GAUSSIAN MEASURE vs LEBESGUE MEASURE AND

ELEMENTS OF MALLIAVIN CALCULUS

λ : Lebesgue measure has the following properties :

1. For every non-empty open set U , $\lambda(U) > 0$;
2. For every bounded Borel set K , $\lambda(K) < \infty$;
3. For every Borel set B , $\lambda(B + x) = \lambda(B)$; (translational invariance) , further
4. λ has inner, outer regularity .

In fact 1), 2) 3) nearly characterize the Lebesgue measure (modulo a multiplicative constant). Does the Lebesgue measure make sense in infinite dimensions ? The answer is negative. To convince ourselves :

H : a separable Hilbert space with an orthonormal basis $\{h_1, h_2, \cdots\}$, ν

: a Borel measure in H

Let $B_{\frac{1}{2}}(h_n)$ be the open ball of radius half, centered at h_n , similarly $B = B_2(0)$;

By 1), 2) and 3) $0 < \nu(B_{\frac{1}{2}}(h_1)) = \nu(B_{\frac{1}{2}}(h_2) = \cdots\cdots < \infty$, but

$\nu(B_2(0)) \geq \sum_n \nu(B_{\frac{1}{2}}(h_n) = \infty$ violates 2).

Gaussian measure in $\mathbb{R}^n$ is absolutely continuous with respect to the Lebesgue measure in n dimensions with Radon-Nikodym derivative (density)

$$\Phi(x) = \frac{1}{(2\pi)^{n/2}\sqrt{DetS}} e^{-\frac{1}{2}\left(S^{-1}(x-m),x-m)\right)}, \quad x \in \mathbb{R}^n$$

m : Mean vector , S : Covariance matrix (symmetric, positive definite) ;

Standard Gaussian Measure : $m = 0, S = I \longrightarrow$ $\Phi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|x\|^2}{2}} \quad x \in \mathbb{R}^n$.

Another way of outlook at the density : random element in $\mathbb{R}^n$

$\exists(\Omega, \mathcal{F}, P)$ and a measurable mapping $X : \Omega \to \mathbb{R}^n$ induces a prob. measure μ in $\mathbb{R}^n$: (assumed to be absolutely continuous)

$\mu(B) = P(X^{-1}(B)) = \int_B \Phi(x)dx \; ; \quad B \in \mathcal{B}_n$

Standard Gaussian measure in finite dimensions is rotationally invariant : $\int_{A^{-1}B} \Phi(x)dx = \int_B \Phi(x)dx$.

Gauss distribution in finite dimensions can also be given by its Fourier transformation (characteristic functional) :

$$\chi(f) = \int_{\mathbb{R}^n} e^{i(f,x)} \Phi(x) dx, \qquad f \in \mathbb{R}^n \; ,$$

$$\chi(f) = e^{i(f,m) - \frac{1}{2}(Sf,f)} \qquad f \in \mathbb{R}^n$$

For $n = 1, f = t$, $\chi(t) = e^{imt - \frac{\sigma^2 t^2}{2}}$.

The char. fn of random variable $(f, x),\ f, x \in \mathbb{R}^n)$:

$$\chi^f(t) = \int_{\mathbb{R}^n} e^{it(f,x)} \Phi(x)dx = \chi(tf) = e^{it(f,m) - \frac{t^2}{2}(Sf,f)}\ , \qquad (*)$$

thus Gauss (in one dimension), converse is also true, i.e. if $(f, .)$ is one dimensional Gauss for all $f \in \mathbb{R}^n,$ then the measure in $\mathbb{R}^n$ is Gaussian (put $t = 1$ in (*)). Hence

$\mu \in \mathbb{R}^n$ is Gaussian $\Longleftrightarrow (f, .)$ has one dmnl Gaussian dist forall $f \in \mathbb{R}^n$

Take this point of view to define Gaussian measure in infinite dimensions :

-A measure on a Hilbert space H is Gaussian $\Longleftrightarrow (x, .)$ is a Gaussian r.v. $\forall x \in H$

-A measure on a Banach space X is Gaussian $\Longleftrightarrow (f, .)$ is a Gaussian r.v. $\forall f \in X^*$.

Two Problems : A) How to characterize the Fourier transform of a finite measure

B) How to characterize the Fourier transform of Gaussian measures ?

Theorem 1 (Bochner) In $\mathbb{R}^n$ a functional $\hat{\mu}(f)$ is the Fourier transform of a finite measure $\iff \hat{\mu}(0) = \mu(\mathbb{R}^n)$, continuous and positive definite.

For infinite dimensional Hilbert spaces these are not sufficient , e.g. $e^{-\frac{1}{2}\|x\|^2}$ satisfy the conditions but not the Fourier transform $\hat{\mu}(x) = \int_H e^{i(x,y)} d\mu(y) \quad x \in H$ of any finite Borel measure on the Hilbert space . (Otherwise for an orthonormal basis $\{h_n\}$, $\int_H e^{i(h_n,y)} d\mu(y) = e^{-\frac{1}{2}}$ which is not compatible with $(h_n, y) \to 0$ as $\sum_n (y, h_n)^2 = \|y\|^2 < \infty$) .

Definite answer

Theorem 2 (Minlos - Sazonov). Let ϕ be a positive-definite fnl on H , then the following are equivalent :

i) ϕ is the Fourier transform of a finite Borel measure on H ,

ii) $\forall\epsilon > 0$, $\exists$ a symmetric, trace class operator S_ϵ , such that

$(S_\epsilon x, x) < 1 \Longrightarrow Re(\phi(0) - \phi(x) < \epsilon$,

iii) $\exists$ a symmetric, trace class operator S on H , such that

ϕ is continuous (or continuous at $x = 0$) w.t. the norm

$\| \, . \, \|_*$ defined by $\|x\|_* \doteq (Sx, x)^{1/2} = \|S^{1/2}x\|$.

If μ is a Borel prob. measure then the mean vector is a vector

$m \in H$ satisfying

$(m, x) = \int_H (x, z) d\mu(z)$.

If $\int_H \|x\| \, d\mu(x) < \infty$, it always exists. On the other hand :

Lemma 1. If μ is a finite Borel measure, then

$\int_H \|x\|^2 \, d\mu(x) < \infty \Longleftrightarrow \exists$ a positive, symmetric, linear, trace class operator , called **S-Operator**, such that $\forall x, y \in H$

$(Sx, y) = \int_H (x, z)(y, z) d\mu(z)$.

If further μ is a probability measure then B, $Bx \doteq Sx - (m, x)x$ satisfies

$(Bx, y) = \int_H (z - m, x)(z - m, y) d\mu(z)$ is the **covariance operator**.

A Gaussian probability on H (i.e. all $(x, .)$ are one-dimensional Gaussian r.v.s) always satisfies the above conditions, thus m and B always exists.

In fact ;

Theorem 3. A Borel probability measure μ on H is Gaussian $\Longleftrightarrow$ Its Fourier transform can be expressed as

$$\hat{\mu}(x) = \exp\{i(m,x) - \frac{1}{2}(Bx,x)\}\ , \quad \forall m, x \in H\ .$$

In a Banach space X the definition of the mean vector is the same. If we use the random element outlook, the mean vector $m \in X$ can be given by a Pettis integral

$$< m, f > \doteq \int_\Omega < f, x(\omega) > dP(\omega)\ , \quad \forall f \in X^*\ . \qquad (\dagger)$$

Thus a necessary condition for the existence of $m \in X$ is that $< f, x(.) > \in L^1(\Omega, \mathcal{F}, P) \ \ \forall f \in X^*$. If the necessary condition is satisfied then the r.h.s. of ($\dagger$) defines a linear , continuous function of f , i.e. an element of X^{**}.

Hence if m exists $m \in X \cap X^{**}$, $(X \hookrightarrow X^{**})$. Therefore for the reflexive spaces , the necessary condition is also sufficient. In the non-reflexive case there are extra sufficient conditions. Leaving them out we know that for Gaussian distribution m always exists.

The covariance operator in X is defined through an S-operator $S : X^* \to X^{**}$, i.e.

$<< Sf, g >>= \int_X < f, z >< g, z > d\mu(z), \quad f, g \in X^*$ and

$Bf = Sf - < m, f > m$ (Covariance operator).

Operator S has properties akin to operators $S \; : H \to H$ e.g.

:

i) Symmetry : $<< Sf, g >>=<< Sg, f >>$;

ii) Positivity : $<< Sf, f >>\geq 0, \;\; \forall f \in X^*$.

The existence of the covariance operator in a Banach space is characterized in terms of the nuclearity of S or B but a modified definition of nuclearity is needed in Banach spaces. Let

$\mathcal{H}(X)$: all symmetric, non-negative , bounded linear mappings $X^* \to X^{**}$;

$\mathcal{H}_1(X)$: the class of covariance operators of distributions on X ;

$\mathcal{H}_2(X)$: the classs of covariance operators of all Gaussian distributions on X. $(\mathcal{H}_2(X) \subset \mathcal{H}_1(X) \subset \mathcal{H}(X))$

If X is separable and reflexive then $\mathcal{H}_1(X) = \mathcal{H}(X)$. Under the same conditions,for $S \in \mathcal{H}(X)$, nuclearity in the first sense is sufficient and the nuclearity in the second sense is necessary for S to be in $\mathcal{H}_2$, (For nuclearity of different senses c.f. [Vakhania]).

If we consider for simplicity $m = 0$,

$\chi(f;\mu) = \exp\{-\frac{1}{2} << Sf, f >>\} \quad \forall f \in X^* \qquad (\surd)$

is the Fourier transform of some Gaussian measure in X

$\Longleftrightarrow S \in \mathcal{H}_2$.

Standard Gaussian Cylinder Measures

The quasi-invariance property which is so important in the differential analysis in infinite dimensions is possessed by the standard Gaussian measure where $m = 0$ and $S = I$. However $S = I$ does not make sense in $(\surd)$ as an operator $X^* \to X^{**}$, and in the Hilbert case I is not nuclear (or trace class) and $\chi(f) = e^{-\frac{1}{2}\|f\|^2}$ obtained by $B = I$ can not be the characteristic functional of a standard type Gaussian distribution in X. Therefore an approach via finite dimensional subspaces is essential.

Let $\mathcal{F}(X^*)$: be the class of all finite dimensional subspaces of X^*;

For any $K \in \mathcal{F}(X^*)$ call

$D = \{x \in X : (< x, y_1 >, < x, y_2 >, \cdots, < x, y_n >) \in E\}$

a cylinder set based on K if $E \in \mathcal{B}_n,\ y_1, \cdots, y_n \in K$.

Let $\Re(X) = \bigcup_{K \in \mathcal{F}(X^*)} \mathcal{C}(K)$, where $\mathcal{C}(K)$ is the σ-algebra generated by cylinder sets with base K.

$\Re(X)$ is only an algebra , but for a separable Banach space X, $\sigma(\Re(X)) = \mathcal{B}_X$ (the Borel σ-alg. in X).

A non-negative set fct μ on $\Re(X)$ is called a "cylinder probability measure" on X, if $\mu(X) = 1$ and a measure when restricted to any $\mathcal{C}(K)$ for $K \in \mathcal{F}(X^*)$. A cylinder probability measure is necessarily compatible.

A real (complex)-valued fct on X is a cylinder fct if it is measurable w.t. $\mathcal{C}(K)$ for some K.

$\hat{\mu}(f) = \int_X e^{i<x,f>} d\mu(x)$, $f \in X^*$: the characteristic fnl of the cylinder measure μ.

Question : What kind of cylinder measures can be extended to $\mathcal{B}_X$?

Answer in a Special Case (Important for Malliavin calculus)

X is the completion of some Hilbert space H w.t. a weaker norm and the cylinder measure on X is lifted from that on H.

Definition 1. (Gross) Let $(H, |\,.\,|)$ be a Hilbert space , μ a cylindrical measure on H, $\|\,.\,\|$ another norm on H weaker than $|\,.\,|$ If for any $\epsilon > 0$ $\exists$ $\pi_\epsilon \in \mathcal{P}$ (the set of all finite dimensional orthogonal projections on H), such that for any $\pi \in \mathcal{P}\,(\pi \perp \pi_\epsilon)$ one has $\mu\{x \in H : \|\pi x\| > \epsilon\}$ $< \epsilon$ then $\|\,.\,\|$ is said to be a **measurable norm** with respect to μ.

(Cylinder measures can also be defined on H : since
$X^* \hookrightarrow H^* \simeq H$ we have $\mathcal{F}(X^*) \subset \mathcal{F}(H)$)

If μ is a cylinder measure on H and $\hat{\mu}(x) = e^{-\frac{1}{2}|x|^2}$, $x \in H$ then μ is called a "standard Gaussian cylinder measure" on H.

Theorem 4.(Gross) Suppose the triplet (X, H, μ) is as above. If μ is a Gaussian cylinder measure on H and $\|\,.\,\|$ is a μ-measurable norm, then the lifting μ^* of μ to X can be extended to a Borel measure on X, called **Standard Gaussian measure** on X. $\mu^*(C) \doteq \mu(C \cap H)$, $C \in \mathcal{R}(X))$.
(X, H, μ) is called an **abstract Wiener space**

Conversely let X be a separable Banach space and let μ be a zero mean Gaussian measure on X, i.e. for any $\alpha \in X^*$ and $\omega \in X$, $< \alpha, \omega >$ is a one-dimensional zero mean Gauss random variable (or for any n and $\alpha_i \in X^*$ $\{< \alpha_i, \omega >, i = 1, 2, \cdots n\}$ is a zero mean Gaussian random vector) .Then there exists a dense Hilbert sub-space $H \subset X$ such that (X, H, μ) is an abstract Wiener space.
H is called the **Cameron-Martin space**.

Classical Wiener space is an example of an abstract Wiener space :

Banach space X : $C_0[0,1]$ with the sup norm,
For $\omega \in C_0[0,1]$, $t \in [0,1]$ coordinate functional on $C_0[0,1]$ is $W_t(\omega) = \omega(t)$.

N. Wiener : There exists a unique probability μ on $\mathcal{B}_{C_0}$ such that the map $(t,\omega) \longrightarrow W_t(\omega)$ is a Wiener process (Brownian motion).

H : absolutely continuous functions in $C_0[0,1]$ with square integrable derivatives.
Then $h \in H \Rightarrow h = \int_0^t \tilde{h}(s)ds$, $\tilde{h} \in L^2[0,1]$.
$\|h\|_{C_0} = \sup_{0\le t\le 1} |\int_0^t \tilde{h}(s)ds| \le \sup_{0\le t\le 1} \left(t\int_0^t |\tilde{h}(s)ds\right)^{\frac{1}{2}} \le \left(\int_0^t |\tilde{h}(s)|^2 ds\right)^{\frac{1}{2}} = (\int_0^t |\dot{h}|^2)^{\frac{1}{2}} \doteq |h|_H$. (defn of norm in H).

With this norm $\tilde{h} \to h$ is a continuous , linear injection from H into C_0, such that its range is dense in C_0 . $\|\,.\,\|$ is weaker than $|\,.\,|$ when restricted to H.

If (X, H, μ) is an abstract Wiener space , considering that the cylindrical measure is on H, we want to construct a process $< h, \omega >$, $h \in H, \omega \in X$.

But since $X^* \subset H$, h may not be in X^*. However X^* is densely imbedded in H. Indicate the injection $X^* \hookrightarrow H^* \simeq H$ by $(\hat{.})$, $\alpha \in X^* \to \hat{\alpha} \in H$. Given $h \in H$ there exists $\alpha_n \in X^*$ s.t. $\hat{\alpha_n} \longrightarrow h$ in H. For $\omega \in X$ the Gaussian sequence $< \alpha_n, \omega >$ is Cauchy in L^p, denote the limit by $W_h(\omega)$, ($\delta h(\omega)$ in some sources), which is $N(0\,, |h|_H^2)$ and also $\mathbb{E}(W_h W_g) = (h, g)_H\,, h, g \in H$.

Thus we have another model which is called **Gaussian probability space** by Malliavin. Namely :

$(\Omega, \mathcal{F}, \mu)$: a complete probability space, H : a real, separable Hilbert space, $\{W_h,\, h \in H\}$ is a family of zero mean Gaussian r.v.'s with $\mathbb{E}[W_h W_g] = (h, g)_H$.

$(\Omega, \mathcal{F}, \mu; H)$ A Gaussian probability space. The classical and abstract Wiener spaces are examples of Gaussian probability spaces. A third example is the **White noise space** : $H = L^2(\mathbb{R})$; $S(\mathbb{R}),\ S^*(\mathbb{R})$: Schwartz spaces of rapidly decreasing C^∞ functions and tempered distributions respectively. ($S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})$ is called a **Gelfand Triplet**). Then by the Minlos theorem there exists a unique Gaussian measure μ on $\mathcal{B}_{S^*(\mathbb{R})}$ such that $\forall \xi \in S(\mathbb{R})$: $\int_{S^*(\mathbb{R})} e^{i<\omega,\xi>} d\mu(\omega) = e^{-\frac{1}{2}\|\xi\|_H^2}$; $<\omega, \xi>$ is the canonical bilinear form on $S^*(\mathbb{R}) \times S(\mathbb{R})$. Then $W_\xi(\omega) =< \omega, \xi >$ $\xi \to W_\xi$ can be extended to a linear isometry $L^2(\mathbb{R}) \longrightarrow L^2(S^*(\mathbb{R}), \mathcal{F}, \mu)$, thus $(S^*(\mathbb{R}), \mathcal{F}, \mu; L^2(\mathbb{R}))$ is a Gaussian probability space.

In an abstract Wiener space or a Gaussian probability space , $\{W_h(\omega),\, h \in H\}$ is like a stochastic process with the index set H .

Some Elements of the Malliavin Calculus

In an abstract Wiener space (or a Gaussian probability space) $F : X \to \mathbb{R}(\mathbb{C})$ is called a **Wiener functional**.

Some examples ; Ito stochastic integrals, solutions of stochastic differential equations. As they are in fact equivalence classes , may not be differentiable in the Fréchet sense, even not continuous. Paul Malliavin , using the quasi-invariance of the Gaussian measure, initiated a kind of weak differential calculus, so that such functionals became smooth in his sense . The quasi-invariance is translational invariance of the Gaussian measure in the directions of the Cameron-Martin space H.

The new kind of differentiation was obtained by perturbing the Wiener paths in the directions of vectors in H, thus taking the name of "stochastic calculus of variation" or as popularly known the Malliavin calculus.

If $F : X \longrightarrow \mathbb{R}$ an attempt to define the derivative by $\lim_{\|\Delta\omega\|\to 0} \dfrac{F(\omega+\Delta\omega)-F(\omega)}{\|\Delta\omega\|}$ will fail since the quotient is not even well-defined as a random variable, (equivalence classes).

This is remedied by the Cameron-Martin Formula (Theorem):

$$\mathbb{E}[F(\omega+h)] = \mathbb{E}[F(\omega)e^{W_h - \frac{1}{2}|h|_H^2}]$$

If in the above quotient the perturbation $\Delta\omega = h$ is taken in the Camerion Martin space, (i.e. $h \in H$) then the limit will be well-defined .

Take two functionals in the same equivalence class :

$$F = G \;\; \mu\text{-}a.s. \Longrightarrow \mu\{\omega | F(\omega+h) \neq G(\omega+h)\} = \mathbb{E}_\mu \left\{ I_{\{\omega | F(\omega+h) \neq G(\omega+h)\}} \right\}$$

$$= \text{(Cameron-Martin thm.)}\; \mathbb{E}_\mu \left\{ I_{\{\omega | F(\omega) \neq G(\omega)\}} e^{W_h - \frac{1}{2}|h|_H^2} \right\} = 0 \,.$$

This allows to define weak differential (Sobolev derivative) starting from cylindrical functionals.

Let (X, H, μ) be an abstract Wiener space.

$$F(\omega) = f\left(W_{h_1}(\omega), \cdots\cdots, W_{h_n}(\omega)\right) \,, \;\; \omega \in X, f \in S(\mathbb{R}^n)$$

is called a **smooth cylindrical functional**, its class is denoted by S_M. Similarly depending on f being a polynomial or L^p function we have a functional in $\mathcal{P}$ or in $L^p(\mu)$. The following inclusions hold

$$\mathcal{P} \subset S_M \subset L^p$$

and $\mathcal{P}$ is dense in L^p. Noticing that $W_{h_j}(\omega + \lambda h) = W_{h_j}(\omega) + \lambda (h_j, h)_H$ we have the following definition of the weak directional derivative (Sobolev derivative) :

$$D_hF(\omega) = \frac{d}{d\lambda}F(\omega+\lambda h)|_{\lambda=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_j}(W_{h_1}(\omega), \cdots, W_{h_n}(\omega))(h_j, h)_H \;\; ; \;\; h \in H.$$

For fixed $\omega, h \to D_h(\omega)$ is linear and continuous , hence it determines (by the Riesz representation thm) an element of $H^* \simeq H$ that we denote by DF (the gradient operator) and

$(DF, h)_H = D_hF$ and also $\mathbb{E}[(Df, h)] = \mathbb{E}[F(W_h))]$.

Also $F(\omega) \longrightarrow DF(\omega)$ is a linear operator from the cylindrical functionals into the space of H-valued Wiener functionals $L^p(\mu; H)$,

(the norm being defined by $\|DF(\omega)\|_p = \left[\int_X |DF(\omega)|_H^p d\mu(\omega)\right]^{\frac{1}{p}}$).

Note. If $G \in L^p(\mu)$ and $F = G$ $(\mu - a.s.)$ then by the Cameron-Martin thm. $G(\omega + \lambda h) = F(\omega + \lambda h)$ $(\mu - a.s.)$, thus DF depends only on the equivalence class that F belongs to.

We want to extend the above operator to a larger class of Wiener functionals. In fact we have

Theorem 5. D is a closable operator from $L^p(\mu)$ into $L^p(\mu; H)$.

Definition 2. $\mathbb{D}_{p,1}$ is the set of equivalence classes of Wiener functionals defined by :

$F \in \mathbb{D}_{p,1} \Leftrightarrow \exists$ a sequence of cylindrical fcts F_n converging to F in $L^p(\mu)$ such that $\{DF_n\,,\, n \in \mathbb{N}\}$ is Cauchy in $L^p(\mu; H)$.

In this case denote $\lim_{n\to\infty} DF_n = DF$.

$(DF$ is independent of the choice of the sequence $F_n)$.

$\mathbb{D}_{p,1}$ is a Banach space under the norm $\|F\|_{p,1}^p = \|F\|_{L^p(\mu)}^p + \|DF\|_{L^p(\mu;H)}^p$.

Generalization to E valued functionals (E is any separable Hilbert space):

$D_h F = (DF, h)$ takes the form :

$\frac{d}{d\lambda}\left[(F(\omega+\lambda h), e)_E\right]|_{\lambda=0} = (DF(\omega), h\otimes e)_{H\otimes E} \quad h \in H\,,\, e \in E$.

$DF \in L^p(\mu; H\otimes E)$. Corresponding Sobolev space is denoted by $\mathbb{D}_{p,1}(E)$.

Higher order derivatives and Sobolev spaces are defined in an inductive manner:

$$D^2F = D(DF) \in L^p(H \otimes H \otimes E) \cdots\cdots, D^kF = D(D^{k-1}F) \in L^p(H^{\otimes(k)} \otimes E).$$

where $\otimes$ denotes the (completed) Hilbert tensor product. Similarly $F \in \mathbb{D}_{p,k}$ if $D^{k-1}F \in \mathbb{D}_{p,1}(H^{\otimes(k-1)} \otimes E)$ and the norms

$$\|F\|_{p,k} = \left(\|F\|_p^p + \sum_{j=1}^{k} \|D^jF\|_p^p \right)^{1/p}.$$

Since the gradient DF is an H-valued functional, H may be considered as tangent space. Hence H-valued functionals are vector fields and the adjoint operator δ is the divergence of vector fields.

For any smooth vector field $V \in S_M(H)$, its **divergence** $\delta V \in S_M(\mathbb{R}) \equiv S_M$ is determined by

$$\mathbb{E}[G.\delta V] = \mathbb{E}[(DG, V)_H], \;\; \forall G \in S_M \,.$$

More generally if $V \in S_M(H \otimes E)$, its divergence in $S_M(E)$ is defined by

$$\mathbb{E}[(G\,, \delta V)_E] = \mathbb{E}[(DG, V)_{H\otimes E}], \quad \forall G \in S_M(E)\,.$$

Explicit expression for the smooth vector fields $V \in S_M(H)$:

$\delta V = \sum_{i=1}^{\infty}[(V, h_i)_H W_{h_i} - (D_{h_i}V, h_i)_H]$, $\{h_i\}$: complete, orthonormal basis in H

δV is also a closable operator.

For a feeling of the divergence operator consider the case $n = 1$.

Example. One dimensional Gaussian space $(\mathbb{R}, \mathcal{B}, \gamma)$; $D = \frac{d}{du}$; $D^* \equiv \delta = -\frac{d}{du} + u.$; and $D^*D = \delta D = -\frac{d^2}{du^2} + u\frac{d}{du}$. For ϕ and ψ real polynomials :

$$(D\phi, \psi)_{L^2(\mathbb{R},\gamma)} = (\phi, \delta\psi)_{L^2(\mathbb{R},\gamma)}\,.$$

D, δ and δD can be extended to closed operators in $L^2(\mathbb{R}, \gamma)$ such that D and δ are mutually adjoint and δD is self adjoint , **number operator** in one dimension . Hermite polynomials are eigen functions of the number operator, i.e.

$$\delta D H_n = n H_n$$

(Hermite polynomials the coefficients of t in the expansion $\exp\{tu - t^2/2\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u)$; $t, u \in \mathbb{R}$, $H_n(u) = (-1)^n e^{u^2/2} \frac{d^n}{du^n} e^{-\frac{u^2}{2}}$, $n \in \mathbb{N}_0$.)

$\mathcal{L} = -\delta D$ is the **Ornstein-Uhlenbeck operator**. $\mathcal{L} H_n = -n H_n$.

In infinite dimensions : $H_\alpha = \prod_j H_{\alpha_j}(W_{h_j}(\omega))$ ($\{h_j\}$ is any orthonormal base in H) , where $\alpha = \{\alpha_j\}_{j=1}^{\infty}$ has only a finite number of non-zero indices . All such indices form a set Γ . and $\Lambda_n = \{\alpha \in \Lambda : \ |\alpha| = n\}$.

Then $\{(\alpha!)^{-\frac{1}{2}} H_\alpha : \alpha \in \Lambda\}$ constitute a base of $L^2(\mu)$. Let $\mathcal{H}_0 \equiv \mathbb{R}$. For $n \geq 1$, let $\mathcal{H}_n$ be the closed subspace generated by $\{H_\alpha : \alpha \in \Lambda_n\}$. Then the infinite direct sum decomposition

$$L^2(\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n .$$

The decomposition is independent of the choice of base in H .

Also isomorphic to the symmetric Fock space (Boson Fock space) over H :

$L^2(\mu) \cong \Gamma(H)$.

$\mathcal{H}_n$: Wiener chaos of order n . J_n : orthogonal projection onto $\mathcal{H}_n$.

$$\mathcal{L} = -\delta D = -\sum_{n=1}^{\infty} n J_n ; \quad \mathcal{L} H_\alpha = -|\alpha| H_\alpha .$$

For $F = f(W_{h_1}, \cdots, W_{h_n}) \in S_M$ we explicitly have

$$\mathcal{L}F = \sum_{j,k=1} \partial_k \partial_j f(W_{h_1}, \cdots, W_{h_n})(h_k, h_j) - \sum_{j=1}^{n} \partial_j f(W_{h_1}, \cdots, W_{h_n}) W_{h_j} .$$

Multiple Wiener-Ito Integral representation

In abstract Wiener space let $H = L^2(T, \mathcal{B}, \lambda)$. where $(T, \mathcal{B})$ is a measurable space, λ : non-atomic, σ-finite, (covers classical Wiener space and the white-noise space). Let $W(A) \doteq W_{I_A}$; $A \in \mathcal{B}$, (i.e. $h = I_A \in H$). We have

$$W(A) \sim N(0, \lambda(A)) \; ; \; \mathbb{E}[(W(A)W(B)] = \lambda(A \cap B) .$$

Then for disjoint $\{A_j\}$, $W(\bigcup_n A_n) = \sum_n W(A_n)$ $(L^2(\mu)$-convergence).

W (a random set function) : **Gaussian orthogonal random measure** , $W_h = \int_T h(t) dW(t)$.

I_n : Multiple Wiener-Ito integral is constructed like multiple Lebesgue integral using this random measure:

$I_n = \int_{T^n} f(t_1, t_2, \cdots, t_n) \, dW(t_1) dW(t_2) \cdots dW(t_n)$.

Relation between Hermite polynomials and multiple Wiener-Ito integrals :

$H_n(W_h) = I_n(h^{\otimes n})$. For $n = 1$, $H_1(u) = u$ then it reduces to $\int_T h(t)dW(t) = H_1(W_h) = W_h$.

More generally if $\{h_j\}_{j\in\mathbb{N}}$ is a base of H we have $H_\alpha = I_{|\alpha|}(\hat{h}_\alpha)$, $\forall\alpha \in \Lambda$ and $\hat{h}_\alpha \equiv \hat{\bigotimes}_j h_j^{\otimes\alpha_j}$.

Also $F \in L^2(\mu)$ has a unique decomposition $F = \sum_{n=0}^{\infty} I_n(f_n)$ where $f_n \in H^{\hat{\otimes}n}$ and $\|F\|^2 = [\mathbb{E}(|F|)^2 + \sum_{n=0}^{\infty} n!\|f_n\|^2$ ($\|f_n\|$ stands for the norm in $L^2(T^n, \mathcal{B}_n, \lambda^n))$.

Another interpretation of the Ornstein-Uhlenbeck Operator $\mathcal{L} = -\delta D$

Ornstein-Uhlenbeck process satisfies the stochastic d.e.

$dX_t = -X_t dt + \sqrt{2}\, dB_t$, $X_0 = x \in \mathbb{R}^n$ (B_t : Brownian motion)

has the solution $X_t = e^{-t}x + \sqrt{2}\int_0^t e^{-(t-s)}dB_s \;\; t \in \mathbb{R}_+$. X_t is a Gaussian process $X_t \sim N(e^{-t}x, 1 - e^{-2t})$. Define the Ornstein-Uhlenbeck semi-group T_t which is a contraction (even e hypercontractive) semi-group.

$$(T_t\phi)(x) \doteq \mathbb{E}[\phi(X_t)] = \int_X \phi(e^{-t}x + \sqrt{1-e^{-2t}}y)\, d\mu(y)\ ;\ \phi \in S_M$$

(μ: standard normal distribution) . Semi-group property follows from the Markov property of diffusion processes and it turns out that the infinitesimal generator of this semi-group is $\mathcal{L}$. Therefore $T_t = e^{t\mathcal{L}} = \sum_{n=0}^{\infty} \frac{t^n \mathcal{L}^n}{n!}$. Using $\mathcal{L}^0 \equiv I = \sum_n J_n$, $\mathcal{L} = \sum_n nJ_n$, $\mathcal{L}^2 = (\sum_n -nJ_n)(\sum_m -mJ_m) = \sum_n n^2 J_n$, (the orthogonality of chaos projections) and by induction $\mathcal{L}^m = \sum_n (-1)^m n^m J_n$ yielding for the representation of the O-U semigroup $T_t = \sum_{n=0}^{\infty} e^{-nt} J_n$.

As each Wiener chaos belongs to the eigen-space of $\mathcal{L}$, we can define non-integer powers, e.g., $(I-\mathcal{L})^a = I - a\mathcal{L} + \frac{a(a-1)}{2}\mathcal{L}^2 - \cdots = \sum J_n + a\sum nJ_n + \frac{(a(a-1)}{2} n^2 J_n + \cdots = \sum (1 + an + \frac{a(a-1)}{2} n^2 + \cdots)J_n = \sum_n (1+n)^a J_n$;

In particular for $p > 1$, $a = -\frac{1}{2}$, the completion of polynomial cylindrical functionals with respect to the norm $\|F\|^{\sim}_{p,k} \doteq \|(I-\mathcal{L})^{\frac{k}{2}}F\|_{L^p}$ is dense in L^p and denoted by $\mathbb{D}^{\sim}_{p,k}$. Now the important Meyer inequalities relating Sobolev and L^p norms

$\forall p > 1, \forall k \in \mathbb{N} \quad \exists c_{p,k}\, ,\, \tilde{c}_{p,k}\, ,$ such that $\forall F \in \mathcal{P}(E)$:

$$c_{p,k}\|(I-\mathcal{L})^{\frac{k}{2}}F\|_{L^p(\mu;E)} \leq \|F\|_{p,k} \leq \tilde{c}_{p,k}\|(I-\mathcal{L})^{\frac{k}{2}}F\|_{L^p(\mu;E)}\, .$$

By these inequalities for $k \in \mathbb{N}$, $\|\,.\,\| \sim \|\,.\,\|^{\sim}$, then we omit $\sim$ sign, but whenever k is non-integer we should know that $\mathbb{D}_{p,k} \equiv \mathbb{D}^{\sim}_{p,k}$. Using Meyer inequalities we show (for E valued functionals) :

i) $\mathbb{D}_{p,k}(E)$ has $\mathbb{D}_{q,-k}(E)$ its continuous dual , $1/p+1/q=1$,

ii) D has a continuous extension from $\mathbb{D}_{p,k}(E)$ into $\mathbb{D}_{p,k-1}(E\otimes H)$ for any $p>1$,

iii) δ has a continuous extension from $\mathbb{D}_{p,k}(E\otimes H)$ into $\mathbb{D}_{p,k-1}(E)$ for any $p>1$.

Generalized Functions

Similar to the test functions-Schwartz distributions dual-pair $(\mathcal{D},\mathcal{D}^*)$ we have :

$$\mathbb{D}^{\infty}(E)=\bigcap_{k>0}\bigcap_{1<p<\infty}\mathbb{D}_{p,k}(E)\ \ ,\ \ \mathbb{D}^{-\infty}(E)=\bigcup_{k>0}\bigcup_{1<p<\infty}\mathbb{D}_{p,-k}(E)\,.$$

$\mathbb{D}^{\infty}$ is equipped with the projective limit topology and $\mathbb{D}^{-\infty}$ is equipped with the inductive limit topology ; the latter is the Meyer-Watanabe distributions.

As an application **Donsker's Delta Function** is $\delta_x(W(t))\in\mathbb{D}^{-\infty}$.

Using Meyer inequalities one can show that :

- D uniquely extends to a continuous operator $\mathbb{D}^{\infty}(E) \longrightarrow \mathbb{D}^{\infty}(E \otimes H)$,
- D uniquely extends to an operator $\mathbb{D}^{-\infty}(E) \longrightarrow \mathbb{D}^{-\infty}(E \otimes H)$,
- Similar extensions for $\delta = D^*$, e.g. $\delta : \mathbb{D}^{-\infty}(E \otimes H) \longrightarrow \mathbb{D}^{-\infty}(E)$,
- $\mathcal{L}$ extends uniquely to an operator $\mathbb{D}^{-\infty}(E) \longrightarrow \mathbb{D}^{-\infty}(E)$ such that $\forall p \in (1, \infty)$, $k \in \mathbb{R}$

  $\mathcal{L} : \mathbb{D}_{p,k+2}(E) \longrightarrow \mathbb{D}_{p,k}(E)$ is continuous, in particular $\mathcal{L} : \mathbb{D}^{\infty}(E) \to \mathbb{D}^{\infty}(E)$

  is continuous.

Densities of Non-degenerate Functionals

One of the main concerns of the Malliavin calculus is the investigation of existence, regularity (smoothness) and other properties of the Wiener (Brownian) functionals.

Let F be an $\mathbb{R}^m$-valued functional (i.e. an m-dimensional random vector). $\mu \circ F^{-1}$ defines a probability measure on $\mathcal{B}_m$. Under which conditions it is absolutely continuous with respect to the Lebesgue meaure λ^m ?

Malliavin Covariance Matrix

Let $F = (F_1, F_2, \cdots, F_m) \in \mathbb{D}_{1,1}(\mathbb{R}^m)$ and $\sigma_{ij} \doteq (DF_i, DF_j)_H$; $1 \le i, j \le m$.

Malliavin Matrix : $\Sigma(\omega) = \langle \sigma_{ij}(\omega) \rangle$.

If $Det\Sigma(\omega) > 0 \ a.s.$ and satisfies $[Det\Sigma(\omega)]^{-1} \in L^{\infty-} \equiv \bigcap_{1<p<\infty} L^p(\mu)$, then we say F is non-degenerate in the sense of Malliavin. The following is a key lemma of harmonic analysis:

Lemma 2. Let ν be a σ-finite measure on $\mathcal{B}_m$. If for $j = 1, 2, \cdots, m$ $\exists c_j$ such that $\forall \phi \in C_0^\infty(\mathbb{R}^m)$:

$$|\int_{\mathbb{R}^m} \partial_j \phi(x) d\nu(x)| \leq c_j \|\phi\|_\infty$$

then $\nu \prec\prec \lambda^m$. When $m > 1$, ν has density $\rho \in L^{m^*}(\mathbb{R}^m)$, $m^* = \dfrac{m}{m-1}$.

Note : Clear for $m = 1$: take ϕ the cumulative D.F. of the uniform $U([a,b])$ r.v. It is not in C_0^∞. But $\exists \phi_n \in C_0^\infty$ such that $\phi_n \to \phi$, $\phi_n' \to \phi'$, $\phi' = \frac{1}{b-a}$, $\|\phi\|_\infty = 1$. Then the condition yields $\nu([a,b]) \leq c_1(b-a) = c_1 \lambda([a,b])$ implying absolute continuity .

Lemma 2 is utilized to prove

Theorem 6. Let $F = (F_1, F_2, \cdots, F_m) \in \mathbb{D}_2^\infty(\mathbb{R}^m) \equiv \bigcap_{1<p<\infty} \mathbb{D}_{p,2}(\mathbb{R}^m)$ be non-degenerate. Then $\exists \Phi_j \in L^{\infty-}(\mu)$ $(j = 1, \cdots, m)$ such that $\forall \phi \in C_0^\infty(\mathbb{R}^m)$, $\mathbb{E}[\partial_j \phi \circ F] = \mathbb{E}[\Phi_j(\phi \circ F)]$ which implies that F has a density ρ. In case $m > 1$, $\rho \in L^{m^*}$. If only the existence of density is sought , the condition can be considerably weakened : For $p > 1$, $F \in \mathbb{D}_{p,1}(\mathbb{R}^m)$, if the Malliavin covariance matrix is invertible, then F has a density.

Smoothness of Densities

The density of F can be formally expressed as $\mathbb{E}[\delta_x \circ F]$, (δ_x : Dirac function with singularity at x).

: (To see this heuristically consider a delta sequence $\delta_{n,x} \to \delta_x$, then $\int_\Omega \delta_{n,x}(F(\omega))d\mu(\omega) = \int_{\mathbb{R}^m} \delta_{n,x}(y)p(y)dy \to p(x)$.)

However following Watanabe we should give a rigorous meaning to the composition of a Schwartz distribution with a functional. If $\phi \in S(\mathbb{R}^m)$ and $F \in \mathbb{D}^\infty(\mathbb{R}^m)$, then the composite functional $\phi \circ F \in \mathbb{D}^\infty$. For fixed F , $\phi \to \phi \circ F$ is a linear map $S(\mathbb{R}^m) \to \mathbb{D}^\infty$. Watanabe's method is to extend it to a linear and (in some sense) continuous map from $S^*(\mathbb{R}^m) \to \mathbb{D}^{-\infty}$. He showed that in this way every Schwartz distribution T can be lifted (pulled-back) to a generalized Wiener functional $T \circ F$. (Note that $S \subset S^*$ and because of $\mathbb{D}_{p,k} \subset \mathbb{D}_{p,-k}$ we have $\mathbb{D}^\infty \subset \mathbb{D}^{-\infty}$). Then $\delta_x \circ F$ can be interpreted as a generalized functional. Using Watanabe's approach we prove

Theorem 7. If $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ is non-degenerate, then F has density ρ which is infinitely differentiable.

Hypoellipticity and Hörmander's condition

Consider the second order partial differential operator

$$L = \frac{1}{2}\sum_{i,j=1}^{m} a^{ij}(.)\partial_i\partial_j + \sum_{i=1}^{m} b^i(.)\partial_i\,.$$

and the Cauchy problem for the heat equation :

$\partial_t u(t,x) = Lu(t,x), \quad t>0, x\in\mathbb{R}^m\,;$

$u(0,x) = \phi(x)\,.$

It is known that if $\phi \in C_b^2(\mathbb{R}^m)$, then

$$u_\phi(t,x) \equiv \mathbb{E}[\phi(X(x,t,\omega))]$$

is the solution of the Cauchy problem . X is the solution of the Ito stochastic differential equation $X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s).dB_s,\ t\geq 0$ where $x\in\mathbb{R}^m, b:\mathbb{R}^m\to\mathbb{R}^m, \sigma:\mathbb{R}^m\to\mathbb{R}^m\otimes\mathbb{R}^d$.

If X_t is non-degenerate $([det\Sigma]^{-1}\in L^{\infty-}),$ then the transition probability $P(t,x,\omega) = \mu\circ X(x,t,\omega)^{-1}$ of diffusion process X has C^∞ density $p(t,x,y) = \mathbb{E}[\delta_y(X(x,t,\omega))]$ which is the fundamental solution of the above Cauchy problem (the heat kernel : $u_\phi(t,x) = \int_{\mathbb{R}^d} p(t,x,y)\phi(y)dy$).

From the theory of p.d.e. if matrix $a(x)$ is uniformly positive definite , i.e. $\exists \eta > 0$ such that $a(.) \geq \eta I$, then the conclusion is true.

Hörmander obtained a much weaker condition through the hypoellipticity of the differential operators, namely the Hörmander condition. To state Hörmander's theorem write L in form of vector fields :

$A_k(.) \equiv \sigma_k^i(.)\partial_i \,, \;\; k = 1, \cdots, d \,,$

$A_0(.) \equiv [b^i(.) - \frac{1}{2}\sum_{k=1}^d \sigma_k^j(.)\partial_j \sigma_k^i(.)]\partial_i$ (with Einstein's summation convention) , we have :

$L = \frac{1}{2}\sum_{k=1}^d A_k^2 + A_0 \,.$

Hörmander's Theorem for Hypoellipticity

If for every $x \in \mathbb{R}^m$ the Lie algebra generated by vector fields $\{A_k, [A_0, A_k], k = 1, \cdots, d)\}$ has dimension m (Hörmander condition) , then L is hypoelliptic , that is , for any open set $U \in \mathbb{R}^m$ and any Schwartz distribution $u \in \mathcal{D}^*$ if $Lu|_U \in C^\infty(U)$, then $u|_U \in C^\infty(U)$

If the Hörmander's condition holds, then (in the elliptic case) the above smooth fundamental solution exists.

($[\,.\,,\,.\,]$ is the Lie bracket : Given two C^1 vector fields V and W on $\mathbb{R}^m$, $[V,W](x) = DV(x)W(x) - DW(x)V(x))$.

The key step in the probabilistic proof of the Hörmander' theorem is to show that under the Hörmander's condition the corresponding Malliavin covariance matrix is non-degenerate.

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