# EIGENVALUE ESTIMATES OF POSITIVE INTEGRAL OPERATORS WITH ANALYTIC KERNELS

*YÜKSEL SOYKAN*

*Bülent Ecevit University, Department of Mathematics,*

*Art and Science Faculty, 67100, Zonguldak, Turkey*

*e-mail: yuksel_soykan@hotmail.com*

**Abstract.** In this talk, we exhibit canonical positive definite integral kernels associated with simply connected domains. We give lower bounds for the eigenvalues of the sums of such kernels.

**2000 Mathematics Subject Classification.**45C05, 45H05 (primary), 42B30, 14Q05, 30D55 (secondary).

**Keywords:** Eigenvalue Problems, Special Kernels, Integral Operators, Hardy Spaces, Smirnov Classes, Closed Analytic Curves, Ahlfors Regular Curves.

## 1. Introduction

Let $\Omega$ be a simply connected domain (with an infinite complement) in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup (\infty)$. Given a Riemann mapping function (RMF) $\varphi$ for $\Omega$, that is a conformal map of $\Omega$ onto the unit disc $\Delta$, define a function $K_\Omega$ on $\Omega \times \Omega$ by

$$K_\Omega(\zeta, z) = \frac{\varphi'(\zeta)^{\frac{1}{2}}\overline{\varphi'(z)^{\frac{1}{2}}}}{1-\varphi(\zeta)\overline{\varphi(z)}}, \qquad (\zeta, z \in \Omega)\,, \tag{1.1}$$

for either of the branches of $\varphi'^{\frac{1}{2}}$. This function is independent of the choice of mapping function $\varphi$, see [9, p.410]. Little's work is on integral operators defined by restricting the function $K_\Omega$ to a square. Specifically, suppose $J \subset \Omega$ is a (bounded) closed real interval then we obtain a compact symmetric operator $T_\Omega$ on $L^2(J)$ defined by

$$T_\Omega f(s) = \int_J K_\Omega(s,t)f(t)dt \qquad (f \in L^2(J), s \in J).$$

This operator is always positive in the sense of operator theory, i.e. $\langle T_\Omega f, f\rangle \geq 0$, for all $f \in L^2(J)$, see [9].

Simple interesting examples can be constructed from standard conformal maps; for instance, if $\Omega = \Delta$ (and $J \subseteq (-1,1)$) we obtain $K_\Omega(s,t) = 1/(1-st)$, and if $\Omega$ is the strip $|\operatorname{Im} z| < \pi/4$ and $\varphi(z) = \tanh z$ we obtain $K_\Omega(s,t) = \operatorname{sech}(s-t)$. For more examples, see [8].

The main object of study in Little's work is the asymptotic behaviour of the eigenvalues $\lambda_n(T_\Omega)$ of $T_\Omega$ and related operators. These are always positive and we assume they are arranged in decreasing order with repetitions to account for multiplicities. This work is most complete in the case when $\Omega$ is symmetric about the real line $\mathbb{R}$, as in the two examples above, for in [8, Proposition 4, p.281] it is shown how to construct a $q$, $0 < q < 1$ such that $\lambda_n(T_\Omega) \simeq q^n$ in the sense that $\lambda_n(T_\Omega) = O(q^n)$ and $q^n = O(\lambda_n(T_\Omega))$.

If we think of operators of the form $T_\Omega$ as "standard examples" then an obvious problem is to consider natural combinations of such operators (they will always be positive). In [9] Little considered distinct simply connected domains $\Omega_1, \Omega_2, ..., \Omega_N$ each of which is bounded by a line or circle. He proves, for instance, that if $J \subset \cap_{i=1}^N \Omega_i$ is a closed interval then

$$\lambda_n(T_+) \simeq \lambda_n(T_\times)$$

where $T_+$ is just the sum $T_{\Omega_1} + T_{\Omega_2} + ... + T_{\Omega_N}$ with kernel $K_{\Omega_1}(s,t) + K_{\Omega_2}(s,t) + ... + K_{\Omega_N}(s,t)$, and $T_\times$ is the integral operator on $L^2(J)$ with whose kernel is the pointwise product $K_{\Omega_1}(s,t)K_{\Omega_2}(s,t)...K_{\Omega_N}(s,t)$.

we will continue in the spirit of [9]. Our main result is

Theorem 1.1. *Let $C = (C_1, C_2, C_3, ..., C_N)$ be an $N$-tuple of closed distinct curves on the sphere $\mathbb{C}_\infty$ and suppose that for each $i$, $1 \le i \le N$, $C_i$ is a circle, a line $\cup\{\infty\}$, an ellipse, a parabola $\cup\{\infty\}$ or a branch of a hyperbola $\cup\{\infty\}$. Let $D$ be a complementary domain of $\cup_{i=1}^N C_i$ and suppose that $D$ is simply connected and contains the closed interval $J$. If $D_i$ is the complementary domain of $C_i$ which contains $D$ and $J$ then there exists some constant $m > 0$ such that*

$$mT_D \le T_{D_1} + T_{D_2} + ... + T_{D_N}$$

*in the operator sense. In particular, for all $n \ge 0$,*

$$m\lambda_n(T_D) \le \lambda_n(T_{D_1} + T_{D_2} + ... + T_{D_N}).$$

We shall prove Theorem 1.1 in sections 4 and 6 using Theorems 4.1 and 4.2 of section 4. Recall that a complementary domain of a closed $F \subseteq \mathbb{C}_\infty$ is a maximal connected subset of $\mathbb{C}_\infty - F$, which must be a domain.

Remark 1.2. *From now on, we shall fix the notation as in Theorem 1.1.*

## 2. An Outline of Proof of Theorem 1.1

An *outline* of our method of proof Theorem 1.1 is as follows. Suppose $\Omega \subseteq \mathbb{C}_\infty$ is a simply connected domain. Then there is a canonical Hilbert Space $E^2(\Omega)$ of analytic functions on $\Omega$. The precise definition of these spaces will be recalled in Section 3; so these spaces will be taken for granted for the moment.

If $\Omega$ contains our closed interval $J$ then there is a natural restriction map $S_\Omega : E^2(\Omega) \to L^2(J)$

$$S_\Omega f(s) = f(s) \qquad\qquad (f \in E^2(\Omega), s \in J).$$

It was shown in section 3 of [9, p.409-410] that $S_\Omega$ is compact and that $S_\Omega S_\Omega^* = T_\Omega$. That is, $S_\Omega S_\Omega^*$ is the positive integral operator on $L^2(J)$ with kernel $K_\Omega$. Similarly if $\Omega_1, \Omega_2, ..., \Omega_N \subseteq \mathbb{C}_\infty$, each containing $J$, are simply connected domains and $S_+ : \bigoplus E^2(\Omega_i) \to L^2(J)$ is given by

$$\begin{aligned} S_+(f_1, f_2, \ldots, f_N)(s) &= S_{\Omega_1} f_1(s) + S_{\Omega_2} f_2(s) + \ldots + S_{\Omega_N} f_N(s) \\ &= f_1(s) + f_2(s) + \ldots + f_N(s) \qquad (f_i \in E^2(\Omega_i), 1 \le i \le N, s \in J) \end{aligned}$$

then (it was shown in [9, p.413] that)

$$S_+ S_+^* = T_{\Omega_1} + T_{\Omega_2} + \ldots + T_{\Omega_N}$$

is the compact positive integral operator on $L^2(J)$ with kernel $K_{D_1} + K_{D_2} + ... + K_{D_N}$ where $T_{\Omega_i}$ is the integral operator on $L^2(J)$ with kernel $K_{\Omega_i}$.

To prove Theorem 1.1 it will sufficient to find a continuous linear operator $A : E^2(D) \to \oplus_{i=1}^N E^2(D_i)$ (see section 4, Theorem 4.1 and Theorem 4.2).

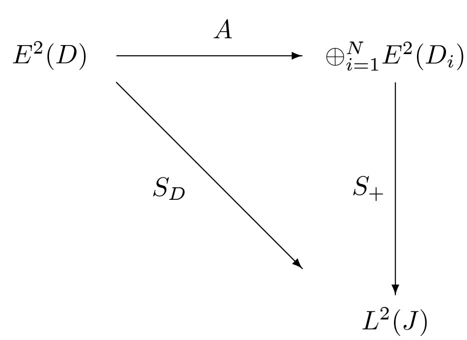


Figure 1. Operator $A$

We shall be able to put $m = \frac{1}{\|A\|^2}$.

Suppose now that $D$ is as in Theorem 1.1 and the operator $A$ as in Figure 1 exists and is continuous. Then, for all $f \in L^2(J)$,

$$\begin{aligned} \langle T_D f, f \rangle &= \langle S_D S_D^* f, f \rangle = \|S_D^* f\|^2 = \left\|A^* S_+^* f\right\|^2 \\ &\le \|A^*\|^2 \left\|S_+^* f\right\|^2 = \|A\|^2 \langle S_+ S_+^* f, f \rangle \\ &= \|A\|^2 \langle (T_{D_1} + T_{D_2} + \ldots + T_{D_N}) f, f \rangle. \end{aligned}$$

Hence

$$T_D \le \|A\|^2 (T_{D_1} + T_{D_2} + \ldots + T_{D_N}). \tag{2.1}$$

Hence, by (2.1)

$$\frac{1}{\|A\|^2}T_D \leq T_{D_1} + T_{D_2} + ... + T_{D_N}$$

in the operator sense. In particular, for all $n \geq 0$,

$$\frac{1}{\|A\|^2}\lambda_n(T_D) \leq \lambda_n(T_{D_1} + T_{D_2} + ... + T_{D_N}).$$

So this proves Theorem 1.1. Although it takes a lot of effort to verify technical details, especially continuity, the operator $A$ is quite easy to understand.

The operator $A$ comes from Cauchy's Integral Formula. The boundary $\partial D$ of $D$ can be decomposed naturally as

$$\partial D = \partial_1 \cup \partial_2 \cup ... \cup \partial_N$$

where $\partial_i \subseteq C_i$. Cauchy's Integral Formula

$$f(z) = \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\partial_j} \frac{\widetilde{f}(\zeta)}{\zeta \ - z} d\zeta \qquad (f \in E^2(D), z \in D)$$

can be validated (see Theorem 3.2). The function $\widetilde{f}$ is the "non-tangential limit function" of $f$.

In Theorem 1.1 it is insisted that the curves $(C_1, C_2, C_3, ..., C_N)$ are distinct. But once Theorem 1.1 has been proved this restriction can be removed. Suppose $(C_1, C_2, C_3, ..., C_R)$ are curves as in Theorem 1.1. Suppose $R > N$, that $(C_1, C_2, C_3, ..., C_N)$ are distinct and that each $C_i$ with $i > N$ already appears in the list $(C_1, C_2, C_3, ..., C_N)$. Clearly

$$\bigcup_{i=1}^{N} C_i = \bigcup_{i=1}^{R} C_i,$$

so that both sets of curves have the same complementary domains. Suppose $D$ is one of them that $D$ is simply connected and contains $J$. Let $k$ be a positive integer such that each $C_i$ appears at most $k$ times in the list $(C_1, C_2, C_3, ..., C_R)$. Then

$$T_{D_1} + T_{D_2} + ... + T_{D_N} \leq T_{D_1} + T_{D_2} + ... + T_{D_R}$$

so

$$m(T_D) \leq T_{D_1} + T_{D_2} + ... + T_{D_R}.$$

## 3. Preliminaries

**3.1. Hardy and Smirnov Classes.** We will need the theory of Hardy and Smirnov classes of analytic functions. The reader is referred to [5], [6], [7] ,[11], [13], and references therein for a basic account of the subject. Here is a synopsis.

If $\gamma$ is a $\sigma-$*rectifiable* arc (see subsection 3.2 for definition) in $\mathbb{C}$ and $f$ is integrable with respect to *arc-length* measure the notations

$$\int_\gamma f(z)dz, \ \int_\gamma f(z)\,|dz|$$

will be used to denote, respectively, the complex line integral of $f$ along $\gamma$ and the integral of $f$ with respect to arc-length measure. In the first case we assume $\gamma$ has an orientation. The notation $L^p(\gamma)$ will denote the $L^p$ space of normalized arc length measure on $\gamma$ with the factor $\frac{1}{2\pi}$.

$\mathbb{T}$ will be used to denote the boundary of the open unit disc $\Delta$. The space $H^\infty(\Delta)$ is just the set of all bounded analytic functions on $\Delta$ with the uniform norm. For $1 \leq p < \infty$, $H^p(\Delta)$ is the set of all functions $f$ analytic on $\Delta$ such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta < \infty. \tag{3.1}$$

For $1 \leq p \leq \infty$, $H^p(\mathbb{T})$ is defined as the set of all $\widetilde{f} \in L^p(\mathbb{T})$ such that

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \widetilde{f}(z) z^m dz = 0, \qquad m = 0, 1, 2, .... \tag{3.2}$$

We have additional definitions of $H^2(\Delta)$ and $H^2(\mathbb{T})$. $H^2(\Delta)$ is the set of all analytic functions $f$ on $\Delta$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{with } a = (a_n) \in \ell^2 \tag{3.3}$$

and $H^2(\mathbb{T})$ is the set of all $\widetilde{f} \in L^2(\mathbb{T})$ of the form

$$\widetilde{f}(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{ni\theta} \qquad \text{with } a = (a_n) \in \ell^2. \tag{3.4}$$

For a more general simply-connected domain $\Omega$ in the extended plane $\mathbb{C}_\infty = \mathbb{C} \cup (\infty)$ and a conformal mapping $\varphi$ from $\Omega$ onto $\Delta$, a function $g$ analytic in $\Omega$ is said to belong to the Smirnov class $E^2(\Omega)$ if and only if $g = (f \circ \varphi)\varphi'^{1/2}$ for some $f \in H^2(\Delta)$ where $\varphi'^{1/2}$ is an analytic branch of the square root of $\varphi'$.

The following Theorem is proved in Duren [5].

THEOREM 3.1. *Let $\Omega$ be the inner domain of a rectifiable Jordan curve $\gamma$ and suppose $\psi : \Delta \to \Omega$ is a conformal equivalence. Then*

**a:** *$\psi' \in H^1(\Delta)$;*

**b:** *Every $f \in E^2(\Omega)$ has a nontangential limit function $\widetilde{f} \in L^2(\partial\Omega)$;*

**c:** *The map $f \to \widetilde{f}$ $\quad (E^2(\Omega) \to L^2(\partial\Omega))$ is an isometric isomorphism onto a closed subspace $E^2(\partial\Omega)$ of $L^2(\partial\Omega)$, so*

$$||f||^2_{E^2(\Omega)} = ||\widetilde{f}||^2_{L^2(\partial\Omega)} = \frac{1}{2\pi} \int_{\partial\Omega} |\widetilde{f}(z)|^2 |dz| \qquad (f \in E^2(\Omega));$$

**d:** *For all $f \in E^2(\Omega)$ and $z \in \Omega$*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\widetilde{f}(\zeta)}{\zeta - z} d\zeta.$$

PROOF. See [5, p.169-170].

We need the following Theorem (see [12]) to show the existence of the operator $A$. When $D$ and the $D_i$'s are as in Theorem 1.1, the boundaries of $D$ and the $D_i$'s are "reasonable" and it will make sense for us to discuss to notion of a nontangential limit function and spaces $E^2(\partial D), E^2(\partial D_i)$ analogous to $H^2(\Delta), H^2(\mathbb{T})$.

THEOREM 3.2. *Let $D$ be as in Theorem 1.1. Then*

**i:** *Every $f \in E^2(D)$ has a non-tangential limit function $\widetilde{f} \in L^2(\partial D)$;*

**ii:** *(Parseval's identity) The map $f \to \widetilde{f}$ is an isometric isomorphism of $E^2(D)$ onto a closed subspace $E^2(\partial D)$ of $L^2(\partial D)$, so*

$$||f||^2_{E^2(D)} = ||\widetilde{f}||^2_{L^2(\partial D)} = \frac{1}{2\pi}\int_{\partial D} |\widetilde{f}(z)|^2 |dz| \qquad (f \in E^2(D));$$

**iii:** *(Cauchy's Integral Formula) For all $f \in E^2(D)$ and $z \in D$*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{\widetilde{f}(\zeta)}{\zeta - z} d\zeta.$$

**3.2. Arcs and Curves .** An arc or closed curve $\gamma$ is called $\sigma-$rectifiable if and only if it is a countable union of rectifiable arcs in $\mathbb{C}$, together with $(\infty)$ in the case when $\infty \in \gamma$. For instance, a parabola without $\infty$ is $\sigma -$ rectifiable arc, and a parabola with $\infty$ is $\sigma -$ rectifiable Jordan curve.

Recall that a compact arc $\sigma$ is called <u>*smooth*</u> if there exists some parametrization $g : [a, b] \to \sigma$ such that $g \in C^1[a, b]$ and $g'(t) \neq 0$, $\forall t \in [a, b]$.

REMARK 3.3. *Note that if $\sigma$ is smooth then it is rectifiable, i.e.,*

$$l(\sigma) = \int_a^b |g'(t)|\, dt < \infty.$$

To define the *arc* length parametrization of $\sigma$ put $s = s(t) = \int_a^t |g'(u)|\, du$ for $a \leq t \leq b$ so that $0 \leq s \leq \ell(\sigma)$. Then $s'(t) = |g'(t)|$ and $t \to s(t)$ $([a, b] \to [0, \ell])$ is $C^1$ with strictly positive derivative. Hence also its inverse $s \to t(s)$ $([0, \ell] \to [a, b])$ is $C^1$ with strictly positive derivative.

The arc length parametrization of the smooth arc $\sigma$ is the map $h : [0, \ell] \to \sigma$ satisfying $h(s) = \{$the point on $\sigma$ length $s$ from the initial point $(g(a))$ $\}$, i.e., $h(s) = g(t(s)) \qquad 0 \leq s \leq \ell.$

Since $h'(s) = g'(t(s))t'(s)$, $h \in C^1[0, \ell]$ with non-zero derivative. Necessarily $|h'(s)| = 1$ since

$$h'(s(t)) = g'(t)t'(s) = \frac{g'(t)}{s'(t)} = \frac{g'(t)}{|g'(t)|}.$$

We need the following Lemma.

LEMMA 3.4. *Let $g \in C^1[0, \infty)$ with $g'(t) \neq 0$ $(t \geq 0)$. Suppose that $c \in \mathbb{C}$ and*

$$\lim_{t\to\infty} g(t) = c \tag{3.5}$$

$$\lim_{t\to\infty} \frac{g'(t)}{|g'(t)|} = \omega \qquad (|\omega| = 1) \tag{3.6}$$

*exist. Define $\sigma = g([0, \infty)) \cup (c)$. Then*

**i:** *$\sigma$ is a compact arc,*

**ii:** *$\sigma$ is rectifiable,*

**iii:** *$\sigma$ is smooth.*

PROOF. The proof is straightforward and will be omitted.

DEFINITION 1. *A simple $\sigma-$rectifiable arc $\gamma \subseteq \mathbb{C}$ is said to satisfy hypothesis-R if and only if, for some (hence all) $a \in \mathbb{C} - \overline{\gamma}$*

$$\int_{\gamma} \frac{|dz|}{|z-a|^2} < \infty.$$

*Equivalently, if the function $f(z) = \dfrac{1}{z-a}$ belongs to $L^2(\gamma)$.*

LEMMA 3.5. *Let $\gamma$ be a simple, $\sigma-$rectifiable arc in $\mathbb{C}$. Suppose that $a \in \mathbb{C} - \overline{\gamma}$ and that $\mu(z) = \dfrac{1}{z-a}$. Then $\gamma$ satisfies hypothesis-R if and only if $\mu(\gamma)$ is rectifiable.*

PROOF. Note that

$$l(\mu(\gamma)) = \int_{\gamma} |\mu'(z)|\,|dz| = \int_{\gamma} \frac{|dz|}{|z-a|^2} < \infty.$$

So the result is clear.

EXAMPLE 1. *An analytic Jordan curve in $\mathbb{C}$ obviously satisfies hypothesis-R. It is also easy to see that if $\gamma$ satisfies hypothesis-R so does $\mu(\gamma)$, where $\mu$ is a linear equivalence: $\mu(z) = \alpha z + \beta$. Obviously $\mathbb{R}$ satisfies hypothesis-R and therefore so does any line; and it can be easily seen that every parabola and hyperbola component satisfies hypothesis-R.*

## 4. The Operator $A$ and Cauchy Integrals and easy cases

The main result in this section is

THEOREM 4.1. *There exists a continuous linear operator*

$$A : E^2(D) \rightarrow \oplus_{i=1}^{N} E^2(D_i), \qquad \text{say } f \rightarrow (f_1, f_2, ...f_N),$$

*such that*

$$f(z) \equiv f_1(z) + f_2(z) + ... + f_N(z) \quad \text{on } D,$$

*for all $f \in E^2(D)$.*

Once this is proved our work will be completed, for we will then have

$$T_D \leq \|A\|^2 \left(T_{D_1} + T_{D_2} + ... + T_{D_N}\right),$$

as proved in Section 3.

Let $\partial_j = \partial D \cap C_j$ $(1 \le j \le N)$ then, by Cauchy's integral formula (Theorem 3.2 iii):

$$f(z) = \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\partial_j}\frac{\widetilde{f}(\zeta)}{\zeta - z}d\zeta \qquad (f \in E^2(D), z \in D).$$

So the natural choice for $f_j$ is

$$f_j(z) = \frac{1}{2\pi i}\int_{\partial_j}\frac{\widetilde{f}(\zeta)}{\zeta - z}d\zeta \qquad (z \in D_j, 1 \le j \le N).$$

Let $F_j$ be the extension of $\widetilde{f}|_{\partial_j}$ to $C_j$ :

$$F_j(z) = \left(\begin{array}{ccc} \widetilde{f}(z), & \text{if} & z \in \partial_j \\ 0, & \text{if} & z \in C_j - \partial_j \end{array}\right. .$$

Of course, if $\partial_j$ is finite or empty $F_j = 0$. In general

$$\frac{1}{2\pi}\int_{C_j}|F_j(z)|^2\,|dz| = \frac{1}{2\pi}\int_{\partial_j}\left|\widetilde{f}(z)\right|^2|dz| \ \le \frac{1}{2\pi}\int_{\partial D}\left|\widetilde{f}(z)\right|^2|dz| = \|f\|^2$$

by Parseval's identity (see Theorem 3.2 ii). So $F_j \in L^2(C_j)$ and the map $f \to F_j$ is continuous. Hence Theorem 4.1 will be proved if the following continuity property of the Cauchy integral can be proved.

THEOREM 4.2. *For each $j$ $(1 \le j \le N)$ the formula*

$$P_j g(z) = \frac{1}{2\pi i}\int_{C_j}\frac{g(\zeta)}{\zeta - z}d\zeta \qquad (g \in L^2(C_j), z \in D_j) \tag{4.1}$$

*defines a continuous linear operator $P_j$ mapping $L^2(C_j)$ into $E^2(D_j)$.*

The operator $P_j$ is a special case of a more general "Cauchy integral operator" or "Cauchy transform".

DEFINITION 2. *Let $\gamma \subseteq \mathbb{C}$ be a simple oriented $\sigma-$rectifiable arc satisfying hypothesis R. For $g \in L^2(\gamma)$ the function $C_\gamma g$ on $\overline{\mathbb{C}} - \overline{\gamma}$ is defined by*

$$C_\gamma g(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{g(\zeta)}{\zeta - z}d\zeta \qquad (g \in L^2(\gamma), z \notin \overline{\gamma}).$$

COROLLARY 4.3. *If $g \in L^2(\gamma)$ then $C_\gamma g$ is analytic on $\mathbb{C} - \overline{\gamma}$. If also $\gamma$ is bounded and rectifiable then $C_\gamma g$ is analytic at $\infty$ and $C_\gamma g(\infty) = 0$.*

PROOF. If $z \in \mathbb{C} - \overline{\gamma}$ then, by Schwarz's inequality

$$\frac{1}{2\pi}\int_{\gamma}\frac{|g(\zeta)|}{|\zeta - z|}|d\zeta| \le \|g\|\left(\frac{1}{2\pi}\int_{\gamma}\frac{|d\zeta|}{|\zeta - z|^2}\right)^{1/2} < +\infty,$$

by hypothesis R, so $C_\gamma g$ is defined on $\mathbb{C}_\infty - \overline{\gamma}$. To prove that $C_\gamma g$ is analytic, let $z \in \mathbb{C} - \overline{\gamma}$ and choose an $r > 0$ such that the open disc $N_{2r}(z)$ centre $z$, radius $2r$ is contained in $\mathbb{C} - \overline{\gamma}$. Let $(h_n)$ be a sequence in $\mathbb{C}$ such that $0 < |h_n| < r$, for all $n$ and $h_n \to 0$. For each $n$, define

$$I_n = \frac{1}{h_n}(C_\gamma g(z + h_n) - C_\gamma g(z)).$$

Then

$$\begin{aligned} I_n &= \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{h_n}\left(\frac{1}{\zeta-z-h_n}-\frac{1}{\zeta-z}\right)d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta-z-h_n)(\zeta-z)}d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta-z-h_n)(\zeta-z)}\omega(\zeta)\,|d\zeta| \end{aligned} \tag{4.2}$$

where $\omega(\zeta)$ is the unit tangent vector of $\gamma$ at $\zeta$ ($|\omega(\zeta)|=1$ a.e.). Now, for all $n$ and almost all $\zeta\in\gamma$,

$$\left|\frac{g(\zeta)\omega(\zeta)}{(\zeta-z-h_n)(\zeta-z)}\right| \le \frac{1}{r}\frac{|g(\zeta)|}{|\zeta-z|}.$$

Since the last function (of $\zeta$) is in $L^1(\gamma)$ it follows from Lebesgue's theorem, applied to (4.2), that

$$(C_\gamma g)'(z) = \lim_{n\to\infty} I_n = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta-z)^2}d\zeta.$$

If $\gamma$ is bounded then $\infty\in\mathbb{C}_\infty-\overline{\gamma}$, and if $\gamma$ is rectifiable then $L^2(\gamma)\subseteq L^1(\gamma)$. So if $g\in L^2(\gamma)$ then, for large values of $z$,

$$|C_\gamma g(z)| \le \frac{1}{2\pi}\int_\gamma \frac{|g(\zeta)|}{|z|-|\zeta|}\,|d\zeta|$$

which tends to 0 as $z\to\infty$, by Lebesgue's theorem. Therefore $C_\gamma g$ is analytic and 0 at $\infty$.

Remark 4.4. *The orientation here is crucial and we must always be clear what it is before starting any manipulations. Obviously $C_{-\gamma}g=-C_\gamma g$. Recall that $C_j$ in Theorem 4.2 is oriented so as to wind positively round $D$ and $D_j$. The function $P_jg$ is then the restriction of $C_\gamma g$ to $D_j$, where $\gamma=C_j$.*

Lemma 4.5. *(Invariance lemma) Let $\gamma_1,\gamma_2$ be simple oriented $\sigma-$rectifiable arcs satisfying hypothesis R, and suppose that $\Omega_1,\Omega_2$ are simply connected complementary domains of $\overline{\gamma_1},\overline{\gamma_2}$, respectively. Suppose that $\mu$ is a fractional linear transformation such that $\mu(\Omega_1)=\Omega_2$ and $\mu(\gamma_1)=\gamma_2$ (with the same orientation). Then the map $f\to C_{\gamma_1}f|_{\Omega_1}$ defines a continuous linear operator mapping $L^2(\gamma_1)$ into $E^2(\Omega_1)$ if and only if the map $g\to C_{\gamma_2}g|_{\Omega_2}$ defines a continuous linear operator mapping $L^2(\gamma_2)$ into $E^2(\Omega_2)$, in which case the two operators are unitarily equivalent.*

Proof. Let us call the first map $C_{\gamma_1\Omega_1}$ and the second $C_{\gamma_2\Omega_2}$; thus $C_{\gamma_1\Omega_1}f$ is the restriction to $\Omega_1$ of the Cauchy integral $C_{\gamma_1}f$ and likewise for $C_{\gamma_2\Omega_2}$.

By symmetry, we need only prove one of the implications. So, suppose $C_{\gamma_2\Omega_2}$ is a continuous linear operator mapping $L^2(\gamma_2)$ into $E^2(\Omega_2)$. Let $V=V_{\mu^{-1}}:L^2(\gamma_1)\to L^2(\gamma_2)$ be the unitary operator

$$Vf(\zeta)=f(\mu^{-1}(\zeta))\mu^{-1\prime}(\zeta)^{1/2} \qquad (f\in L^2(\gamma_1),\zeta\in\gamma_2)$$

and let $U=U_\mu:E^2(\Omega_2)\to E^2(\Omega_1)$ be the unitary operator

$$Ug(z)=g(\mu(z))\mu'(z)^{1/2} \qquad (g\in E^2(\Omega_2),z\in\Omega_1).$$

Then $UC_{\gamma_2\Omega_2}V : L^2(\gamma_1) \to E^2(\Omega_1)$ is a unitarily equivalent copy of $C_{\gamma_1\Omega_1}$. To elucidate this operator, let $f \in L^2(\gamma_1)$ and $z \in G_2$, then

$$\begin{aligned} C_{\gamma_2\Omega_2}Vf(z) &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\mu^{-1}(\zeta))\mu^{-1\prime}(\zeta)^{1/2}}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(u)\mu'(u)^{1/2}}{\mu(u) - z}du, \end{aligned}$$

using the substitution $\zeta = \mu(u)$. So if $w \in \Omega_1$

$$UC_{\gamma_2\Omega_2}Vf(w) = \frac{1}{2\pi i}\int_{\gamma_1} f(u)\frac{\mu'(u)^{1/2}\mu'(w)^{1/2}}{\mu(u) - \mu(w)}du.$$

Now because $\mu$ is fractional linear it satisfies the functional equation

$$\frac{\mu'(u)^{1/2}\mu'(w)^{1/2}}{\mu(u) - \mu(w)} = \frac{1}{u - w} \tag{4.3}$$

as the reader will be able to verify. Hence $C_{\gamma_1\Omega_1} = UC_{\gamma_2\Omega_2}V : L^2(\gamma_1) \to E^2(\Omega_1)$ is continuous and unitarily equivalent to $C_{\gamma_2\Omega_2}$.

Lemma 4.6. *(see [9]) Theorem 4.2 is true if $C_j$ is a circle or a line.*

Proof. We can map $D_j$ onto $\Delta$ and $C_j$ onto the positively oriented unit circle using a fractional linear function. Thus in this case $P_j$ is unitarily equivalent to the operator $C : L^2(\mathbb{T}) \to H^2(\Delta)$ :

$$Cf(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}d\zeta,$$

which is continuous, because if

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{ni\theta}$$

with $a = (a_n) \in \ell^2(\mathbb{Z})$ then

$$Cf(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \Delta);$$

it is unitarily equivalent to the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

## 5. David's Theorem

The remaining cases of Theorem 4.2 will be proved using David's theorem [10] (and theorem 5.3 below). This has to do with the $L^2-$space of a rectifiable Jordan curve. Suppose that $\gamma$ is a positively oriented rectifiable Jordan curve with inner domain $\Omega$ and outer domain $G$. We have two spaces of analytic functions $E^2(\Omega)$ on $\Omega$ and $E^2(G)$ on $G$. Associated with these two we have closed subspaces $E^2(\partial\Omega), E^2(\partial G)$ of $L^2(\gamma)$ ($\gamma = \partial\Omega = \partial G$) consisting of all nontangential limit functions of elements of $E^2(\Omega), E^2(G)$, respectively. An obvious question arises " is $L^2(\gamma) = E^2(\partial\Omega) \oplus E^2(\partial G)$, not necessarily orthogonally ? ". David's theorem gives a sufficient condition for the answer to be yes.

The reason the question is obvious is that, when $\gamma = \mathbb{T}$, $E^2(\Omega) = H^2(\Delta)$ and so $E^2(\partial\Omega)$ is all functions $f$ on $\mathbb{T}$ of the form

$$f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{ni\theta}$$

with $(a_n) \in \ell^2$. And, because $z \to \dfrac{1}{z}$ is an RMF for $G = \mathbb{C}_\infty - \overline{\Delta}$, $E^2(\partial G)$ is all functions $f$ on $\mathbb{T}$ of the form

$$f(e^{i\theta}) = \sum_{n=0}^{\infty} b_n e^{-(n+1)i\theta}$$

with $(b_n) \in \ell^2$, so David's theorem is true when $\gamma = \mathbb{T}$.

Suppose now that our rectifiable Jordan curve $\gamma$ does satisfy $L^2(\gamma) = E^2(\partial\Omega) \oplus E^2(\partial G)$. It follows immediately from Banach's theorem that the (not necessarily orthogonal) projection $(f_1, f_2) \to f_1 \quad (L^2(\gamma) \to E^2(\partial\Omega))$ is continuous. It will be easy to show (Lemma 5.4 ii below) that the Cauchy integral satisfies

$$C_{\gamma\Omega}(f_1 + f_2)(z) = C_{\gamma\Omega} f_1(z) \qquad (f_1 \in E^2(\partial\Omega), f_2 \in E^2(\partial G), z \in \Omega),$$

so that, by b and c of Theorem 3.1, $C_{\gamma\Omega}(L^2(\gamma)) \subseteq E^2(\Omega)$. And if write $f = f_1 + f_2$ as above we see that, for all $f \in L^2(\gamma)$,

$$\|C_{\gamma\Omega} f\|_{E^2(\Omega)} = \|C_{\gamma\Omega} f_1\|_{E^2(\Omega)} = \|f_1\|_{L^2(\gamma)} \leq M_\Omega \|f\|_{L^2(\gamma)}$$

for some $M_\Omega > 0$, so that the Cauchy integral $C_{\gamma\Omega}$ defines a continuous linear operator mapping $L^2(\gamma)$ into $E^2(\Omega)$. There is a complete symmetry here and so $C_{\gamma G}$ defines a continuous linear operator mapping $L^2(\gamma)$ into $E^2(G)$ as well.

Now the hypothesis of David's theorem involves the curve $\gamma$ only. So *in order to prove Theorem 4.2 in the case where $C_j$ is an ellipse it is sufficient to show that an ellipse satisfies the hypothesis of David's theorem*, and because of the invariance Lemma 4.5, *in order to verify Theorem 4.2 when $C_j$ is a parabola or hyperbola component it is sufficient to show that some fractional linear image $\mu(C_j)$ of $C_j$ satisfies the hypothesis of David's theorem.*

The key hypothesis in David's theorem is that the Jordan curve is "Ahlfors regular".

DEFINITION 3. *An arc or closed curve $\gamma$ is called Ahlfors regular if there is a constant $k > 0$ such that*

$$l(N_r(z_0) \cap \gamma) \leq kr, \qquad \text{for all } r > 0 \text{ and } z_0 \in \gamma,$$

*where $l$ is arc length measure.*

COROLLARY 5.1. *If $\gamma$ and $k$ are as above then*

$$l(N_r(z) \cap \gamma) \leq 2kr, \qquad \text{for all } r > 0 \text{ and } z \in \mathbb{C}.$$

PROOF. We only need to consider the case where $N_r(z) \cap \gamma \neq \emptyset$. So choose $z_0 \in N_r(z) \cap \gamma$. Then $N_r(z) \subseteq N_{2r}(z_0)$. Hence the result.

Corollary 5.2. *If $\gamma \subseteq \mathbb{C}$ is an Ahlfors regular Jordan curve and $\mu$ is a fractional linear function which is finite on $\gamma$ then $\mu(\gamma)$ is Ahlfors regular.*

Proof. Choose $\alpha > 0$ such that, for all $z_0 \in \gamma$ and $r_0 > 0$,

$$l(N_{r_0}(z_0) \cap \gamma) \leq \alpha r_0.$$

The pole $a$ of $\mu^{-1}$ is not on $\mu(\gamma)$ so

$$\delta = \inf_{\zeta \in \mu(\gamma)} |\zeta - a| > 0.$$

Let $K$ be the closure of the $\frac{\delta}{2}$ neighbourhood of $\mu(\gamma)$ then

$$M = \sup_{u \in K} \left|\mu^{-1\prime}(u)\right| < +\infty.$$

If $z \in \mu(\gamma)$ and $|w - z| < \frac{\delta}{2}$ then the line segment $[z, w] \subseteq K$. Integrating $\mu^{-1\prime}$ along this arc shows that

$$\left|\mu^{-1}(w) - \mu^{-1}(z)\right| \leq M\,|w - z| \qquad (z \in \mu(\gamma), |w - z| < \frac{\delta}{2}).$$

Hence also, if $r < \frac{\delta}{2}$

$$\mu^{-1}(N_r(z)) \subseteq N_{Mr}(\mu^{-1}(z)).$$

Now

$$N_r(z) \cap \mu(\gamma) \subseteq \mu(N_{Mr}(\mu^{-1}(z)) \cap \gamma),$$

so

$$\begin{aligned} l(N_r(z) \cap \mu(\gamma)) &\leq \int_{N_{Mr}(\mu^{-1}(z)) \cap \gamma} |\mu'(\zeta)|\,|d\zeta| \\ &\leq \|\mu'\|_\infty\, l(N_{Mr}(\mu^{-1}(z)) \cap \gamma) \\ &\leq (\|\mu'\|_\infty\, \alpha M)\, r \qquad\qquad (r < \frac{\delta}{2}), \end{aligned}$$

where $\|\mu'\|_\infty$ is the uniform norm of $\mu'$ on $\gamma$. Hence for all $z \in \mu(\gamma)$ and $r > 0$

$$l(N_r(z) \cap \mu(\gamma)) \leq \max\left(\|\mu'\|_\infty\, \alpha M, \frac{2l(\mu(\gamma))}{\delta}\right) r.$$

Theorem 5.3. *David's theorem (([10], theorem 8, chapter 12) or [4]) Let $\gamma$ be an Ahlfors regular Jordan curve with inner domain $\Omega$ and outer domain $G$. Let $R_\Omega$ be the set of all rational functions with no poles in $\overline{\Omega}$ and let $R_G$ be the set of all rational functions which vanish at $\infty$ and have no poles in $\overline{G}$. Then*

$$L^2(\gamma) = \overline{R_\Omega} \oplus \overline{R_G}$$

*where $\overline{R_\Omega}, \overline{R_G}$ are the closures of $R_\Omega, R_G$ respectively in $L^2(\gamma)$.*

To make use of David's theorem one more lemma is needed. It applies to a general rectifiable Jordan curve.

LEMMA 5.4. *Let $\gamma$ be a rectifiable Jordan curve with inner domain $\Omega$ and outer domain $G$,and let $R_\Omega, R_G$ be as in Theorem 5.3.*

**i:** $\overline{R_\Omega} \subseteq E^2(\partial\Omega)$ *and* $\overline{R_G} \subseteq E^2(\partial G)$.

**ii:** *If* $f_1 \in E^2(\partial\Omega)$ *and* $f_2 \in E^2(\partial G)$ *then*

$$C_\gamma f_1(z) \equiv 0 \text{ on } G \text{ and } C_\gamma f_2(z) \equiv 0 \text{ on } \Omega.$$

**iii:** $E^2(\partial\Omega) \cap E^2(\partial G) = (0)$.

PROOF. We show first that $H^\infty(\Omega) \subseteq E^2(\Omega)$. Let $g \in H^\infty(\Omega)$ and let $\varphi : \Omega \to \Delta$ be an RMF for $\Omega$ with inverse $\psi : \Delta \to \Omega$. By definition, $g \in E^2(\Omega)$ iff $g = (f \circ \varphi)\varphi'^{1/2}$ for some $f \in H^2(\Delta)$, equivalently iff $(g \circ \psi)\psi'^{1/2} \in H^2(\Delta)$. But $g \circ \psi \in H^\infty(\Delta)$ and $\psi'^{1/2} \in H^2(\Delta)$ by Theorem 3.1 a. Taking non-tangential limits gives $R_\Omega \subseteq H^\infty(\partial\Omega) \subseteq E^2(\partial\Omega)$, so $\overline{R_\Omega} \subseteq E^2(\partial\Omega)$, because $E^2(\partial\Omega)$ is closed in $L^2(\gamma)$.

To prove that $\overline{R_G} \subseteq E^2(\partial G)$ choose $a \in \Omega$ and let $X$ be the set of all function on $G$ of the form

$$g(z) = f(z)\frac{1}{z-a} \quad \text{with } f \in H^\infty(G).$$

It is sufficient to show that $X \subseteq E^2(\partial G)$ because $R_G \subseteq X$. Choose $a \in \Omega$ and let $\mu(z) = \dfrac{1}{z-a}$. Then $\mu(\gamma)$ is a rectifiable Jordan curve with inner domain $\mu(G)$, so that $H^\infty(\mu(G)) \subseteq E^2(\mu(G))$, as above. Let $U = U_\mu : E^2(\mu(G)) \to E^2(G)$ be the unitary operator

$$Uh(z) = h(\mu(z))\mu'(z)^{1/2} \qquad\qquad (g \in E^2(\mu(G)), z \in G).$$

The map $h \to h \circ \mu$ is obviously an isomorphism of $H^\infty(\mu(G))$ onto $H^\infty(G)$, and $\mu'(z)^{1/2}$ is a multiple of $\dfrac{1}{z-a}$. So $X \subseteq E^2(G)$ and i is proved.

To prove ii let $f_1 \in E^2(\partial\Omega)$ and $z \in G$. Let $R(\zeta) = \dfrac{1}{\zeta - z}$, then $R \in E^2(\partial\Omega)$, by i, and

$$\begin{aligned} C_\gamma f_1(z) &= \frac{1}{2\pi i}\int_\gamma f_1(\zeta)R(\zeta)d\zeta \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} f_1(\psi(u))R(\psi(u))\psi'(u)du. \end{aligned}$$

The last integrand here is in $H^1(\mathbb{T})$, because it is the product of two elements of $H^2(\mathbb{T}) : (f_1 \circ \psi)\psi'^{1/2}$ and $(R \circ \psi)\psi'^{1/2}$. Hence $C_\gamma f_1(z) = 0$ by equation (3.2) with $m = 0$. The same argument shows that $C_\gamma f_2 \equiv 0$ on $\Omega$.

The proof of iii is more subtle. Suppose that $m$ is a (finite) regular Borel measure with compact support $X \subseteq \mathbb{C}$. The "Cauchy transform" $\widehat{m}$ of $m$ is defined by

$$\widehat{m}(z) = \frac{1}{2\pi i}\int_X \frac{dm(\zeta)}{\zeta - z}.$$

It is defined almost everywhere on $\mathbb{C}$. The Cauchy transform is one of the main tools in the study of function algebras. One of its key properties is

$$m = 0 \text{ if and only if } \widehat{m}(z) = 0, \text{ for almost all } z \in \mathbb{C} \tag{5.1}$$

(see [3, p.155]). Now suppose $f \in E^2(\partial\Omega) \cap E^2(\partial G)$ and let $m$ be the measure on $\gamma$ :

$$dm(\zeta) = f(\zeta)d\zeta = f(\zeta)\omega(\zeta)\,|d\zeta|\,,$$

where $\omega(\zeta)$ is the unit tangent vector of $\gamma$ at $\zeta$. Clearly $\widehat{m}(z) = C_\gamma f(z)$, for all $z \in \mathbb{C} - \gamma$. Now $\gamma$ is rectifiable, so it has plane measure 0. It follows from ii that $\widehat{m} = 0$ a.e. Therefore $m = 0$, so $f = 0$.

THEOREM 5.5. *Let $\gamma$ be an Ahlfors regular Jordan curve with inner domain $\Omega$ and outer domain $G$.*

**i:** $L^2(\gamma) = E^2(\partial\Omega) \oplus E^2(\partial G)$.

**ii:** *The map $f \to C_\gamma f|_\Omega$ is a continuous linear operator mapping $L^2(\gamma)$ into $E^2(\Omega)$ and the map $f \to C_\gamma f|_G$ is a continuous linear operator mapping $L^2(\gamma)$ into $E^2(G)$.*

PROOF. Let $R_\Omega, R_G$ be as before. Let $f \in E^2(\partial\Omega)$. Then, by David's theorem $f = g + h$, where $g \in \overline{R_\Omega} \subseteq E^2(\partial\Omega)$ and $h \in \overline{R_G} \subseteq E^2(\partial G)$. Therefore $h = f - g \in E^2(\partial\Omega) \cap E^2(\partial G)$ and so $h = 0$ by Lemma 5.4 iii. So $E^2(\partial\Omega) = \overline{R_\Omega}$ and likewise $E^2(\partial G) = \overline{R_G}$. Hence $L^2(\gamma) = E^2(\partial\Omega) \oplus E^2(\partial G)$.

Now, for $f \in L^2(\gamma)$, let us write $f = f_1 + f_2$ with $f_1 \in E^2(\partial\Omega), f_2 \in E^2(\partial G)$. By Banach's theorem the map

$$f \to f_1 \qquad (L^2(\gamma) \to E^2(\partial\Omega))$$

is continuous. By Lemma 5.4 ii

$$C_\gamma f|_\Omega = C_\gamma f_1|_\Omega.$$

By b and c of Theorem 3.1, $C_\gamma f_1|_\Omega \in E^2(\Omega)$ and

$$\|C_\gamma f_1|_\Omega\|_{E^2(\Omega)} \le M_\Omega \|f\|_{L^2(\gamma)}$$

for some constant $M_\Omega > 0$.

If $a \in \Omega$ and $\mu(z) = \dfrac{1}{z-a}$ then $\mu(G)$ is the inner domain of $\mu(\gamma)$, an Ahlfors regular curve, by Corollary 5.2. Our previous argument shows that

$$f \to C_{\mu(\gamma)} f|_{\mu(G)}$$

is a continuous linear operator mapping $L^2(\mu(\gamma))$ into $E^2(\mu(G))$. Hence, by the invariance Lemma 4.5

$$f \to C_\gamma f|_G$$

is a continuous linear operator mapping $L^2(\gamma)$ into $E^2(G)$.

## 6. Completion of the proof of Theorem 4.2

We shall find sufficient conditions for a Jordan curve to be Ahlfors regular and show that they apply to $C_j$ or to some fractional linear image $\mu(C_j)$ of $C_j$ (see Lemma 4.5).

DEFINITION 4. *A (simple) rectifiable arc $\gamma$ with arc length parametrization $h \in C^1[0, L]$ is called a Lavrentiev arc if there is an $\alpha > 0$ such that*

$$|h(s) - h(t)| \ge \alpha\,|s - t| \qquad (s, t \in [0, L]).$$

THEOREM 6.1. **i:** *A simple smooth arc is a Lavrentiev arc.*

**ii:** *A Lavrentiev arc is Ahlfors regular.*

**iii:** *If the arc $\gamma$ is the union $\gamma_1 \cup \gamma_2 \cup ... \cup \gamma_n$ of Ahlfors regular arcs then $\gamma$ itself is Ahlfors regular.*

**iv:** *A piecewise smooth Jordan curve is Ahlfors regular.*

PROOF. Recall, from Remark 3.3, that $\gamma$ is rectifiable with total length $L$, say, and that its arc length parametrization $h \in C^1[0, L]$. Now suppose $\gamma$ is not a Lavrentiev arc. Then there are sequences $(s_n), (t_n) \subseteq [0, L]$ such that

$$|h(s_n) - h(t_n)| \leq \frac{1}{n} |s_n - t_n| \qquad (n \geq 1). \tag{6.1}$$

Assume that these two sequences are convergent, say $s_n \to s_0, t_n \to t_0$. It follows that $h(s_0) - h(t_0) = 0$, so that $s_0 = t_0$. Now, for all $n$

$$\begin{aligned} h(t_n) - h(s_n) &= \int_{s_n}^{t_n} h'(u)du \\ &= (t_n - s_n) \int_0^1 h'(s_n + v(t_n - s_n))dv \\ &\sim (t_n - s_n)h'(t_0), \end{aligned}$$

by Lebesgue's bounded convergence theorem. But $|h'(t_0)| = 1$, contradicting (6.1).

To prove ii let $z_0 = h(t) \in \gamma$, with $t \in [0, L]$. If $z = h(s)$ is a general point of $\gamma$ then, for $r > 0$,

$$\begin{aligned} z &\in N_r(z_0) \Leftrightarrow |h(s) - h(t)| < r \\ &\Rightarrow |s - t| < \frac{r}{\alpha} \\ &\Leftrightarrow s \in (t - \frac{r}{\alpha}, t + \frac{r}{\alpha}). \end{aligned}$$

Remembering that $s$ is arc length we see that

$$\begin{aligned} l(N_r(z_0) \cap \gamma) &\leq \text{arc length from } h(t - \frac{r}{\alpha}) \text{ to } h(t + \frac{r}{\alpha}) \\ &= \frac{2}{\alpha} r. \end{aligned}$$

To prove iii suppose that $\gamma = \gamma_1 \cup \gamma_2 \cup ... \cup \gamma_n$ where each $\gamma_i$ is Ahlfors regular, say

$$l(N_r(z_i) \cap \gamma_i) \leq k_i r \quad (z_i \in \gamma_i, r > 0). \tag{6.2}$$

Let $z_0 \in \gamma$, say $z_0 \in \gamma_i$. Then we certainly have

$$l(N_r(z_0) \cap \gamma_i) \leq k_i r \quad (r > 0)$$

and by Corollary 5.1

$$l(N_r(z_0) \cap \gamma_j) \leq 2k_j r \quad (j \neq i) \ .$$

Varying $i$ gives

$$l(N_r(z_0) \cap \gamma) \leq 2(k_1 + k_2 + ... + k_n)r \qquad (z_0 \in \gamma, r > 0).$$

Item iv is now clear and therefore so is.

Corollary 6.2. *If $C_j$ is an ellipse then Theorem 4.2 is true.*

Now suppose $C_j$ is a parabola. By the invariance Lemma 4.5, we can assume that $C_j$ is the parabola

$$y^2 = 4(1-x).$$

Now let $\mu(z) = \dfrac{1}{z}$. Another application of the same lemma and Theorem 6.1 iv shows that it is sufficient to show that $\mu(C_j)$ is piecewise smooth. Now $\mu(C_j)$ has a cusp at $0 = \mu(\infty)$. But we can show that the parts of $\mu(C_j)$ in the upper and lower half planes are smooth. The upper part of $\mu(C_i)$ has parametric function $g \in C^1[0,\infty)$

$$g(t) = \frac{1}{(1+it)^2} \qquad (t \geq 0),$$

so that $g'(t) = \dfrac{-2i}{(1+it)^3}$. Now $g(t) \to 0$ as $t \to +\infty$ and

$$\frac{g'(t)}{|g'(t)|} = -i\frac{|1+it|^3}{(1+it)^3} = -i\frac{\left|i+\frac{1}{t}\right|^3}{(i+\frac{1}{t})^3} \to 1$$

as $t \to +\infty$. So by Lemma 3.4, $g[0,\infty) \cup g(\infty)$ is smooth and, similarly $g(-\infty,0] \cup g(-\infty)$ is smooth.

Finally, let $C_j$ be a hyperbola component, say

$$C_j = \sin(\alpha + i\mathbb{R}), \text{ where } 0 < \alpha < \frac{\pi}{2}.$$

Again put $\mu(z) = \dfrac{1}{z}$. The upper part of $\mu(C_j)$ is parametrized by

$$g(t) = \frac{1}{\sin\alpha\cosh t + i\cos\alpha\sinh t} \qquad (t \geq 0).$$

Again $g(t) \to 0$ as $t \to +\infty$ and

$$\begin{aligned}\frac{g'(t)}{|g'(t)|} &= -\frac{(\sin\alpha\sinh t + i\cos\alpha\cosh t)}{|\sin\alpha\sinh t + i\cos\alpha\cosh t|}\frac{|\sin\alpha\cosh t + i\cos\alpha\sinh t|^2}{(\sin\alpha\cosh t + i\cos\alpha\sinh t)^2} \\ &= -\frac{(\sin\alpha\tanh t + i\cos\alpha)}{|\sin\alpha\tanh t + i\cos\alpha|}\frac{|\sin\alpha\coth t + i\cos\alpha|^2}{(\sin\alpha\coth t + i\cos\alpha)^2} \\ &\to \frac{-ie^{-i\alpha}}{1}\frac{1}{(ie^{-i\alpha})^2} = ie^{i\alpha}.\end{aligned}$$

Arguing as before we see that $\mu(C_j)$ is piecewise smooth.

Remark 6.3. *Theorem 1.1 creates the impression that an important role is to be played by the very specific curves mentioned; but on closer inspection of proof it can be seen that the argument runs on the lines*

*conic section ⇒ piecewise smooth ⇒ Lavrentief arc ⇒ Ahlfors regular*

*and then Ahlfors regularity is used to prove continuity of the mapping A.*

## 7. Conclusions and further work

In this paper we proved Theorem 1.1 in detail by showing the continuity of the operator $A$ which is followed by the continuity of the Cauchy integral operators (Theorem 4.2). Also we need Theorem 3.2 which shows that the operator $A$ exists.

There are some difficulties in generalizing Theorem 1.1 for more general case than conic sections. We raise some of them in the following questions. Under what conditions on an $N$-tuple $C = (C_1, C_2, C_3, ..., C_N)$ of closed distinct curves in the extended plane

(a) are the results of Theorem 3.2 true? (so that the operator $A$ exists)

and

(b) is Theorem 4.2 true?, i.e., is each Cauchy integral operators continuous?

i, ii and iii of Theorem 3.2 are consequences of being $\psi' \in H^1(\Delta)$ for the bounded case of $D$ and $\dfrac{\psi'}{(\psi - a)^2} \in H^1(\Delta)$ for the unbounded case of $D$ where $a \in \mathbb{C}\backslash\overline{D}$, see [12, Theorems 25 and 30]. Arsove [1, 2] gives a complete description of bounded simply connected domains $\Omega$ for which the derivative of a conformal mapping of the unit disk onto $\Omega$ belongs to $H^1$ . In fact, according to Theorem 1 of [1], if $\Omega$ is a bounded simply connected domain and $\chi$ is a Riemann mapping function for $\Omega$ then $\chi'$ is in the Hardy class $H^1$ if and only if $\partial\Omega$ can be parametrized as a rectifiable closed curve. This description shows that there is no real need to restrict oneself to domains bounded by pieces of quadrics so that Theorem 3.2 can be improved.

Therefore, it looks that the more difficult part of future work is to prove the continuity of the Cauchy integral operators (Theorem 4.2). Further work includes investigating more complex scenarios such as choosing all closed curves as analytic $\sigma$-rectifiable Jordan curve.

## References


[1] Arsove M.G., Some boundary properties of the Riemann mapping function, *Proc. AMS*, 19, No. 3, (1968), 560-568 .

[2] Arsove M.G., A correction to “Some boundary properties of the Riemann mapping function”, *Proc. AMS*, 22, No. 3, (1969), 711-712 .

[3] Browder A., *Introduction to Function Algebras* (W.A.Benjamin, Inc., New York, Amsterdam, 1969).

[4] David J., Opérateurs intégraux singuliers sur certaines courbes du plan complexe. *Ann. Sci. E.N.S.* **17** (1984), 157-189.

[5] Duren P. L., *Theory of $H^p$ Spaces* (Academic Press, New York, 1970).

[6] Goluzin G.M., *Functions of a Complex Variable* (Amer. Math. Soc., Providence, RI, 1969 (translated from Russian).

[7] Khavinson D., Factorization theorems for certain classes of analytic functions in multiply connected domains. *Pacific. J. Math.* **108** (1983), 295-318.

[8] Little G., Asymptotic estimates of the eigenvalues of certain positive Fredholm operators. *Math. Proc. Camb.Phil. Soc.* **91** (1982), 267-284.

[9] Little G., Equivalences of positive integral operators with rational kernels. *Proc. London. Math. Soc.* (3) **62** (1991), 403-426.

[10] Meyer Y., and Coifman R., *Wavelets, Calderon-Zygmund and multilinear operators* (Cambridge studies in advanced mathematics, 48, Translation of vols. 2-3 of : Ondelettes et operateurs by D.H. Salinger, 1997).

[11] Privalov I.I., *Boundary Properties of Analytic Functions* (Moscow, (1950)) (Russian; German translation: Randeigenschaften analytischer Funktionen, Deutscher Verlag der Wiss., Berlin, 1956).

[12] Soykan Y., Boundary Behaviour of Analytic Curves. *Int. Journal of Math. Analysis.* Vol. 1 no. **3** (2007), 133-157. [Online: http://www.m-hikari.com/ijma/ijma-password-2007/ijma-password1-4-2007/index.html].

[13] Soykan Y., On Equivalent Characterisation of Elements of Hardy and Smirnov Spaces. *Int. Math. Forum*, Vol. 2 no. **24** (2007), 1185-1191. [Online: http://www.m-hikari.com/imf-password2007/21-24-2007/index.html].