Bergman Kernel and Pluripotential Theory

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Ω is called Bergman complete if it is complete w.r.t. the Bergman metric $B_\Omega = i\partial\bar{\partial} \log K_\Omega$

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Since ι is distance decreasing,

$$dist_\Omega^B(z, w) \geq \arccos \frac{|K_\Omega(z, w)|}{\sqrt{K_\Omega(z) K_\Omega(w)}}.$$

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True if $\partial\Omega \in C^2$ (Herbort, 2000)

Demailly (1985) If Ω is hyperconvex then $G_w = G_\Omega(\cdot, w)$ is the unique solution to

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega} \setminus \{w\}) \\ (dd^c u)^n = (2\pi)^n \delta_w \\ u = 0 \text{ on } \partial\Omega \\ u \leq \log|\cdot - w| + C \end{cases}$$

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B. (1995) If Ω is hyperconvex then $\exists!\ u = u_\Omega$ s.th.

$$\begin{cases} u \in PSH(\Omega) \cap C(\bar{\Omega}) \\ (dd^c u)^n = 1\, d\lambda \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

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Proposition If Ω is hyperconvex then

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Sketch of proof $||G_w||_n^n = \int_\Omega |G_w|^n (dd^c u_\Omega)^n$

$$\leq n! ||u_\Omega||_\infty^{n-1} \int_\Omega |u_\Omega| (dd^c G_w)^n \leq C(n, \lambda(\Omega))\, |u_\Omega(w)|$$

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Open Problem $dist_\Omega^B(\cdot, w) \geq \frac{1}{C} \log \delta_\Omega^{-1}$

From Herbort's estimate

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B. (2005) $c_n = (1 + 4/Ei(n))^2$, where $Ei(t) = \displaystyle\int_t^\infty \frac{ds}{se^s}$

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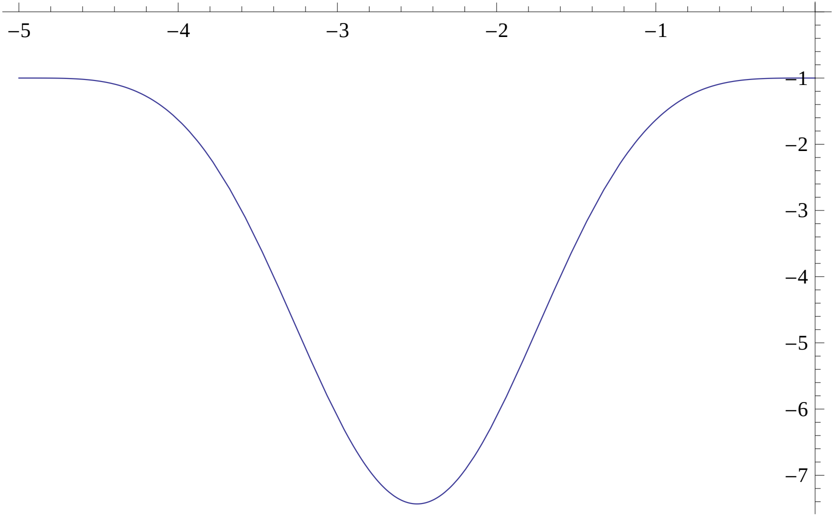
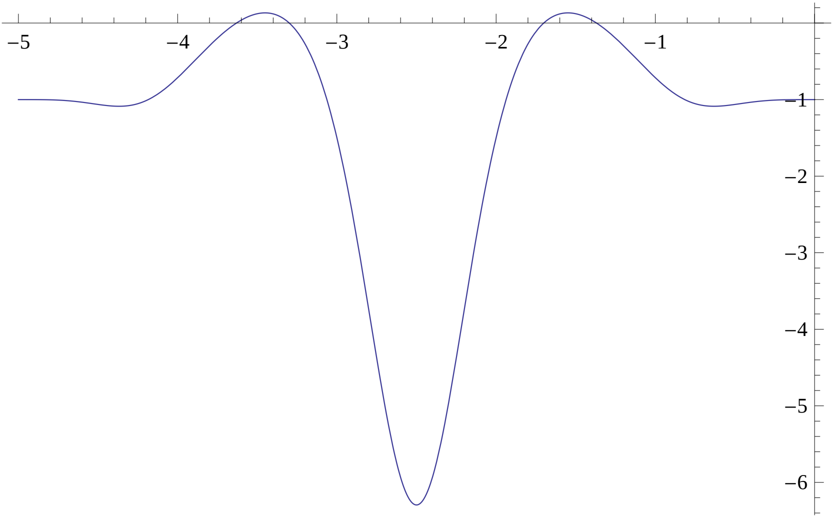
Suita Conjecture (1972): $\quad Curv_{c_D|dz|} \leq -1$

- "$=$" if D is simply connected
- "$<$" if D is an annulus (Suita)
- Enough to prove for D with smooth boundary
- "$=$" on ∂D if D has smooth boundary



$Curv_{c_D|dz|}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$



$Curv_{(\log K_D)_{z\bar{z}}|dz|^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $\log|z|$

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Using the methods of the Ohsawa-Takegoshi extension theorem he showed the estimate

$$c_D^2 \leq C\pi K_D$$

with $C = 750$.

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$C = 2$ (B., 2007)
$C = 1.95388\ldots$ (Guan-Zhou-Zhu, 2011)

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B.-Y. Chen (2011) proved that the Ohsawa-Takegoshi theorem (without optimal constant) follows form Hörmander's estimate.

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$$K_{\widetilde{\Omega}}(\widetilde{w}) = (K_\Omega(w))^m, \quad \lambda_{2nm}(\{G_{\widetilde{w}} < -1\}) = (\lambda_{2n}(\{G_w < -1\})^m.$$

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but

$$\lim_{m \to \infty} c_{nm}^{1/m} = e^{2n}.$$

Repeating this argument for any sublevel set we get

Theorem 1 Assume Ω is pseudoconvex in $\mathbb{C}^n$. Then for $t \leq 0$ and $w \in \Omega$

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where $\nu : \partial K \to S^{n-1}$ is the Gauss map.

By the Schwarz formula

$$\varphi(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta} \, \nu^{-1} \left(\frac{b + 2\mathrm{Re}\,(e^{it} w)}{|b + 2\mathrm{Re}\,(e^{it} w)|} \right) dt + i \mathrm{Im}\,\varphi(0).$$

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