Compactness of the $\overline{\partial}$-Neumann operator

F. Haslinger

Fakultät für Mathematik, University of Vienna

Istanbul, November 22, 2013

Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and $\varphi : \Omega \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function and define the space

$$L^2(\Omega, e^{-\varphi}) = \{f : \Omega \longrightarrow \mathbb{C} \ : \ \|f\|_\varphi^2 = \int_\Omega |f|^2 \, e^{-\varphi} \, d\lambda < \infty\},$$

where λ denotes the Lebesgue measure.
Let $1 \leq q \leq n$ and

$$f = \sum_{|J|=q} {}' \, f_J \, d\overline{z}_J,$$

where $J = (j_1, \dots, j_q)$ and $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ and $f_J \in L^2(\Omega, e^{-\varphi})$.

We write $f \in L^2_{(0,q)}(\Omega, e^{-\varphi})$ and define

$$\overline{\partial} f = \sum_{|J|=q}{}' \sum_{j=1}^{n} \frac{\partial f_J}{\partial \overline{z}_j} \, d\overline{z}_j \wedge d\overline{z}_J$$

for $1 \leq q \leq n-1$ and

$$\mathrm{dom}\overline{\partial} = \{f \in L^2_{(0,q)}(\Omega, e^{-\varphi}) \, : \, \overline{\partial} f \in L^2_{(0,q+1)}(\Omega, e^{-\varphi})\}.$$

We consider the weighted $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\Omega, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\Omega, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\Omega, e^{-\varphi})$$

and we set

$$\Box_{\varphi,q} = \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial}.$$

Under suitable conditions $\Box_{\varphi,q}$ is surjective and there exists a bounded inverse to $\Box_{\varphi,q}$, which is denoted by

$$N_{\varphi,q} : L^2_{(0,q)}(\Omega, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\Omega, e^{-\varphi}).$$

For $\overline{\partial} u = f$ we get the canonical solution $\overline{\partial}^*_\varphi N_{\varphi,q} f$.

For many questions (for instance regularity) compactness of $N_{\varphi,q}$ is of special interest.

We will formulate a criterion for compactness. Let

$$j_{\varphi,q} : dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi) \longrightarrow L^2_{(0,q)}(\Omega, e^{-\varphi})$$

be the embedding, where $dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$ is endowed with the graph norm $u \mapsto (\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}$.

Then $N_{\varphi,q} = j_{\varphi,q} \circ j^*_{\varphi,q}$ and $N_{\varphi,q}$ is compact if and only if $j_{\varphi,q}$ is compact.
$j_{\varphi,q}$ is compact if and only if the unit ball in $dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$ is a compact subset in $L^2_{(0,q)}(\Omega, e^{-\varphi})$.

We use a characterization of compact subsets in L^2-spaces: a bounded subset $A \subset L^2(\Omega, e^{-\varphi})$ is precompact if and only if for each $\epsilon > 0$ there exist $\delta > 0$ and $\omega \subset\subset \Omega$ such that for each $u \in A$ and for each $h \in \mathbb{C}^n$ with $|h| < \delta$ we have

- (i) $\int_\omega |\tilde{u}(z+h) - \tilde{u}(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z) < \epsilon^2$, where $\tilde{u} = u$ in Ω and $\tilde{u} = 0$ outside of Ω,
- (ii) $\int_{\Omega \setminus \overline{\omega}} |u(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z) < \epsilon^2$.

Remark: condition (i) corresponds to Gårding's inequality in the interior of Ω.

Theorem

$N_{\varphi,q}$ is compact if and only if for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$(*) \quad \int_{\Omega \setminus \overline{\omega}} |u(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z) \leq \epsilon(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2)$$

for each $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}_\varphi^*)$.

Special cases: (a)

$q = 1$ and $\Omega = \mathbb{C}^n$. Suppose that φ is a plurisubharmonic $\mathcal{C}^2$-function such that

$$\lim_{|z|\to\infty} \mu_\varphi(z) = +\infty,$$

where μ_φ is the lowest eigenvalue of the Levi matrix M_φ of φ. For $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$ we have by the Kohn-Morrey formula and by our assumption

$$\begin{aligned}\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi &\geq \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, u_j \overline{u}_k\, e^{-\varphi}\, d\lambda \\ &\geq \int_{\mathbb{C}^n} \mu_\varphi(z)\, |u(z)|^2\, e^{-\varphi}\, d\lambda.\end{aligned}$$

Hence we get

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z)$$
$$\leq \int_{\mathbb{C}^n \setminus \mathbb{B}_R} \frac{\mu_\varphi(z) \, |u(z)|^2}{\inf\{\mu_\varphi(z) : |z| \geq R\}} \, e^{-\varphi(z)} \, d\lambda(z)$$
$$\leq \frac{\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2}{\inf\{\mu_\varphi(z) : |z| \geq R\}} < \epsilon(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2),$$

if R is large enough.
Hence condition (*) is satisfied and $N_{\varphi,1}$ is compact.

For $\varphi(z) = (|z_1|^2 + \cdots + |z_n|^2)^\alpha$, $\alpha > 1$ it turns out that

$$\lim_{|z|\to\infty} \mu_\varphi(z) = +\infty,$$

hence $N_{\varphi,1}$ is compact.

Let $\varphi(z_1, z_2) = |z_1|^2 + |z_2|^2$. We will investigate the following sequence of $(0,1)$-forms

$$u_k(z_1, z_2) = \psi_k(z_1)\, d\overline{z}_2,$$

where $\psi_k(z_1) = \frac{z_1^k}{\sqrt{\pi k!}}$, for $k \in \mathbb{N}$. It follows that $\overline{\partial} u_k = 0$ for each $k \in \mathbb{N}$ and

$$\overline{\partial}^*_\varphi u_k(z_1, z_2) = \overline{z}_2 \psi_k(z_1).$$

This implies

$$\Box_{\varphi,1} u_k = u_k \ \text{ and } \ N_{\varphi,1} u_k = u_k$$

for each $k \in \mathbb{N}$. The set $\{u_k : k \in \mathbb{N}\}$ is a bounded set of mutually orthogonal $(0,1)$-forms in $L^2_{(0,1)}(\mathbb{C}^2, e^{-\varphi})$. As $N_{\varphi,1} u_k = u_k$, it follows that $N_{\varphi,1}$ fails to be compact.

It can also directly be shown that condition (*) of the Theorem is not satisfied in this case.

(b)
$q = 1, \varphi = 0$ and Ω is a smoothly bounded pseudoconvex domain. Suppose that Ω satisfies
Property (P) (Catlin): For each $M > 0$ there exists a neighborhood U of $b\Omega$ and a plurisubharmonic function $\varphi_M \in \mathcal{C}^2(U)$ with $0 \leq \varphi_M \leq 1$ on U such that for each $p \in b\Omega$ and for each $t \in \mathbb{C}^n$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(p)\, t_j \bar{t}_k \geq M|t|^2.$$

By a similar reasoning as before one can show that property (P) implies condition (*) of the Theorem. Hence N_1 is compact.

Christ and Fu showed:
If $\Omega \subset \mathbb{C}^2$ is a smoothly bounded pseudoconvex Hartogs domain (i.e. with $(z, w) \in \Omega$ we have $(z, e^{i\theta} w) \in \Omega$ for every $\theta \in \mathbb{R}$), then the $\overline{\partial}$-Neumann operator N is compact if and only if property (P) holds.

(c)

$q > 1$ and $\Omega = \mathbb{C}^n$. Let s_q be the sum of the q smallest eigenvalues of M_φ. Suppose that

$$\lim_{|z|\to\infty} s_q(z) = +\infty.$$

For a $(0,q)$-form $u = \sum'_{|J|=q} u_J \, d\overline{z}_J \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$ we have by the Kohn-Morrey formula

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi$$
$$= \sum_{|J|=q}{}' \sum_{j=1}^n \left\| \frac{\partial u_J}{\partial \overline{z}_j} \right\|^2_\varphi + \sum_{|K|=q-1}{}' \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \, u_{jK} \overline{u}_{kK} \, e^{-\varphi} \, d\lambda.$$

The last term can be estimated from below

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_{jK} \overline{u}_{kK} \, e^{-\varphi} \, d\lambda$$

$$\geq \int_{\mathbb{C}^n} s_q(z) |u(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z),$$

by a similar reasoning as in (a) one gets that $\lim_{|z|\to\infty} s_q(z) = +\infty$ implies compactness of $N_{\varphi,q}$.

Example: We consider the plurisubharmonic weight function $\varphi(z, w) = |z|^2|w|^2 + |w|^4$ in $\mathbb{C}^2$. The Levi matrix of φ has the form

$$\begin{pmatrix} |w|^2 & \overline{z}w \\ \overline{w}z & |z|^2 + 4|w|^2 \end{pmatrix}$$

and the eigenvalues are

$$\mu_{\varphi,1}(z, w) = \frac{1}{2}\left(5|w|^2 + |z|^2 - \sqrt{9|w|^4 + 10|z|^2|w|^2 + |z|^4}\right)$$

and

$$\mu_{\varphi,2}(z, w) = \frac{1}{2}\left(5|w|^2 + |z|^2 + \sqrt{9|w|^4 + 10|z|^2|w|^2 + |z|^4}\right).$$

It follows that the lowest eigenvalue $\mu_{\varphi,1}$ does not tend to $+\infty$ as $(|z|^2 + |w|^2)^{1/2} \to \infty$, we even have

$$\lim_{|z|\to\infty} |z|^2 \mu_{\varphi,1}(z,0) = 0$$

but

$$s_2(z,w) = \frac{1}{4}\,\Delta\varphi(z,w) = |z|^2 + 5|w|^2 \to +\infty$$

as $(|z|^2 + |w|^2)^{1/2} \to \infty$, which implies that the corresponding $\overline{\partial}$-Neumann operator $N_{\varphi,2}$ is compact.

Remark:

It is interesting to compare the Sobolev imbedding

$$W^1(\Omega) \subset L^r(\Omega)\ ,\ r \in [1, 2n/(n-2))$$

where the derivatives are taken with respect of the real variables $x_j = \Re z_j$ and $y_j = \Im z_j$ for $j = 1, \dots, n$, with the imbedding of the space

$$\mathcal{W}(\Omega) := \{u \in L^2_{(0,1)}(\Omega) : u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*)\}$$

endowed with graph norm, into $L^2_{(0,1)}(\Omega)$. We have the following result

If $\Omega \subset\subset \mathbb{C}^n$ is a smoothly bounded pseudoconvex domain and the inequality

$$\|u\|_r \leq C(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$$

for some $r > 2$ and for all $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*)$ holds, then the $\overline{\partial}$-Neumann operator

$$N : L^2_{(0,1)}(\Omega) \longrightarrow L^2_{(0,1)}(\Omega)$$

is compact.

To show this we have to check that the unit ball in $\mathcal{W}(\Omega)$ is precompact in $L^2_{(0,1)}(\Omega)$. By the theorem from above, we have to show that for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\int_{\Omega \setminus \omega} |u(z)|^2 \, d\lambda(z) < \epsilon^2,$$

for all u in the unit ball of $\mathcal{W}(\Omega)$.

By the basic estimates

$$\|u\| \leq C(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$$

and Hölder's inequality we have

$$\begin{aligned}\left(\int_{\Omega\setminus\omega} |u(z)|^2 \, d\lambda(z)\right)^{\frac{1}{2}} &\leq \left(\int_{\Omega\setminus\omega} |u(z)|^q \, d\lambda(z)\right)^{\frac{1}{r}} \cdot |\Omega \setminus \omega|^{\frac{1}{2}-\frac{1}{r}} \\ &\leq C \, |\Omega \setminus \omega|^{\frac{1}{2}-\frac{1}{r}}.\end{aligned}$$

Now we can choose $\omega \subset\subset \Omega$ such that the last term is $< \epsilon$.

Determination of the spectrum

$1 \leq q \leq n$ and $\Omega = \mathbb{C}^n$ and $\varphi(z) = \sum_{j=1}^n |z_j|^2$.
Then the spectrum of $\Box_{\varphi,q}$ consists of all integers $\{q, q+1, q+2, \dots\}$ each of which is of infinite multiplicity. The essential spectrum is non-empty, therefore $N_{\varphi,q}$ fails to be compact.

For $n > 1$ and $1 \leq q \leq n-1$ the $\overline{\partial}$-Neumann Laplacian $\Box_{\varphi,q}$ acts diagonally: for

$$u = \sum_{|J|=q} {}' u_J \, d\overline{z}_J \in \operatorname{dom} \Box_{\varphi,q} \subseteq L^2_{(0,q)}(\mathbb{C}^n, e^{-|z|^2})$$

we have

$$\Box_{\varphi,q} \, u = (\overline{\partial}\, \overline{\partial}^*_{\varphi} + \overline{\partial}^*_{\varphi}\, \overline{\partial})\, u = \sum_{|J|=q} {}' \left(-\frac{1}{4} \triangle u_J + \sum_{j=1}^{n} \overline{z}_j \, u_{J\overline{z}_j} + q\, u_J \right) d\overline{z}_J .$$

Let $n = 1$ and $\varphi(z) = |z|^2$. The spectrum of $\Box_{\varphi,0}$ consists of all non-negative integers $\{0, 1, 2, \dots\}$ each of which is of infinite multiplicity, so 0 is the bottom of the essential spectrum. The spectrum of $\Box_{\varphi,1}$ consists of all positive integers $\{1, 2, 3, \dots\}$ each of which is of infinite multiplicity.

$$\Box_{\varphi,0}\, u = \overline{\partial}^*_\varphi\, \overline{\partial} u = -\frac{1}{4}\, \triangle u + \sum_{j=1}^n \overline{z}_j\, u_{\overline{z}_j},$$

where $u \in \operatorname{dom} \Box_{\varphi,0} \subseteq L^2(\mathbb{C}^n, e^{-|z|^2})$.

$$\Box_{\varphi,1}\, u = \overline{\partial}\, \overline{\partial}^*_\varphi u = -\frac{1}{4}\, \triangle u + \sum_{j=1}^n \overline{z}_j\, u_{\overline{z}_j} + u,$$

where $u \in \operatorname{dom} \Box_{\varphi,1} \subseteq L^2_{(0,1)}(\mathbb{C}^n, e^{-|z|^2})$.

Lemma

Let $n = 1$. For $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$ the functions

$$u_{k,m}(z,\overline{z}) = \overline{z}^{k+m}\, z^m + \sum_{j=1}^{m} \frac{(-1)^j (k+m)!\, m!}{j!\,(k+m-j)!\,(m-j)!}\, \overline{z}^{k+m-j}\, z^{m-j}$$

are eigenfunctions for the eigenvalue $k+m$ of the operator $\Box_{\varphi,0}\, u = -u_{z\,\overline{z}} + \overline{z} u_{\overline{z}}$.
For $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$ the functions

$$v_{k,m}(z,\overline{z}) = \overline{z}^{k}\, z^{k+m} + \sum_{j=1}^{k} \frac{(-1)^j (k+m)!\, k!}{j!\,(k+m-j)!\,(k-j)!}\, \overline{z}^{k-j}\, z^{k+m-j}$$

are eigenfunctions for the eigenvalue k of the operator $\Box_{\varphi,0}\, u = -u_{z\,\overline{z}} + \overline{z} u_{\overline{z}}$.

If $\varphi(z) = (\sum_{j=1}^{n} |z_j|^2)^\alpha, \ \alpha > 1$, then condition (*) is satisfied and $N_{\varphi,q}$ is compact.

For $\varphi(z) = |z_1|^{2k} + |z_2|^{2k}, \ k = 2, 3, \dots$ the operator $N_{\varphi,2}$ is compact, but $N_{\varphi,1}$ fails to be compact.

Witten Laplacians

Let $Z_k = \frac{\partial}{\partial \overline{z}_k} + \frac{1}{2}\frac{\partial \varphi}{\partial \overline{z}_k}$ and $Z_k^* = -\frac{\partial}{\partial z_k} + \frac{1}{2}\frac{\partial \varphi}{\partial z_k}$.
We consider $(0,q)$-forms $h = \sum_{|J|=q}{}' h_J \, d\overline{z}_J$, and define

$$\overline{D}_{q+1} h = \sum_{k=1}^{n} \sum_{|J|=q}{}' Z_k(h_J)\, d\overline{z}_k \wedge d\overline{z}_J$$

$$\overline{D}_q^* h = \sum_{k=1}^{n} \sum_{|J|=q}{}' Z_k^*(h_J)\, d\overline{z}_k \rfloor d\overline{z}_J,$$

where $d\overline{z}_k \rfloor d\overline{z}_J$ denotes the contraction, or interior multiplication by $d\overline{z}_k$, i.e. we have

$$\langle \alpha, d\overline{z}_k \rfloor d\overline{z}_J \rangle = \langle d\overline{z}_k \wedge \alpha, d\overline{z}_J \rangle$$

for each $(0,q-1)$-form α.

The complex Witten-Laplacian on $(0, q)$-forms is then given by

$$\Delta_\varphi^{(0,q)} = \overline{D}_q \, \overline{D}_q^* + \overline{D}_{q+1}^* \, \overline{D}_{q+1},$$

for $q = 1, \dots, n-1$.
The general $\overline{D}$-complex has the form

$$L^2_{(0,q-1)}(\mathbb{C}^n) \underset{\overline{D}_q^*}{\overset{\overline{D}_q}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n) \underset{\overline{D}_{q+1}^*}{\overset{\overline{D}_{q+1}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n) \, .$$

For a $(0,1)$-form $g = \sum_{\ell=1}^n g_\ell \, d\overline{z}_\ell \in L^2_{(0,1)}(\mathbb{C}^n)$ we obtain

$$\begin{aligned}\Delta_\varphi^{(0,1)} g &= (\overline{D}_1 \, \overline{D}_1^* + \overline{D}_2^* \, \overline{D}_2) g \\ &= (\Delta_\varphi^{(0,0)} \otimes I) g + M_\varphi g,\end{aligned}$$

where we set

$$M_\varphi g = \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j} \, g_k \right) \, d\overline{z}_j$$

and

$$(\Delta_\varphi^{(0,0)} \otimes I) \, g = \sum_{k=1}^n \Delta_\varphi^{(0,0)} \, g_k \, d\overline{z}_k .$$

In general we have

$$\Delta_\varphi^{(0,q)} = e^{-\varphi/2}\,\Box_\varphi^{(0,q)}\, e^{\varphi/2}.$$

Hence, the condition

$$\lim_{|z|\to\infty} s_q(z) = +\infty,$$

implies that $\Delta_\varphi^{(0,q)}$ is with compact resolvent.

Special cases: (a)

$$\Delta_\varphi^{(0,0)} = \overline{D}_1^* \, \overline{D}_1 = e^{-\varphi/2} \, \overline{\partial}_\varphi^* \, \overline{\partial} \, e^{\varphi/2}.$$

Shikegawa used results of Ohsawa to show that

$$\lim_{|z| \to \infty} |z|^2 \mu_\varphi(z) = +\infty$$

implies that

$$A^2(\mathbb{C}^n, e^{-\varphi}) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ entire } : \, f \in L^2(\mathbb{C}^n, e^{-\varphi})\}$$

has infinite dimension.
Hence 0 belongs to the essential spectrum of $\Delta_\varphi^{(0,0)}$ and $\Delta_\varphi^{(0,0)}$ fails to be with compact resolvent.

(b)

$n = 1$, let φ be a subharmonic $\mathcal{C}^2$-function on $\mathbb{C}$. We consider Schrödinger operators $\mathcal{S}$ with magnetic fields

$$\mathcal{S} = \frac{1}{4}(-\Delta_A + B),$$

where the 1-form $A = A_1\, dx + A_2\, dy$ is related to the weight φ by

$$A_1 = \partial_y \varphi/2\,,\; A_2 = -\partial_x \varphi/2\,,$$

$$\Delta_A = \left(\frac{\partial}{\partial x} + iA_1\right)^2 + \left(\frac{\partial}{\partial y} + iA_2\right)^2 \,,$$

and the magnetic field $Bdx \wedge dy$ satisfies

$$B(x,y) = \frac{1}{2}\Delta\varphi(x,y)\,.$$

Both operators $\overline{D}\,\overline{D}^*$ and $\overline{D}^*\,\overline{D}$ from above are non-negative, self-adjoint operators and we have

$$4\overline{D}\,\overline{D}^* = -\Delta_A + \frac{1}{2}\Delta\varphi \ , \ \ 4\overline{D}^*\,\overline{D} = -\Delta_A - \frac{1}{2}\Delta\varphi$$

Theorem (H.- Helffer)

The operator $\mathcal{S} = \overline{D}\,\overline{D}^*$ is with compact resolvent if and only if $\Delta\varphi(z) \to +\infty$ as $|z| \to \infty$.

The Pauli operators are defined by

$$P_+ = 4\overline{D}\,\overline{D}^* \ , \ \ P_- = 4\Delta_\varphi^{(0,0)} = 4\overline{D}^*\,\overline{D}.$$

The Dirac operator $\mathcal{D}$ is given by

$$\mathcal{D} = (-i\frac{\partial}{\partial x} - A_1)\,\sigma_1 + (-i\frac{\partial}{\partial y} - A_2)\,\sigma_2,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

and

$$\mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}.$$

Theorem

If

$$\lim_{|z|\to\infty} |z|^2 \, \Delta\varphi(z) = +\infty,$$

then the Dirac operator $\mathcal{D}$ *fails to have a compact resolvent.*