On *m*-rectangle Characteristics and Isomorphims of Mixed (F)-, (DF)- Spaces

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- Mixed (F)-, (DF)- Spaces
- Criteria for Quasidiagonal Isomorphisms
- *m*-rectangle Characteristics and Quasidiagonal Isomorphisms
- Invariance of *m*-rectangle Characteristics

Definition

Two locally convex spaces X and Y are called *isomorphic*, denoted by $X \simeq Y$, if there exists an operator from X into Y that is a linear bijective homeomorphism.
If $\mathcal{X}$ is a class of locally convex spaces and Γ is a set with an equivalence relation $\sim$, then $\gamma : \mathcal{X} \to \Gamma$ is called a *linear topological invariant* if $X \simeq Y$ implies $\gamma(X) \sim \gamma(Y)$ for all $X, Y \in \mathcal{X}$.

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Classical invariants for non-normable locally convex spaces:

- Approximative dimension (Kolmogorov 1958, Pełczyński 1957)
- Diametral Dimension (Bessaga, Pełczyński, Rolewicz 1961, Mityagin 1961)

Definition

A matrix $A = (a_{i,p})_{i,p\in\mathbb{N}}$ of non-negative numbers satisfying

(i) for each $i \in \mathbb{N}$ there exists $p = p(i)$ such that $a_{i,p} > 0$,

(ii) $a_{i,p} \leq a_{i,p+1}$ for all $i, p \in \mathbb{N}$,

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is called a *Köthe Matrix*. For a Köthe matrix A, the locally convex space $K(A)$ of all sequences $\xi = (\xi_i)_{i \in \mathbb{N}}$ with the locally convex topology generated by the system of seminorms $\{||.||_p : p \in \mathbb{N}\}$, where

$$||\xi||_p = \sum_{i \in \mathbb{N}} |\xi_i| a_{i,p} < \infty,$$

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is called *the Köthe space* defined by A.

$K(A)$ is a Fréchet space for any Köthe matrix $A = (a_{i,p})_{i,p \in \mathbb{N}}$.

An important subclass of Köthe spaces are the power series spaces.

Definition

For any positive sequence $a = (a_i)_{i \in \mathbb{N}}$,

$$E_\alpha(a) = \mathrm{proj}_{\lambda < \alpha} l_1(\exp(\lambda a))$$

where $-\infty < \alpha \leq \infty$, is called a *power series space of finite type* if $\alpha < \infty$, or a *power series space of infinite type* if $\alpha = \infty$.

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Without loss of generality, we only need to consider

$$E_0(a) = \mathrm{proj}_{\leftarrow p}\ l_1(\exp(-\frac{1}{p}a)),\ \ E_\infty(a) = \mathrm{proj}_{\leftarrow p}\ l_1(\exp(pa))$$

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Examples of power series spaces:

- The spaces of analytic functions $A(\mathbb{D})$, $A(\mathbb{C})$
- The space of rapidly decreasing sequences

$$s = E_\infty((\log i)_{i\in\mathbb{N}})$$

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Theorem (Mityagin 1961)

For positive sequences $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$ both increasing to infinity, the following statements are equivalent:

(i) $E_0(a) \simeq E_0(b)$.

(ii) $E_\infty(a) \simeq E_\infty(b)$.

(iii) *There exists a constant $C > 1$ such that $\frac{1}{C}a_i \leq b_i \leq Ca_i$ for all $i \in \mathbb{N}$.*

If a and b satisfy condition (*iii*), then we denote it by $a_i \asymp b_i$.

The isomorphic classification of non-Schwartz power series spaces were also investigated by Mityagin by analysing the counting functions $N_a(u,v) = |\{i \in \mathbb{N} : u \leq a_i \leq v\}|$, $0 \leq u \leq v < \infty$.

The isomorphic classification of non-Schwartz power series spaces were also investigated by Mityagin by analysing the counting functions $N_a(u,v) = |\{i \in \mathbb{N} : u \leq a_i \leq v\}|$, $0 \leq u \leq v < \infty$.

Theorem (Mityagin 1970, 1983)

For positive sequences $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$, the following conditions are equivalent:

(i) $E_0(a) \simeq E_0(b)$.

(ii) $E_\infty(a) \simeq E_\infty(b)$.

(iii) *There exists a constant $R > 0$ such that for any u, v,* $0 \leq u \leq v < \infty$,

$$N_a(u,v) \leq N_b(Ru, \frac{v}{R}), \quad N_b(u,v) \leq N_a(Ru, \frac{v}{R}).$$

A related question in isomorphic classification of locally convex spaces is whether a locally convex space has the *quasi-equivalence property*, that is, if any two bases in a locally convex space are quasiequivalent.

Definition

Two bases (e_n) and (f_n) of a locally convex space X are called *quasiequivalent* if the operator $T : X \to X$ where $Te_n = t_n f_{\sigma(n)}$ for some sequence of scalars (t_n) and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ for every $n \in \mathbb{N}$ is an isomorphism.

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Examples of spaces with the quasi-equivalence property:

- $A(\mathbb{D})$ (Dragilev 1958)
- Power series spaces (Mityagin 1961)
- Nuclear Fréchet spaces in classes (d_1) and (d_2) with regular bases (Dragilev 1965)
- Cartesian products of spaces in classes (d_1) and (d_2) (Zakharyuta 1973)

A sequence (x_n) in a locally convex space X is called a *(Schauder) basis*, if for each x in X there is a unique sequence of scalars (t_n) such that $x = \sum t_n x_n$, where the sum converges in the topology of X.

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$$\frac{||e_i||_p}{||e_i||_{p+1}} \geq \frac{||e_{i+1}||_p}{||e_{i+1}||_{p+1}}.$$

Theorem (Crone - Robinson 1974, Kondakov 1974)

Every nuclear Fréchet space with a regular basis has the quasi-equivalence property.

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In distinguishing spaces without a regular basis, the diametral dimensions are not very efficient.

Example (Mityagin 1961, Rolewicz 1962)

The cartesian product $A(\mathbb{D}) \times A(\mathbb{C})$ has no regular basis and $A(\mathbb{D})$ and $A(\mathbb{D}) \times A(\mathbb{C})$ are non-isomorphic. However, $\Gamma'(A(\mathbb{D})) = \Gamma'(A(\mathbb{D}) \times A(\mathbb{C}))$.

To investigate spaces without a regular basis, generalized linear topological invariants called *compound invariants* were introduced by Zakharyuta, and used in several joint papers by Chalov, Djakov, Terzioğlu, Yurdakul and Zakharyuta in the isomorphic classification of cartesian and tensor products of power series spaces, and the *power Köthe spaces of first type*

$$E(\lambda, a) = K\left(\exp\left(\left(-\frac{1}{p} + p\lambda_i\right) a_i\right)\right),$$

where $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $a = (a_i)_{i\in\mathbb{N}}$ are positive sequences.

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where $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $a = (a_i)_{i\in\mathbb{N}}$ are positive sequences. An important invariant in this consideration is the m-rectangle characteristics, which compute the number of the points (λ_i, a_i) that are inside the union of m-rectangles.

Compound invariants were also used in several joint papers by Goncharov, Terzioğlu and Zakharyuta in isomorphic classification of complete projective tensor products of power series spaces with the (DF)- power series spaces.

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Definition

For a sequence of positive numbers $a = (a_i)_{i \in \mathbb{N}}$,

$$E_0'(a) = \mathrm{ind}_{q\to} \; l_1(\exp(\frac{1}{q}a))$$

is called a *dual power series space of finite type*, and

$$E_\infty'(a) = \mathrm{ind}_{q\to} \; l_1(\exp(-qa))$$

is called a *(DF)- power series space of infinite type*.

Problems on isomorphic classification and quasi-equivalence of bases of the class of mixed (F)-, (DF)- spaces

$$G(\lambda, a) = \mathrm{proj}_{\leftarrow p}\big(\mathrm{ind}_{q\rightarrow}\ l_1\left(\omega(p,q)\right)\big),$$

where $\omega_i(p,q) = \exp\left((p - q\lambda_i)\, a_i\right)$ for sequences of positive numbers $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$, which includes the basis subspaces of the tensor products

$$E_\infty(c)\hat{\otimes}E'_\infty(d),$$

were investigated by Chalov, Terzioğlu and Zakharyuta (1998), and it was shown that for each $m \in \mathbb{N}$, the m-rectangle characteristic is a linear topological invariant for this class under some equivalence.

Mixed (F)-, (DF)- Spaces

We consider the classes of mixed (F)-, (DF)- spaces

$$G_{\alpha,\beta}(\lambda, a) = \mathrm{proj}_{\leftarrow p} \left(\mathrm{ind}_{q\rightarrow}\ l_1 \left(\omega^{\alpha,\beta}(p,q) \right) \right)$$

for $\alpha, \beta \in \{0, \infty\}$ with $p, q \in \mathbb{N}$ and $\omega^{\alpha,\beta}(p,q) = (\omega_i^{\alpha,\beta}(p,q))_{i\in\mathbb{N}}$ when

1. $\omega_i^{\infty,\infty}(p,q) = \exp\left(\left(p - q\lambda_i\right) a_i\right)$,
2. $\omega_i^{0,\infty}(p,q) = \exp\left(\left(-\frac{1}{p} - q\lambda_i\right) a_i\right)$,
3. $\omega_i^{0,0}(p,q) = \exp\left(\left(-\frac{1}{p}\lambda_i + \frac{1}{q}\right) a_i\right)$,
4. $\omega_i^{\infty,0}(p,q) = \exp\left(\left(p\lambda_i + \frac{1}{q}\right) a_i\right)$,

where $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ are sequences of positive numbers.

Mixed (F)-, (DF)- Spaces

Here, $l_1\left(\omega^{\alpha,\beta}(p,q)\right)$ denote the weighted l_1 spaces

$$l_1\left(\omega^{\alpha,\beta}(p,q)\right)=\left\{x=(\xi_i)_{i\in\mathbb{N}}:||x||_{p,q}=\sum_{i=1}^{\infty}|\xi_i|\omega_i^{\alpha,\beta}(p,q)<\infty\right\}.$$

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For each $p\in\mathbb{N}$, we put $X_p:=\bigcup_{q\in\mathbb{N}}l_1\left(\omega_i^{\alpha,\beta}(p,q)\right)$, equipped with the inductive limit topology, that is, the finest locally convex topology for which the inclusion maps $i_q:l_1\left(\omega^{\alpha,\beta}(p,q)\right)\rightarrow X_p$ are continuous.

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Then, $G_{\alpha,\beta}(\lambda, a) = \operatorname{proj}_{\leftarrow p} X_p$ endowed with the projective limit topology, that is, the coarsest topology for which the inclusion maps $\pi_p : G_{\alpha,\beta}(\lambda, a) \to X_p$ are continuous.

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Then, $G_{\alpha,\beta}(\lambda,a)=\operatorname{proj}_{\leftarrow p} X_p$ endowed with the projective limit topology, that is, the coarsest topology for which the inclusion maps $\pi_p: G_{\alpha,\beta}(\lambda,a)\to X_p$ are continuous.

For any space $G_{\alpha,\beta}(\lambda,a)$ in this class, the coordinate basis $\{e_n : n\in\mathbb{N}\}$ is an absolute basis. A subspace of $G_{\alpha,\beta}(\lambda,a)$ that is generated by a subset of the coordinate basis is called a *basis (step) subspace*.

Given two Hausdorff locally convex spaces X and Y, we denote by $X\hat{\otimes}_\pi Y$ the complete projective tensor product of X and Y, that is, the completion of the finest locally convex topology on $X \otimes Y$ for which the canonical bilinear map $\otimes : X \times Y \to X \otimes Y$ is continuous.

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Proposition

The classes of spaces $(1) - (4)$, *up to isomorphisms, consist of basis subspaces of projective tensor products*

1. $E_\infty(c)\hat{\otimes}E'_\infty(d)$
2. $E_0(c)\hat{\otimes}E'_\infty(d)$
3. $E_0(c)\hat{\otimes}E'_0(d)$
4. $E_\infty(c)\hat{\otimes}E'_0(d)$

respectively, where $c = (c_i)_{i\in\mathbb{N}}$ *and* $d = (d_i)_{i\in\mathbb{N}}$ *are sequences of positive numbers.*

Two locally convex topological vector spaces X, Y, with respective absolute bases $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$, are called *quasidiagonally isomorphic*, denoted by $X \overset{qd}{\simeq} Y$, if there exists a locally convex space isomorphism $T : X \to Y$ such that

$$Tx_i = t_i y_{\sigma(i)}$$

for a sequence of scalars (t_i), and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

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$$Tx_i = t_i y_{\sigma(i)}$$

for a sequence of scalars (t_i), and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$.
If T is a quasidiagonal isomorphism such that $t_i = 1$ for all $i \in \mathbb{N}$, then X and Y are called *permutationally isomorphic*, denoted by $X \overset{p}{\simeq} Y$. If T is a quasidiagonal isomorphism such that $\sigma(i) = i$ for all $i \in \mathbb{N}$, then X and Y are called *diagonally isomorphic*, denoted by $X \overset{d}{\simeq} Y$.

For Montel spaces $G_{\infty,\infty}(\lambda, a)$ belonging to class (1) where $\omega_i^{\infty,\infty}(p,q) = \exp\left((p - q\lambda_i)\, a_i\right)$, we have the following criteria for quasidiagonal isomorphisms.

Proposition (Chalov, Terzioğlu, Zakharyuta 1998)

For Montel spaces $G_{\infty,\infty}(\lambda, a)$ and $G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$, TFAE:

(i) $G_{\infty,\infty}(\lambda, a) \overset{p}{\simeq} G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$

(ii) $G_{\infty,\infty}(\lambda, a) \overset{qd}{\simeq} G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$

(iii) *there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$a_i \asymp \tilde{a}_{\sigma(i)},$$

and for any subsequence (i_k) of $\mathbb{N}$,

$$(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0.$$

For Montel spaces $G_{0,\infty}(\lambda, a)$ that are in class (2) where $\omega_i^{0,\infty}(p,q) = \exp\left(\left(-\frac{1}{p} - q\lambda_i\right) a_i\right)$, the same criteria hold for quasidiagonal isomorphisms.

Proposition

For Montel spaces $G_{0,\infty}(\lambda, a)$ and $G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, TFAE:

(i) $G_{0,\infty}(\lambda, a) \overset{p}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a})$.

(ii) $G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a})$.

(iii) *There exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$a_i \asymp \tilde{a}_{\sigma(i)},$$

and for any subsequence (i_k) of $\mathbb{N}$,

$$(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0.$$

For the spaces $G_{0,0}(\lambda, a)$ belonging to class (3) where $\omega_i^{0,0}(p,q) = \exp\left((-\frac{1}{p}\lambda_i + \frac{1}{q})a_i\right)$, we have slightly different criteria for quasidiagonal isomorphisms.

Proposition

For Montel spaces $G_{0,0}(\lambda, a)$ and $G_{0,0}(\tilde{\lambda}, \tilde{a})$, TFAE:

(i) $G_{0,0}(\lambda, a) \overset{p}{\simeq} G_{0,0}(\tilde{\lambda}, \tilde{a})$

(ii) $G_{0,0}(\lambda, a) \overset{qd}{\simeq} G_{0,0}(\tilde{\lambda}, \tilde{a})$.

(iii) *There exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, a constant $\Delta > 1$, and a strictly decreasing function $\Psi : [1, \infty) \to \mathbb{R}^+$, $\Psi(t) \to 0$ as $t \to \infty$ such that $a_i \asymp \tilde{a}_{\sigma(i)}$, $(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, and*

$$\frac{1}{\Delta}\lambda_i \leq \tilde{\lambda}_{\sigma(i)} \leq \Delta\lambda_i \text{ for } \lambda_i \geq \Psi(a_i).$$

The same criteria for quasidiagonal isomorphisms hold for spaces $G_{\infty,0}(\lambda, a)$ in the class (4) where $\omega_i^{\infty,0}(p,q) = \exp\left((p\lambda_i + \frac{1}{q})a_i\right)$.

Proposition

For Montel spaces $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$, TFAE:

(i) $G_{\infty,0}(\lambda, a) \overset{p}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a})$,

(ii) $G_{\infty,0}(\lambda, a) \overset{qd}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a})$,

(iii) *There exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, a constant $\Delta > 1$, and a strictly decreasing function $\Psi : [1, \infty) \to \mathbb{R}^+$, $\Psi(t) \to 0$ as $t \to \infty$ such that $a_i \asymp \tilde{a}_{\sigma(i)}$, $(\lambda_{i_k}) \to 0 \iff (\tilde{\lambda}_{\sigma(i_k)}) \to 0$ for any subsequence (i_k) of $\mathbb{N}$, and*

$$\frac{1}{\Delta}\lambda_i \leq \tilde{\lambda}_{\sigma(i)} \leq \Delta\lambda_i \text{ for } \lambda_i \geq \Psi(a_i).$$

m-rectangle characteristics

Definition

If $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $a = (a_i)_{i\in\mathbb{N}}$ be sequences of positive numbers and $m \in \mathbb{N}$, then the function

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) \;=\; \left|\bigcup_{k=1}^{m}\{i \;:\; \delta_k \le \lambda_i \le \varepsilon_k \;,\; \tau_k \le a_i \le t_k\}\right|$$

defined for $\delta = (\delta_k)_{k=1}^m$, $\varepsilon = (\varepsilon_k)_{k=1}^m$, $\tau = (\tau_k)_{k=1}^m$ and $t = (t_k)_{k=1}^m$ such that $0 \le \delta_k \le \varepsilon_k \le 2$ and $0 < \tau_k \le t_k < \infty$ where $k = 1, 2, \cdots, m$, is called the *m-rectangle characteristic* of the pair (λ, a).

Here, $|S|$ denotes the number of elements of a set S if the S is finite, and equal to ∞ if S is infinite.

Given another couple of positive sequences $\tilde{\lambda} = (\tilde{\lambda}_i)$ and $\tilde{a} = (\tilde{a}_i)$, and $m \in \mathbb{N}$, the functions $\mu_m^{(\lambda,a)}$ and $\mu_m^{(\tilde{\lambda},\tilde{a})}$ are said to be *equivalent*, denoted by $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$, if there exists a strictly increasing function $\varphi : [0,2] \to [0,1]$ with $\varphi(0) = 0$ and $\varphi(2) = 1$, and a positive constant α such that the inequalities

$$\begin{aligned} \mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) &\leq \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right) \\ \mu_m^{(\tilde{\lambda},\tilde{a})}(\delta,\varepsilon;\tau,t) &\leq \mu_m^{(\lambda,a)}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right) \end{aligned}$$

hold with $\varphi(\delta) = (\varphi(\delta_k))$, $\varphi^{-1}(\varepsilon) = (\varphi^{-1}(\varepsilon_k))$, $\frac{\tau}{\alpha} = (\frac{\tau_k}{\alpha})$, $\alpha t = (\alpha t_k)$ for all collections of parameters $\delta, \varepsilon, \tau, t$.

Given another couple of positive sequences $\tilde{\lambda} = (\tilde{\lambda}_i)$ and $\tilde{a} = (\tilde{a}_i)$, and $m \in \mathbb{N}$, the functions $\mu_m^{(\lambda,a)}$ and $\mu_m^{(\tilde{\lambda},\tilde{a})}$ are said to be *equivalent*, denoted by $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$, if there exists a strictly increasing function $\varphi : [0,2] \to [0,1]$ with $\varphi(0) = 0$ and $\varphi(2) = 1$, and a positive constant α such that the inequalities

$$
\begin{aligned}
\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) &\leq \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right) \\
\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta,\varepsilon;\tau,t) &\leq \mu_m^{(\lambda,a)}\left(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t\right)
\end{aligned}
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The systems of characteristics $(\mu_m^{(\lambda,a)})_{m\in\mathbb{N}}$ and $(\mu_m^{(\tilde{\lambda},\tilde{a})})_{m\in\mathbb{N}}$ are said to be *equivalent*, denoted by $\left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$, if the function φ and the constant α can be chosen so that the inequalities above hold for all $m \in \mathbb{N}$.

This equivalence of systems of *m*-rectangle characteristics completely characterize quasidiagonal isomorphisms between the Montel spaces $G_{\infty,\infty}(\lambda, a)$ in class (1).

Theorem (Chalov, Terzioğlu, Zakharyuta 1998)

For Montel spaces $G_{\infty,\infty}(\lambda, a)$ and $G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$,

$$G_{\infty,\infty}(\lambda, a) \overset{qd}{\simeq} G_{\infty,\infty}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

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We obtain an analogous result for the class of Montel spaces $G_{0,\infty}(\lambda, a)$ in class (2).

Theorem

For Montel spaces $G_{0,\infty}(\lambda, a)$ and $G_{0,\infty}(\tilde{\lambda}, \tilde{a})$,

$$G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \sim \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

For any $m \in \mathbb{N}$, we again call the m-rectangle characteristics equivalent, and this time denote by $\mu_m^{(\lambda,a)} \approx \mu_m^{(\tilde{\lambda},\tilde{a})}$, if there exists a constant $c > 1$, a strictly decreasing function $\Psi : [1,\infty) \to (0,\infty)$ where $\Psi(\xi) \to 0$ as $\xi \to \infty$, and a strictly increasing function $\varphi : [0,2] \to [0,1]$ where $\varphi(0) = 0$, $\varphi(2) = 1$ and $\varphi(\xi) < \frac{\xi}{c}$ for all $\xi \in [0,2]$, such that the inequalities

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}(\Phi_1(\delta,\tau),\Phi_2(\delta,\varepsilon,\tau);\frac{\tau}{c},ct) \qquad (1)$$

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta,\varepsilon;\tau,t) \leq \mu_m^{(\lambda,a)}(\Phi_1(\delta,\tau),\Phi_2(\delta,\varepsilon,\tau);\frac{\tau}{c},ct) \qquad (2)$$

hold for all collections of parameters $\delta,\varepsilon,\tau,t$, where $\varphi(\delta) = (\varphi(\delta_k))$, $\varphi^{-1}(\varepsilon) = (\varphi^{-1}(\varepsilon_k))$, $\frac{\tau}{\alpha} = (\frac{\tau_k}{\alpha})$, $\alpha t = (\alpha t_k)$,

and the functions $\Phi_1(\delta, \tau) = (\Phi_1(\delta_k, \tau_k))$,
$\Phi_2(\delta, \varepsilon, \tau) = (\Phi_2(\delta_k, \varepsilon_k, \tau_k))$ defined as

$$\Phi_1(\delta_k, \tau_k) = \begin{cases} \frac{\delta_k}{c} & \text{if } \delta_k \geq \Psi(\tau_k), \\ \varphi(\delta_k) & \text{if } \delta_k < \Psi(\tau_k), \end{cases}$$

$$\Phi_2(\delta_k, \varepsilon_k, \tau_k) = \begin{cases} c\varepsilon_k & \text{if } \delta_k \geq \Psi(\tau_k), \\ \varphi^{-1}(\varepsilon_k) & \text{if } \delta_k < \Psi(\tau_k). \end{cases}$$

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The systems of characteristics $(\mu_m^{(\lambda,a)})_{m\in\mathbb{N}}$ and $(\mu_m^{(\tilde{\lambda},\tilde{a})})_{m\in\mathbb{N}}$ are then called *equivalent*, denoted by $\left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right)$ if the constant c and the functions Ψ, φ can be chosen so that the inequalities above hold for all $m \in \mathbb{N}$.

With this equivalence of systems of *m*-rectangle characteristics, we completely characterize the quasidiagonal isomorphisms between the Montel spaces $G_{0,0}(\lambda, a)$ in class (3) and the Montel spaces $G_{\infty,0}(\lambda, a)$ in class (4).

Theorem

For Montel spaces $G_{0,0}(\lambda, a)$ and $G_{0,0}(\tilde{\lambda}, \tilde{a})$,

$$G_{0,0}(\lambda, a) \overset{qd}{\simeq} G_{0,0}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

Theorem

For Montel spaces $G_{\infty,0}(\lambda, a)$ and $G_{\infty,0}(\tilde{\lambda}, \tilde{a})$,

$$G_{\infty,0}(\lambda, a) \overset{qd}{\simeq} G_{\infty,0}(\tilde{\lambda}, \tilde{a}) \iff \left(\mu_m^{(\lambda,a)}\right) \approx \left(\mu_m^{(\tilde{\lambda},\tilde{a})}\right).$$

Invariance of m-rectangle characteristics

For each $m \in \mathbb{N}$, the m-rectangle characteristic is a linear topological invariant for the class (1) where $\omega_i^{\infty,\infty}(p,q) = \exp\left((p - q\lambda_i)a_i\right)$.

Theorem (Chalov, Terzioğlu, Zakharyuta 1998)

If $G_{\infty,\infty}(\lambda, a) \simeq G_{\infty,\infty}(\tilde{\lambda}, \tilde{a})$, then $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$ for every $m \in \mathbb{N}$.

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We obtain an analogous result for the spaces in class (2) where $\omega_i^{0,\infty}(p,q) = \exp\left((-\frac{1}{p} - q\lambda_i)a_i\right)$.

Theorem

If $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, then $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$ for every $m \in \mathbb{N}$.

For this purpose, we need the following definitions and results:

Definition

Let X be a locally convex space space and U, V be absolutely convex sets in X. Then, the β-characteristics of V and U is defined by

$$\beta(V, U) := \sup_{L \in L_n} \{\dim L : U \cap L \subset V\}$$

where L_n is the collection of all finite dimensional subspaces of $\overline{\mathrm{span} V}$.

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where L_n is the collection of all finite dimensional subspaces of $\overline{\mathrm{span}\, V}$.

Remark: For absolutely convex sets U, V, $\tilde{U}$, $\tilde{V}$ of X, and $\alpha > 0$,

(a) $\beta(\alpha V, U) = \beta(V, \frac{1}{\alpha} U)$,

(b) if $V \subset \tilde{V}$ and $\tilde{U} \subset U$, then $\beta(V, U) \leq \beta(\tilde{V}, \tilde{U})$.

Definition

For a locally convex space X with an absolute basis $e = \{e_i\}_{i\in\mathbb{N}}$ and a sequence $a = (a_i)_{i\in\mathbb{N}}$ of positive numbers, the set

$$B^e(a) = \left\{ x = \sum_{i=1}^{\infty} \xi_i e_i \in X : \sum_{i=1}^{\infty} |\xi_i| a_i \leq 1 \right\}$$

is called *the weighted l_1-ball* with the weight sequence a with respect to the basis e.

Proposition

Let X be a locally convex space with an absolute basis $e = \{e_i\}_{i\in\mathbb{N}}$ and $a^{(j)} = (a_i^{(j)})$ be sequences of positive numbers for $j = 1, \cdots, m$. Then,

$$B^e(c) \subset \bigcap_{j=1}^{m} B^e(a^{(j)}) \subset mB^e(c), \quad B^e(d) = conv\left(\bigcup_{j=1}^{m} B^e(a^{(j)})\right),$$

where $c = (c_i)_{i\in\mathbb{N}}$ and $d = (d_i)_{i\in\mathbb{N}}$ are sequences such that $c_i = \max\{a_i^{(j)} : j = 1, \ldots, m\}$, $d_i = \min\{a_i^{(j)} : j = 1, \ldots, m\}$.

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Proposition

For sequences of positive numbers $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$,

$$\beta(B^e(a), B^e(b)) = |\{i \in \mathbb{N} : a_i \le b_i\}|.$$

Given an isomorphism $T : G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a}) \to G_{\alpha,\beta}(\lambda, a)$, consider the coordinate basis $e = (e_i)_{i\in\mathbb{N}}$ of $G_{\alpha,\beta}(\lambda, a)$ and the image of the coordinate basis $\tilde{e} = (\tilde{e}_i)_{i\in\mathbb{N}}$ of $G_{\alpha,\beta}(\tilde{\lambda}, \tilde{a})$ under T, and denote it by $f = (f_i)$, where $f_i = T\tilde{e}_i$, $i \in \mathbb{N}$.

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We define, for all $p, q \in \mathbb{N}$, the sets

$$A_{p,q} = \left\{ x = \sum_{i=1}^{\infty} \xi_i e_i \in G_{0,\infty}(\lambda, a) : \sum_{i=1}^{\infty} |\xi_i| \omega_i^{\alpha,\beta}(p, q) \leq 1 \right\},$$

$$\tilde{A}_{p,q} = \left\{ x = \sum_{i=1}^{\infty} \eta_i f_i \in G_{0,\infty}(\lambda, a) : \sum_{i=1}^{\infty} |\eta_i| \tilde{\omega}_i^{\alpha,\beta}(p, q) \leq 1 \right\},$$

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Lemma

For every $r \in \mathbb{N}$ there exists $p \geq r$ such that for every $q \in \mathbb{N}$ there exists $s \geq q$ and a constant $C > 1$ so that

$$A_{p,q} \subset C\tilde{A}_{r,s}, \quad \tilde{A}_{p,q} \subset CA_{r,s}.$$

Lemma

Let $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, and $m \in \mathbb{N}$. Then, there exists a strictly increasing function $\gamma : [0,2] \to [0,1]$ where $\gamma(0) = 0$ and $\gamma(2) = 1$, a decreasing function $M : (0,1] \to (0,\infty)$, and a constant $\alpha > 1$ such that

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right)$$

for all $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$ and $t = (t_k)$ where $0 < \delta_k \leq \varepsilon_k \leq 1$ and $0 < \tau_k \leq t_k < \infty$, $k = 1, \cdots, m$.

Sketch of proof: Choosing specific weighted l_1-balls $A_{p,q}$ and $\tilde{A}_{r,s}$ as building blocks, we geometrically construct absolutely convex sets U, V, $\tilde{U}$ and $\tilde{V}$ so that $\tilde{U} \subset U$ and $V \subset \tilde{V}$ by the previous lemma, and we have

$$\beta(V, U) \leq \beta(\tilde{V}, \tilde{U}).$$

Sketch of proof: Choosing specific weighted l_1-balls $A_{p,q}$ and $\tilde{A}_{r,s}$ as building blocks, we geometrically construct absolutely convex sets U, V, $\tilde{U}$ and $\tilde{V}$ so that $\tilde{U} \subset U$ and $V \subset \tilde{V}$ by the previous lemma, and we have

$$\beta(V, U) \leq \beta(\tilde{V}, \tilde{U}).$$

These sets may not be weighted l_1-balls, so we find weight sequences $c = (c_i)_{i\in\mathbb{N}}$, $d = (d_i)_{i\in\mathbb{N}}$, $\tilde{c} = (\tilde{c}_i)_{i\in\mathbb{N}}$, $\tilde{d} = (\tilde{d}_i)_{i\in\mathbb{N}}$ so that

$$B^e(c) \subset V,\ U \subset mB^e(d),\ \tilde{V} \subset 4B^f(\tilde{c}),\ B^f(\tilde{d}) \subset \tilde{U}$$

which implies

$$\beta(B^e(c), B^e(d)) \leq \beta(4mB^f(\tilde{c}), B^f(\tilde{d})).$$

Now we can compute the β-characteristics in terms of the weight sequences, that is,

$$\begin{aligned}\beta(B^e(c), B^e(d)) &= \left|\{i \in \mathbb{N} : c_i \leq d_i\}\right| \\ \beta(4mB^f(\tilde{c}), B^f(\tilde{d})) &= \left|\{i \in \mathbb{N} : \tilde{c}_i \leq 4m\tilde{d}_i\}\right|\end{aligned}$$

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Using these computations, we show that

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \beta(B^e(c), B^e(d))$$

and

$$\beta(4mB^f(\tilde{c}), B^f(\tilde{d})) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right)$$

for some strictly increasing function $\gamma : [0,2] \to [0,1]$ where $\gamma(0) = 0$ and $\gamma(2) = 1$, decreasing function $M : (0,1] \to (0,\infty)$, and constant $\alpha > 1$.

Using the main lemma for $2m$-rectangles, we obtain the following result for the spaces in class (2) where $\omega_i^{0,\infty}(p,q) = \exp\left((-\frac{1}{p} - q\lambda_i)a_i\right)$.

Theorem

If $G_{0,\infty}(\lambda, a) \simeq G_{0,\infty}(\tilde{\lambda}, \tilde{a})$, then $\mu_m^{(\lambda,a)} \sim \mu_m^{(\tilde{\lambda},\tilde{a})}$ for every $m \in \mathbb{N}$.

As an application of the invariance of *m*-rectangle characteristics, we obtain the following result for certain Montel spaces $G_{0,\infty}(\lambda, a)$ that are in class (2).

Proposition

If $G_{0,\infty}(\lambda, a)$ is Montel and $G_{0,\infty}(\lambda, a) \overset{qd}{\simeq} G_{0,\infty}(\lambda, a) \times G_{0,\infty}(\lambda, a)$, then the absolute bases in $G_{0,\infty}(\lambda, a)$ are pairwise quasiequivalent.

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