalent.

\( A(\Omega) \) is a complete, separable, ultrabornological, nuclear, reflexive space.

The closed graph theorem and the open mapping theorem hold in \( A(\Omega) \).

(Domański, Vogt 2000) \( A(\Omega) \) has no Schauder basis.
Definition

Given a sequence of nonzero scalars $\omega = (\omega_n)_{n \in \mathbb{N}}$, a linear continuous operator

$$B_\omega : A(\Omega) \rightarrow A(\Omega),$$

that sends

- the monomials $x^n$ to $\omega_n x^{n-1}$ for all $n \in \mathbb{N}$,
- the unit function to the zero function,

is called a **weighted backward shift** with the weight sequence $\omega$. 
A linear continuous operator

\[ M : A(\Omega) \to A(\Omega) \]

is called a \textit{multiplier} whenever every monomial is an eigenvector. The corresponding sequence of eigenvalues \((m_n)_{n \in \mathbb{N}}\) is called the \textit{multiplier sequence} for \(M\).
Multipliers on $A(\Omega)$

A linear continuous operator

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is called a *multiplier* whenever every monomial is an eigenvector. The corresponding sequence of eigenvalues $(m_n)_{n \in \mathbb{N}}$ is called the *multiplier sequence* for $M$.

**Theorem (Domański, Langenbruch 2012)**

Any multiplier sequence $(m_n)_{n \in \mathbb{N}}$ is a sequence of Laurent coefficients of some function $g$ which is holomorphic at infinity, that is,

$$g(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}.$$
Proposition

There is a one-to-one correspondence between the weighted backward shifts and the multipliers on $A(\Omega)$. 
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Proof If $B_\omega : A(\Omega) \to A(\Omega)$ is a weighted backward shift with the weight sequence $\omega = (\omega_n)_{n \in \mathbb{N}}$, then the map $M : A(\Omega) \to A(\Omega)$ defined by

$$M(f)(x) = B_\omega(xf(x)), \quad f \in A(\Omega), \quad x \in \Omega,$$

is a multiplier with the multiplier sequence $\omega$. 
Multipliers on $A(\Omega)$

**Proposition**
There is a one-to-one correspondence between the weighted backward shifts and the multipliers on $A(\Omega)$.

**Proof** If $B_\omega : A(\Omega) \to A(\Omega)$ is a weighted backward shift with the weight sequence $\omega = (\omega_n)_{n \in \mathbb{N}}$, then the map $M : A(\Omega) \to A(\Omega)$ defined by

$$M(f)(x) = B_\omega(xf(x)), \quad f \in A(\Omega), \quad x \in \Omega,$$

is a multiplier with the multiplier sequence $\omega$.

Similarly, if $M : A(\Omega) \to A(\Omega)$ is a multiplier, then the map $T : A(\Omega) \to A(\Omega)$ defined by

$$T(f)(x) = M \left( \frac{f(x) - f(0)}{x} \right), \quad f \in A(\Omega), \quad x \in \Omega,$$

is a weighted backward shift.
Let $H(\mathbb{C})$ denote the space of entire functions, and $H(\{0\})$ denote the space of germs of holomorphic functions at zero.
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**Lemma**

If $B_\omega$ is a weighted backward shift on $A(\Omega)$, then $B_\omega$ is also a weighted backward shift on $H(\{0\})$ and $H(\mathbb{C})$.
Density Arguments

Let $H(\mathbb{C})$ denote the space of entire functions, and $H(\{0\})$ denote the space of germs of holomorphic functions at zero.

**Lemma**

If $B_\omega$ is a weighted backward shift on $A(\Omega)$, then $B_\omega$ is also a weighted backward shift on $H(\{0\})$ and $H(\mathbb{C})$.

**Proof** Let $\omega = (\omega_n)$ be a weight sequence. Then, $\omega$ is also a multiplier sequence, and it can be represented as a sequence of Laurent coefficients of some function which is holomorphic at infinity. Hence, $\exists r > 0$ such that

$$\sup_n |\omega_n| r^n < \infty.$$  

We can then show that the maps $B_\omega : H(\{0\}) \to H(\{0\})$ and $B_\omega : H(\mathbb{C}) \to H(\mathbb{C})$ are well-defined linear continuous maps.
As $H(\mathbb{C})$ is dense in $A(\Omega)$, and $A(\Omega)$ is dense in $H(\{0\})$ whenever $0 \in \Omega$, we have the following observation.

Lemma

- If $B_\omega$ is hypercyclic on $H(\mathbb{C})$, then it is also hypercyclic on $A(\Omega)$.
- If $B_\omega$ is hypercyclic on $A(\Omega)$, then it is also hypercyclic on $H(\{0\})$. 
Proposition

For an open set $\Omega$ in $\mathbb{R}$ with $0 \in \Omega$, and a weighted backward shift $B_\omega : A(\Omega) \to A(\Omega)$ with the weight sequence $\omega = (\omega_n)_{n \in \mathbb{N}}$,

(a) if there is an increasing sequence $(n_k)$ of positive integers such that for all $R > 0$,

$$\lim_{k \to \infty} \left( \left( \prod_{\nu=1}^{n_k} \omega_\nu \right)^{-1} R^{n_k} \right) = 0,$$

then $B_\omega$ is hypercyclic on $A(\Omega)$,
Conditions on the Hypercyclicity of $B_\omega$

Main Proposition

Proposition

For an open set $\Omega$ in $\mathbb{R}$ with $0 \in \Omega$, and a weighted backward shift $B_\omega : A(\Omega) \to A(\Omega)$ with the weight sequence $\omega = (\omega_n)_{n \in \mathbb{N}}$,

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then $B_\omega$ is hypercyclic on $A(\Omega)$,

(b) if $B_\omega$ is hypercyclic on $A(\Omega)$, then there exist an increasing sequence $(n_k)$ of positive integers and $R > 0$ such that

$$\lim_{k \to \infty} \left( \left( \prod_{\nu=1}^{n_k} \omega_\nu \right)^{-1} R^{n_k} \right) = 0.$$
Proof

(a) Let $B_\omega$ be a weighted backward shift on $A(\Omega)$. Then, it is also a weighted backward shift on $H(\mathbb{C})$. Since $H(\mathbb{C})$ is a Fréchet sequence space, the given condition implies that $B_\omega$ is hypercyclic on $H(\mathbb{C})$. Hence, $B_\omega$ is hypercyclic on $A(\Omega)$. 
Proof

(a) Let \( B_\omega \) be a weighted backward shift on \( A(\Omega) \). Then, it is also a weighted backward shift on \( H(\mathbb{C}) \). Since \( H(\mathbb{C}) \) is a Fréchet sequence space, the given condition implies that \( B_\omega \) is hypercyclic on \( H(\mathbb{C}) \). Hence, \( B_\omega \) is hypercyclic on \( A(\Omega) \).

(b) Let \( B_\omega \) be hypercyclic on \( A(\Omega) \). Then, it is also hypercyclic on \( H(\{0\}) \).
Proof

(a) Let $B_\omega$ be a weighted backward shift on $A(\Omega)$. Then, it is also a weighted backward shift on $H(\mathbb{C})$. Since $H(\mathbb{C})$ is a Fréchet sequence space, the given condition implies that $B_\omega$ is hypercyclic on $H(\mathbb{C})$. Hence, $B_\omega$ is hypercyclic on $A(\Omega)$.

(b) Let $B_\omega$ be hypercyclic on $A(\Omega)$. Then, it is also hypercyclic on $H(\{0\})$. The space $H(\{0\})$ is isomorphic to the nuclear Köthe co-echelon space $k_p(V)$, $1 \leq p \leq \infty$, where

$$k_p(V) = \text{ind}_{n \to l_p(v_n)}$$

with $V = (v_{nk})$, $v_{nk} = e^{-kn}$. 
(Bierstedt, Meise, Summers 1982) For $1 \leq p < \infty$, $k_p(V)$ is topologically isomorphic to the space

$$K_p(\bar{V}) = \text{proj}_{\bar{v} \in \bar{V}}/p(\bar{v})$$

$$= \left\{ x = (x_k) : \forall \bar{v} \in \bar{V} \parallel x \parallel_{\bar{v}} = \left( \sum_{k=1}^{\infty} |x_k|^p \bar{v}_k^p \right)^{1/p} \right\},$$

where $\bar{V} = \{ \bar{v} = (\bar{v}_k) \in \mathbb{R}_+^\mathbb{N} : \sup_k \frac{\bar{v}_k}{v_{nk}} < \infty \ \forall n \in \mathbb{N} \}$. 
**Conditions on the Hypercyclicity of \( B_\omega \)**

**Proof of the Main Proposition**

*(Bierstedt, Meise, Summers 1982)* For \( 1 \leq p < \infty \), \( k_p(V) \) is topologically isomorphic to the space

\[
K_p(\bar{V}) = \text{proj}_{\bar{v} \in \bar{V}} l_p(\bar{v})
\]

\[
= \left\{ x = (x_k) : \forall \bar{v} \in \bar{V} \left\| x \right\|_{\bar{v}} = \left( \sum_{k=1}^{\infty} |x_k|^p \bar{v}_k^p \right)^{1/p} \right\},
\]

where \( \bar{V} = \{ \bar{v} = (\bar{v}_k) \in \mathbb{R}_+^N : \sup_k \frac{\bar{v}_k}{\bar{v}_{nk}} < \infty \ \forall n \in \mathbb{N} \} \).

Therefore, \( H(\{0\}) \) is topologically isomorphic to \( K_p(\bar{V}) \), and \( B_\omega \) is hypercyclic on \( K_p(\bar{V}) \) by our assumption.
Since $B_\omega : K_p(\bar{V}) \to K_p(\bar{V})$ is continuous, given $\bar{v}^{(0)} \in \bar{V}$, we can obtain inductively that for every $n \in \mathbb{N}$, there exists $\bar{v}^{(n)} \in \bar{V}$ and constant $C_n$ so that

$$\|B_\omega x\|_{\bar{v}^{(n-1)}} \leq C_n \|x\|_{\bar{v}^{(n)}}, \quad x \in K_p(\bar{V}).$$

Hence, $B_\omega$ is continuous on $K_p(\bar{V})$ equipped with the topology given by the sequence of norms $(\| \cdot \|_{\bar{v}^{(n)}})_{n \in \mathbb{N}}$. 
Since $B_\omega : K_p(\bar{V}) \to K_p(\bar{V})$ is continuous, given $\bar{v}^{(0)} \in \bar{V}$, we can obtain inductively that for every $n \in \mathbb{N}$, there exists $\bar{v}^{(n)} \in \bar{V}$ and constant $C_n$ so that

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Hence, $B_\omega$ is continuous on $K_p(\bar{V})$ equipped with the topology given by the sequence of norms $\left(\|\cdot\|_{\bar{v}^{(n)}}\right)_{n \in \mathbb{N}}$. By completing this space, we obtain a Fréchet space $X$ with the following properties:

- $X$ is isomorphic to the Köthe sequence space $\lambda_p((v^{(n)})_{n \in \mathbb{N}})$,
- $X$ contains $K_p(\bar{V})$ continuously and densely,
- $B_\omega$ is a weighted backward shift on $X$. 
Since $B_\omega$ is hypercyclic on $K_p(\bar{V})$, and $K_p(\bar{V})$ is dense in $X$, $B_\omega$ is also hypercyclic on $X$. As $X$ is a Fréchet sequence space, there is an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$\left( \frac{n_k}{\prod_{\nu=1}^{\omega} \omega_\nu} \right)^{-1} e_{n_k} \to 0$$

in $X$ as $k \to \infty$. 

Since $B_\omega$ is hypercyclic on $K_p(\tilde{V})$, and $K_p(\tilde{V})$ is dense in $X$, $B_\omega$ is also hypercyclic on $X$. As $X$ is a Fréchet sequence space, there is an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$\left( \prod_{\nu=1}^{n_k} \omega_{\nu} \right)^{-1} e_{n_k} \to 0$$

in $X$ as $k \to \infty$.

Then, we can show that there exists $R > 0$ satisfying

$$\lim_{k \to \infty} \left( \left( \prod_{\nu=1}^{n_k} \omega_{\nu} \right)^{-1} R^{n_k} \right) = 0.$$
Problem

Clearly, there are weight sequences satisfying the condition

$$\lim_{k \to \infty} \left( \left( \prod_{\nu=1}^{n_k} \omega_{\nu} \right)^{-1} R^{n_k} \right) = 0$$

for some $R > 0$, but not all $R > 0$. 

Example

The usual backward shift on $A(\Omega)$, that is, $\omega = (\omega_n)$ where $\omega_n = 1$ for all $n \in \mathbb{N}$. 

Question

Is the usual backward shift on $A(\Omega)$ hypercyclic?
Clearly, there are weight sequences satisfying the condition

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**Example** The usual backward shift on \( A(\Omega) \), that is, \( \omega = (\omega_n) \) where \( \omega_n = 1 \) for all \( n \in \mathbb{N} \).
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**Example** The usual backward shift on \( A(\Omega) \), that is, \( \omega = (\omega_n) \) where \( \omega_n = 1 \) for all \( n \in \mathbb{N} \).

**Question** Is the usual backward shift on \( A(\Omega) \) hypercyclic?
The usual backward shift on $A(\Omega)$ is hypercyclic.

**Proof**

The usual backward shift $B: A(\mathbb{R}) \to A(\mathbb{R})$, where $B(x_n) = x_{n-1}$ for all $n \in \mathbb{N}$ and $B(1) = 0$, coincides with the function $T(f)(x) = f(x) - f(0)$ on polynomials. Since the polynomials are dense in $A(\mathbb{R})$, we have $B = T$. 

**Theorem**

The usual backward shift on $A(\mathbb{R})$ is hypercyclic.
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$$T(f)(x) = \frac{f(x) - f(0)}{x}$$

on polynomials. Since the polynomials are dense in $A(\mathbb{R})$, we have $B = T$. 
If we take the strip

\[ S = \{ z \in \mathbb{C} : |\text{Im} \ z| < 1/2 \}, \]

then \( H(S) \) is dense in \( A(\mathbb{R}) \), so it is enough to show that \( T \) is hypercyclic on \( H(S) \). For this purpose, we need the following criterion.

**Godefroy-Shapiro criterion**

Let \( T \) be an operator on a separable Fréchet space \( X \). If the subspaces \( X_0 := \text{span} \{ x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| < 1 \} \) and \( Y_0 := \text{span} \{ x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| > 1 \} \) are dense in \( X \), then \( T \) is hypercyclic.
If we take the strip

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\[ Y_0 := \text{span}\{ x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| > 1 \} \]

are dense in \( X \), then \( T \) is hypercyclic.
Solving the equation $Tf = \lambda f$, we can observe that for any $\zeta \in \hat{C}\setminus S$, the function

$$f_\zeta(z) = \frac{1}{\zeta - z}$$

is an eigenfunction of $T$ with eigenvalue $1/\zeta$. 
Solving the equation $Tf = \lambda f$, we can observe that for any $\zeta \in \hat{C}\setminus S$, the function

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is an eigenfunction of $T$ with eigenvalue $1/\zeta$. Using a variation of Runge's Theorem and the Grothendieck - Köthe - Silva duality, we show that

$$\text{span}\{f_\zeta : \zeta \in \hat{C}\setminus S\}$$

are dense in $H(S)$ for the separate cases $|\zeta| < 1$ and $|\zeta| > 1$. 
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are dense in $H(S)$ for the separate cases $|\zeta| < 1$ and $|\zeta| > 1$. Therefore, by the Godefroy-Shapiro criterion, $T$ is hypercyclic on $H(S)$, which implies that $T$ is hypercyclic on $A(\mathbb{R})$. 


The End