Asymptotic properties of the Delannoy numbers and similar arrays

Christer O. Kiselman

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Abstract

The Delannoy numbers were introduced and studied by Henri-Auguste Delannoy (1833–1915). He investigated the possible moves on a chessboard: the numbers under consideration appear when one studies “la marche de la Reine,” i.e., how the queen moves (the binomial coefficients appear similarly for the moves of the rook).

The asymptotic behavior of the array of Delannoy numbers is studied. The regularized upper and lower radial indicators of the array are determined, proved to coincide and to be concave. We also describe the radial indicator as an infimum of linear functions, which amounts to determining its Fenchel transform.

Since the methods developed for this study apply to more general convolution equations, we prove results also for these equations.

1. Introduction

The Delannoy numbers $d(x, y)$, $(x, y) \in \mathbf{Z}^2$, are defined as 0 when $x \leqslant -1$ or when $y \leqslant -1$, as 1 when $(x, y) = (0, 0)$, and for $(x, y) \in \mathbf{N}^2 \smallsetminus \{(0, 0)\}$ by the recursion formula

$$d(x, y) = d(x - 1, y) + d(x - 1, y - 1) + d(x, y - 1). \tag{1.1}$$

They are named for Henri-Auguste Delannoy (1833–1915). He investigated the possible moves on a chessboard. The numbers under consideration here appear when one studies “la marche de la Reine.” For biographies of Delannoy, see Banderier & Schwer (2005) and Schwer & Autebert (2006).

The purpose of the present note is to determine the growth at infinity of the array of Delannoy numbers, more precisely to determine the upper and lower radial indicators (as defined in Definition 5.1) of the array (Theorem 12.5). They are proved to coincide and to be concave. We also describe the radial indicator as an infimum of linear functions, which amounts to determining its Fenchel transform (Corollary 11.1).

The Delannoy numbers solve a convolution equation on $\mathbf{Z}^2$; we shall put them into a somewhat wider framework and study similar convolution equations on $\mathbf{Z}^n$. See Theorem 8.2.

The Delannoy numbers appear in many problems in mathematics; see Sulanke (2003), who lists 29 different examples. To mention just one, $d(n,r) = d(r,n)$ is the cardinality of the ball of radius r in $\mathbf{Z}^n$ equipped with the l^1 metric (also known as the hyperoctahedron),

$$\Big\{x \in \mathbf{Z}^n;\ \|x\|_1 = |x_1| + \cdots + |x_n| \leqslant r\Big\};$$

Vassilev & Atanasov (1987), quoted here from Sulanke (2003, note 18).

To Sulanke’s examples I added a thirtieth (2008:609): for $(a,b) \in \mathbf{Z}^2$, $a+b \geqslant 0$, the number of Khalimsky-continuous functions $[0, a+b]_{\mathbf{Z}} \to \mathbf{Z}$ satisfying $f(0) = 0$ and $f(a+b) = a-b$ is equal to $d(a,b)$. For a detailed proof, see Samieinia (2010: Theorem 2.2). And then a thirty-first: a fundamental solution

$$E(x+iy) = i^{y-x} d(x,y), \qquad x+iy \in \mathbf{Z}[i],$$

for a discrete analogue of the Cauchy–Riemann operator (2008:608). Thus I came to the Delannoy numbers along two paths: digital geometry, where the Khalimsky topology is a useful structure; and discrete complex analysis.

To get a rough idea of the growth of this array we note the following easy result.

Proposition 1.1. *The Delannoy numbers satisfy*

$$3^{x\wedge y} \leqslant d(x,y) \leqslant \Big(\sqrt{2}+1\Big)^{x+y}, \qquad (x,y) \in \mathbf{N}^2.$$

In particular the diagonal numbers satisfy

$$3^x \leqslant d(x,x) \leqslant \Big(3+\sqrt{8}\Big)^x, \qquad x \in \mathbf{N}.$$

Proof. We shall perform an induction on $x+y=m$. For $m=0$ and $m=1$ the estimates hold.

If $d(x,y) \leqslant \Lambda^{x+y}$ when $x+y<m$, then for $x+y=m$ we get from the recursion formula (1.1)

$$d(x,y) \leqslant \Lambda^{x+y-1} + \Lambda^{x+y-2} + \Lambda^{x+y-1} \leqslant \Lambda^{x+y},$$

where the last inequality holds for $\Lambda = \sqrt{2}+1$.

For the estimate from below, we shall prove that

$$d(x,y) \geqslant \lambda^{(x-1)\wedge y} + \lambda^{(x-1)\wedge(y-1)} + \lambda^{x\wedge(y-1)} \geqslant \lambda^{x\wedge y}.$$

By symmetry, it is enough to consider the two cases $x = y$ and $x < y$. It is easy to verify that this works with $\lambda = 3$ in both cases. □

We note that the estimate from below shows that the array is not temperate: it grows faster than any power of $\|(x,y)\|$.

The Delannoy numbers are explicitly given by

$$d(x,y) = \sum_{j=0}^{x} \binom{x}{j}\binom{y}{j} 2^j = \sum_{j=0}^{x} \binom{y}{j}\binom{x+y-j}{y}, \qquad (x,y) \in \mathbf{N}^2,$$

Delannoy (1895:77), Comtet (1974:81). The array has the generating function

$$G_d(z,w) = \sum_{(x,y)\in\mathbf{N}^2} d(x,y) z^x w^y = \frac{1}{1-z-w-zw}, \tag{1.2}$$

Comtet (1974:81).

A lot is known about the *diagonal* (or *central*) *Delannoy numbers* $d(x,x)$; see Comtet (1974:81), Stanley (2001:185) and Sulanke (2003). Actually $d(x,x) = P_x(3)$, where $P_x(t)$ is the Legendre polynomial (Comtet 1974:81). The sequence $(d(x,x))_{x\in\mathbf{N}}$ has a generating function

$$\sum_{x\in\mathbf{N}} d(x,x)t^x = \frac{1}{\sqrt{1-6t+t^2}}$$

(Comtet 1974:81, Stanley 2001:185). We have

$$d(x,x) = 3\left(2-\tfrac{1}{x}\right) d(x-1,x-1) - \left(1-\tfrac{1}{x}\right) d(x-2,x-2) \sim \left(\sqrt{2}+1\right)^{2x}.$$

Not so much is known about the general numbers $d(x,y)$. I conjectured in (2008:609–610) that

$$d(x,y) \sim \left(\frac{r+y}{x}\right)^x \left(\frac{r+x}{y}\right)^y,$$

more precisely that

$$\begin{aligned}\lim_{\substack{t\in\dot{\mathbf{N}}\\ t\to+\infty}} \frac{1}{t}\log d(tx,ty) &= x\log\frac{r+y}{x} + y\log\frac{r+x}{y} \\ &= x\log\left(\frac{1+\sin\theta}{\cos\theta}\right) + y\log\left(\frac{1+\cos\theta}{\sin\theta}\right), \qquad (x,y)\in\dot{\mathbf{N}}^2,\end{aligned}$$

where (r,θ) are the usual polar coordinates in $\mathbf{R}^2$ and $\dot{\mathbf{N}} = \mathbf{N} \smallsetminus \{0\}$. The right-hand side is a concave function of $(x,y) \in \mathbf{R}^2_+$ and is positively homogeneous of degree one. Here we define $\mathbf{R}_+ = \{x \in \mathbf{R};\ x > 0\}$.

We shall prove this conjecture (Theorem 12.5).

For a fixed y this implies that we have polynomial growth of degree y,

$$d(x,y) \sim C_y(2x)^y, \qquad x \to +\infty.$$

Examples are: $d(x,0)=1$, $d(x,1)=2x+1$, $d(x,2)=2x^2+2x+1$, $x \geqslant 0$.

For reference we include a table with some of the Delannoy numbers.

Table 1. The Delannoy numbers $d(x,y)$ with $0 \leqslant x, y \leqslant 9$.

9	1	19	181	1159	5641	22,363	75,517	224,143	598,417	1,462,563
8	1	17	145	833	3649	13,073	40,081	108,545	265,729	598,417
7	1	15	113	575	2241	7183	19,825	48,639	108,545	224,143
6	1	13	85	377	1289	3653	8989	19,825	40,081	75,517
5	1	11	61	231	681	1683	3653	7183	13,073	22,363
4	1	9	41	129	321	681	1289	2241	3649	5641
3	1	7	25	63	129	231	377	575	833	1159
2	1	5	13	25	41	61	85	113	145	181
1	1	3	5	7	9	11	13	15	17	19
0	1	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8	9

2. Convexity properties of the array of Delannoy numbers

In the proofs of this section we shall need to calculate with differences, so we make the following definition.

Definition 2.1. Let G be an abelian group (in this note only $G=\mathbf{Z}$ and $G=\mathbf{Z}^2$ will occur), and let $a \in G$. We define a difference operator D_a by

$$(D_a f)(x) = f(x+a) - f(x), \qquad x \in G, \quad f \in \mathbf{R}^G. \qquad \square$$

Definition 2.2. We shall say that a function $f\colon \mathbf{Z}^n \to [-\infty,+\infty]$ is *convex extensible* if it is the restriction to $\mathbf{Z}^n$ of some convex function $F\colon \mathbf{R}^n \to [-\infty,+\infty]$. We say that f is *concave extensible* if $-f$ is convex extensible. $\square$

For $n=1$ it is easy to see that $f\colon \mathbf{Z} \to \mathbf{R}$ is convex extensible if and only if $D_1^2 f \geqslant 0$. It is equivalent to require that $D_b D_a f \geqslant 0$ for all $a, b \in \mathbf{N}$.

Proposition 2.3. *For every $y \in \mathbf{Z}$, the partial function $x \mapsto d(x,y)$ is increasing and convex extensible. Moreover $x \mapsto d(x,y)$ is more strongly convex than both $x \mapsto d(x,y-1)$ and $x \mapsto d(x-1,y)$. By symmetry we get the corresponding results for the partial functions $y \mapsto d(x,y)$, $x \in \mathbf{Z}$.*

Proof. We have

$$(D_{(1,0)}d)(x,y) = d(x+1,y) - d(x,y) = d(x,y-1) + d(x+1,y-1) \geqslant 0, \quad (x,y) \in \mathbf{Z}^2.$$

This shows that the partial function $x \mapsto d(x, y)$ is increasing. (The inequality is strict when $x \geqslant -1$ and $y \geqslant 1$.)

Moreover,

$$(D^2_{(1,0)}d)(x, y) = (D_{(2,0)}d)(x, y - 1) \geqslant 0, \qquad (x, y) \in \mathbf{Z}^2,$$

which shows that the partial function is convex extensible. (The inequality is strict when $x \geqslant -2$ and $y \geqslant 2$.)

We also note that $D_{(0,1)}D_{(1,0)}d = 2d \geqslant 0$: the function is supermodular. This gives

$$(D_{(0,1)}D^2_{(1,0)}d)(x, y) = 2(D_{(1,0)}d)(x, y) \geqslant 0, \qquad (x, y) \in \mathbf{Z}^2.$$

This shows that $x \mapsto d(x, y)$ is more strongly convex than $x \mapsto d(x, y - 1)$. (The inequality is strict when $x \geqslant -1$ and $y \geqslant 1$.)

Finally,

$$(D^3_{(1,0)}d)(x, y) = (D_{(1,0)}D_{(2,0)}d)(x, y - 1) \geqslant 0, \qquad (x, y) \in \mathbf{Z}^2.$$

This shows that $x \mapsto d(x, y)$ is more strongly convex than $x \mapsto d(x - 1, y)$. (The inequality is strict when $x \geqslant -3$ and $y \geqslant 3$.) ☐

Proposition 2.4. *The restriction of d to any bidiagonal is increasing and convex extensible, i.e., the function $\mathbf{Z} \ni x \mapsto f_m(x) = d(x, m + x)$ is increasing and convex extensible for all choices of $m \in \mathbf{Z}$.*

Proof. We find that, with $m = y - x$,

$$(D_1 f_m)(x) = (D_{(1,1)}d)(x, y) = d(x, y + 1) + d(x + 1, y) \geqslant 0,$$

which shows that f_m is increasing. Combining with the same formula for $x + 1$ we get

$$(D^2_1 f_m)(x) = (D_1 f_{m+1})(x) + (D_1 f_{m-1})(x + 1),$$

where the last expression is nonnegative since f_{m+1} and f_{m-1} are increasing. This shows that f_m is convex extensible. ☐

Proposition 2.5. *The restriction of d to any antidiagonal is concave on an interval close to the main diagonal; more precisely, the function*

$$J_m = [s_m, m - s_m] \ni x \mapsto g_m(x) = d(x, m - x)$$

is concave, where $s_m = 0$ for $m = 0, 1, 2$, $s_m = \frac{1}{2}(m - 4)$ for m even, $m \geqslant 4$, and $s_m = \frac{1}{2}(m - 3)$ for m odd, $m \geqslant 3$.

Proof. The interval J_m is not long; in fact $\mathbf{card}(J_m) = m - 2s_m + 1$ is 4 for odd $m \geqslant 3$ and 5 for even $m \geqslant 4$.

We see from Table 1 that the statement is true for $m = 0, 1, 2, 3, 4$. We find that

$$\begin{aligned} &d(x - 1, y + 1) - 2d(x, y) + d(x + 1, y - 1) \\ &\qquad = [d(x - 2, y + 1) - d(x - 1, y) - d(x, y - 1) + d(x + 1, y - 2)] \\ &\qquad + [d(x - 1, y) - 2d(x - 1, y - 1) + d(x, y - 2)]. \end{aligned} \tag{2.1}$$

This can work as the induction step in an induction over $x+y$, for if the sum of the coordinates is $x+y=m$ in the left-hand side, then it is $m-1$ in the first bracket on the right, and $m-2$ in the second bracket on the right.

If m is odd, this induction step works, for then the values of x in the first bracket belong to the interval J_{m-1} and those in the second bracket to J_{m-2}. We can start with $m=5$.

If m is even, a special argument is needed, for if $x-1=s_m$, then $x-2=s_m-1=s_{m-1}-1<s_{m-1}$, so that $x-2$ falls outside the interval J_{m-1}. However, then $d(x,y-1)=d(x+1,y)$, so that the first bracket in (2.1) is equal to

$$d(x-2,y+1)-d((x-1,y-2)\leqslant 0.$$

The second bracket causes no problem. The induction works also in this case. □

3. The logarithms of the Delannoy numbers

The logarithms of the Delannoy numbers seem to display interesting concavity properties, but are less easy to calculate with. I list a few conjectures concerning them. First a table with some of the values of the function $\log d$.

Table 2. The natural logarithms of the Delannoy numbers with $0\leqslant x,y\leqslant 9$, rounded off to three decimal places.

9	0	2.944	5.198	7.055	8.638	10.015	11.232	12.320	13.302	14.196
8	0	2.833	4.977	6.725	8.202	9.478	10.599	11.594	12.490	13.302
7	0	2.708	4.727	6.354	7.715	8.879	9.895	10.792	11.595	12.320
6	0	2.565	4.443	5.932	7.162	8.203	9.104	9.895	10.599	11.232
5	0	2.398	4.111	5.442	6.524	7.428	8.203	8.879	9.478	10.015
4	0	2.197	3.714	4.860	5.771	6.524	7.162	7.715	8.202	8.638
3	0	1.946	3.219	4.143	4.860	5.442	5.932	6.354	6.725	7.055
2	0	1.609	2.565	3.289	3.714	4.111	4.443	4.727	4.977	5.198
1	0	1.099	1.609	1.946	2.197	2.398	2.565	2.708	2.833	2.944
0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

Conjecture 3.1. *For every $y\in\mathbf{Z}$, the partial function $x\mapsto\log d(x,y)$ is concave extensible.* □

The conjectured concavity of $x\mapsto\log d(x,y)$ and the proved convexity of $x\mapsto d(x,y)$ can be expressed by a double inequality

$$\sqrt{d(x-1,y)d(x+1,y)}\leqslant d(x,y)\leqslant\tfrac12 d(x-1,y)+\tfrac12 d(x+1,y),\qquad (x,y)\in\mathbf{Z}^2.$$

Svante Janson (personal communication 2012-03-01) made an interesting comment in support of this conjecture: let us consider paths from $(0,0)$ to $(2x,2y)$, which are $d(2x,2y)$ in number, and let $h(s)$ be the number of paths that pass through the point (s,y). They are $d(s,y)d(2x-s,y)$ in number; in particular, $h(x)=d(x,y)^2$ and $h(x-1)=h(x+1)=d(x-1,y)d(x+1,y)$. It seems to be highly plausible that $h(x-1)\leqslant h(x)$, which is equivalent to Conjecture 3.1.

Conjecture 3.2. *The function* $\mathbf{Z} \ni x \mapsto \log g_m(x) = \log d(x, m-x) \in [-\infty, +\infty[$ *is concave extensible for all choices of* $m \in \mathbf{Z}$. □

Here one could make a comment like the one for Conjecture 3.1 in support of this conjecture.

From the conjectured concavity of these functions one might guess that the function $\log d$ would be concave extensible as a function of two integer variables. However, as is evident from Table 2, this is not true. Instead I make the following conjecture.

Conjecture 3.3. *The function* $\mathbf{N} \ni x \mapsto \log d(x, x) \in [0, +\infty[$ *is convex extensible.* □

The function $[-m^-, +\infty[\ni x \mapsto \log f_m(x) = \log d(x, m+x) \in [-\infty, +\infty[$ seems to be convex extensible close to the diagonal (for small values of $|m|$) and concave extensible in certain regions away from the diagonal.

4. The Fenchel transformation

The Fenchel transformation is a tropical counterpart of the Fourier transformation.

The Fenchel transform of a function $f\colon \mathbf{R}^n \to [-\infty, +\infty]$ is defined as

$$\tilde{f}(\xi) = \sup_{x \in \mathbf{R}^n} (\xi \cdot x - f(x)), \qquad \xi \in \mathbf{R}^n.$$

The second transform $\tilde{\tilde{f}}$ satisfies $\tilde{\tilde{f}} \leqslant f$ with equality if and only if f is convex, lower semicontinuous, and takes the value $-\infty$ only if it is $-\infty$ everywhere.

If f is only defined on the integer points, we extend it as $+\infty$ on $\mathbf{R}^n \smallsetminus \mathbf{Z}^n$.

If f takes only the values 0 and $+\infty$, then $\tilde{f}$ is positively homogeneous of degree one as the supremum of a family of linear functions:

$$\tilde{f}(\xi) = \sup_{\substack{x \in \mathbf{R}^n \\ f(x) = 0}} \xi \cdot x, \qquad \xi \in \mathbf{R}^n.$$

Conversely, if f is positively homogeneous of degree one, then $\tilde{f}$ can take only the values $0, +\infty, -\infty$. Indeed, if $f(tx) = tf(x)$ for all $t > 0$, then $t\tilde{f} = \tilde{f}$ for all $t > 0$, and this is only true for the three values $0, +\infty, -\infty$. The value $-\infty$ will not occur for the functions we are studying, so then $\tilde{f}$ is an indicator function, $\tilde{f} = \mathbf{ind}_M$ for some set M, where we define generally $\mathbf{ind}_M$ to take the value 0 in M and $+\infty$ in its complement.

5. The radial indicators

Definition 5.1. Given a function $f\colon \mathbf{R}^n \to [0, +\infty[$ we define its *upper radial indicator* as

$$p_f(x) = \limsup_{t \to +\infty} \frac{1}{t} \log f(tx), \qquad x \in \mathbf{R}^n,$$

and its *lower radial indicator* as

$$q^f(x) = \liminf_{t\to+\infty} \frac{1}{t}\log f(tx), \qquad x \in \mathbf{R}^n.$$

We also define the *regularized upper radial indicator* as

$$p_f^*(x) = \limsup_{y\to x} p_f(y), \qquad x \in \mathbf{R}^n,$$

the smallest upper semicontinuous majorant of p_f, and the *regularized lower radial indicator* as

$$q_*^f(x) = \liminf_{y\to x} q_f(y), \qquad x \in \mathbf{R}^n,$$

the largest lower semicontinuous minorant of q^f. □

If f is only defined in $\mathbf{Z}^n$, we have to restrict t in the definition of p_f and q^f to those t for which tx belongs to $\mathbf{Z}^n$. This means that p_f and q^f will be defined only in $\mathbf{RZ}^n$. Similarly, in the definitions of the regularized indicators, we have to restrict y to $\mathbf{RZ}^n$. But then p_f^* and q_*^f will be well defined everywhere, since $\mathbf{RZ}^n \supset \mathbf{Q}^n$ is dense in $\mathbf{R}^n$.

Proposition 5.2. *Let $f\colon \mathbf{N}^n \to [0,+\infty[$ be any function and let $\sigma = (\sigma_1,\dots,\sigma_n) \in \mathbf{R}^n$. Then the following four properties are equivalent.*
(A). *For every positive ε there exists a constant C_ε such that*

$$f(x) \leqslant C_\varepsilon e^{-(\sigma-\varepsilon\mathbf{1})\cdot x}, \qquad x \in \mathbf{N}^n,$$

where $\mathbf{1} = (1,1,\dots,1) \in \mathbf{N}^n$.
(B). *The regularized upper radial indicator of f satisfies*

$$p_f^*(x) \leqslant -\sigma\cdot x, \qquad x \in \mathbf{R}_+^n.$$

(C). *The Fenchel transform of $-p_f^*$ satisfies $(-p_f^*)^\sim(\sigma) \leqslant 0$.*
(D). *$\sigma \in M_f$, where M_f is the set such that $(-p_f^*)^\sim = \mathbf{ind}_{M_f}$.*

Proof. That (B), (C) and (D) are all equivalent is obvious from the definitions.

If (A) holds, we get

$$\frac{1}{t}\log f(tx) \leqslant \frac{1}{t}\log C_\varepsilon - (\sigma-\varepsilon\mathbf{1})\cdot x \to -(\sigma-\varepsilon\mathbf{1})\cdot x.$$

Hence $p_f(x) \leqslant -\sigma\cdot x + \varepsilon\mathbf{1}\cdot x$ for all $x \in \mathbf{N}^n$. The same inequality holds for p_f^*. Letting $\varepsilon \to 0$ we get (B).

Conversely, let us assume that (A) does not hold. Then there exists a positive number ε such that for every $k \in \mathbf{N}$ there is a point $x^{(k)} \in \mathbf{N}^n$ such that

$$f(x^{(k)}) > ke^{-(\sigma-\varepsilon\mathbf{1})\cdot x^{(k)}}.$$

It follows that $t_k = \sum_{j=1}^n x_j^{(k)}$ must tend to $+\infty$. Define $y^{(k)} = x^{(k)}/t_k$. The points $y^{(k)}$ belong to the compact set $S = \{s \in \mathbf{R}^n;\ 0 \leqslant s_j \text{ and } \sum s_j = 1\}$, so there exists a subsequence which converges to some point $y \in S$. After a change of notation we may assume that the sequence $(y^{(k)})_k$ itself converges to y. We obtain

$$\begin{aligned} p_f^*(y) &\geqslant \limsup \frac{1}{t_k} \log f(x^{(k)}) \geqslant \limsup \frac{1}{t_k}(-(\sigma - \varepsilon \mathbf{1}) \cdot x^{(k)}) \\ &= -(\sigma - \varepsilon \mathbf{1}) \cdot y > -\sigma \cdot y. \end{aligned}$$

Hence (B) does not hold. We are done. □

Conjecture 5.3. *For all* $(x, y) \in \mathbf{N}^2$*, the function* $\dot{\mathbf{N}} \ni t \mapsto t^{-1} \log d(tx, ty)$ *is increasing.* □

If this is true, the upper radial indicator satisfies

$$p_d(x, y) = \lim_{\substack{t \in \dot{\mathbf{N}} \\ t \to +\infty}} \frac{1}{t} \log d(tx, ty) = \sup_{t \in \dot{\mathbf{N}}} \frac{1}{t} \log d(tx, ty), \qquad (x, y) \in \mathbf{N}^2.$$

However, we shall prove that the first equality holds, i.e., that the limit superior is actually a limit.

It is easy to see that

$$\log d(x, y) \leqslant \frac{1}{t} \log d(tx, ty), \qquad t \in \dot{\mathbf{N}}, \quad (x, y) \in \mathbf{N}^2.$$

This is implied by the conjecture, but is not enough: it does not imply that

$$\frac{1}{2} \log d(2x, 2y) \leqslant \frac{1}{3} \log d(3x, 3y).$$

Consider the paths that go from $(0, 0)$ to $(6x, 6y)$ and pass through $(3x, 3y)$ (they are $d(3x, 3y)^2$ in number), and then those that pass through both $(2x, 2y)$ and $(4x, 4y)$ (they are $d(2x, 2y)^3$ in number). It seems very likely that one restriction decreases the number of paths less than two restrictions.

More generally, consider the paths that go from $(0, 0)$ to $((t^2 + t)x, (t^2 + t)y)$ and pass through the $t - 1$ points $(j(t + 1)x, j(t + 1)y)$, $j = 1, \ldots, t - 1$, (they are $d((t + 1)x, (t + 1)y)^t$ in number), and then those that pass through the t points (ktx, kty), $k = 1, \ldots, t$, (they are $d(tx, ty)^{t+1}$ in number). It seems very likely that $t - 1$ restrictions decrease the number of paths less than t restrictions. This is equivalent to the conjecture.

6. Convolution

We define the convolution product $h = f * g$ of two functions $f, g \colon \mathbf{Z}^n \to \mathbf{R}$ by

$$h(x) = \sum_{y \in \mathbf{Z}^n} f(x - y) g(y), \qquad x \in \mathbf{Z}^n,$$

provided the sum is finite for all x. We can define two kinds of algebras satisfying this provision.

(1) The first is the algebra of all functions with finite support. (The support of a function is here just the set where it is nonzero.)

(2) Given a vector $\alpha \neq 0$ we consider the algebra of all functions with support contained in a translate of the cone $K_\alpha = \{x \in \mathbf{R}^n;\ \alpha \cdot x \geqslant \|x\|\}$.

However, sometimes we need to define a convolution product in other situations.

(3) We can define a convolution product $f_1 * \cdots * f_k$ when all factors except one have finite support.

(4) We can define a convolution product $f_1 * \cdots * f_k$ when all factors except one have their support contained in translates of a cone K_α and the remaining one has its support contained in a translate of the half space $\{x \in \mathbf{R}^n;\ \alpha \cdot x \geqslant 0\}$ with the same vector α.

In these cases, the associative law holds.

Example 6.1. That some care is needed is shown by a simple example: take $f(x) = 1$ for all $x \in \mathbf{Z}$; $g = \delta_{-1} - \delta_0$ (a difference operator); and $h(x) = 1$ for all $x \in \mathbf{N}$, $h(x) = 0$ for $x \leqslant -1$.

Then $f * g = 0$ (case (3)) and $(f * g) * h = 0$, while $g * h = \delta_{-1}$ (case (3)) and $f * (g * h) = 1 \neq 0$.

Note that neither $f * h$ nor $f * g * h$ can be defined in accordance with any of the cases (1)–(4). □

The recursion formula (1.1) can be written

$$d * (\delta - \delta_{(1,0)} - \delta_{(1,1)} - \delta_{(0,1)}) = \delta, \tag{6.1}$$

where δ_a is the Kronecker delta placed at a and $\delta = \delta_0$. For the binomial coefficients we have a similar formula,

$$b * (\delta - \delta_{(1,0)} - \delta_{(1,0)}) = \delta. \tag{6.2}$$

Both equations are thus of the form

$$f * (\delta - \mu) = \delta, \tag{6.3}$$

which is why we shall study this equation in the next two sections.

7. The Fourier transformation

We define the Fourier transform $\hat{f}$ of a function $f\colon \mathbf{Z}^n \to \mathbf{C}$ by

$$\hat{f}(\zeta) = \sum_{x \in \mathbf{Z}^n} f(x) e^{i\zeta x}, \qquad \zeta \in \mathbf{C}^n,$$

for those ζ for which the sum has a good sense. Since $d(x, y) \leqslant (\sqrt{2} + 1)^{x+y}$, we see that $\hat{d}$ is well defined for $\operatorname{Im}\zeta_1,\ \operatorname{Im}\zeta_2 > \log(\sqrt{2} + 1)$.

If f satisfies condition (A) in Proposition 5.2, then for every positive ε, $\hat{f}$ is holomorphic for $\operatorname{Im}\zeta_j > -\sigma_j + \varepsilon$, $j = 1, \ldots, n$, and satisfies

$$|\hat{f}(\zeta)| \leqslant \prod_{j=1}^{n} \left(1 - e^{-\sigma_j - \operatorname{Im}\zeta_j + \varepsilon}\right)^{-1} \tag{7.1}$$

there. Letting ε tend to zero we see that $\hat{f}$ is holomorphic for $\operatorname{Im}\zeta_j > -\sigma_j$, $j = 1, \dots, n$.

We have adapted the signs to the usual conventions concerning Fourier series. The Fourier inversion formula therefore is the formula for retrieving the coefficients of the Fourier series, i.e.,

$$f(x) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{f}(\xi) e^{-i\xi\cdot x}\, d\xi_1 \cdots d\xi_n, \qquad x \in \mathbf{Z}^n.$$

Here $\xi = (\xi_1, \dots, \xi_n)$ are n real variables.

The Fourier transform of a convolution product is given by

$$(f * g)\hat{} = \hat{f}\,\hat{g}.$$

Since d grows fast, we cannot apply the inversion formula to $\hat{d}$, but to $\hat{d}_a$, the Fourier transform of $d_a(x, y) = d(x, y)a^{x+y}$, for any positive constant $a < \sqrt{2} - 1$. We obtain

$$d(x, y)a^{x+y} = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \hat{d}_a(\xi) e^{-i(\xi_1 x + \xi_2 y)}\, d\xi_1\, d\xi_2, \qquad (x, y) \in \mathbf{Z}^2,$$

where $\hat{d}_a(\zeta) = \hat{d}(\zeta_1 - i\log a, \zeta_2 - i\log a)$, which means that for $\hat{d}$, the integral goes over a square in $\mathbf{R}^2$ translated in $\mathbf{C}^2$ by the imaginary vector $-i(\log a, \log a)$.

The convolution formula (6.3) yields

$$\hat{f}(\zeta) = \frac{1}{1 - \hat{\mu}(\zeta)} \qquad \zeta \in \mathbf{C}^n, \quad \operatorname{Im}\zeta_j \text{ large.} \tag{7.2}$$

In the case $f = d$ the formula reads

$$\hat{d}(\zeta) = \frac{1}{1 - e^{i\zeta_1} - e^{i\zeta_2} - e^{i(\zeta_1+\zeta_2)}}, \qquad \zeta \in \mathbf{C}^2, \quad \operatorname{Im}\zeta_j > \log\left(\sqrt{2} + 1\right). \tag{7.3}$$

This is of course equivalent to the formula (1.2) for the generating function. In fact, $\hat{d}(\zeta) = G_d(e^{i\zeta_1}, e^{i\zeta_2})$.

With this expression for $\hat{d}$ we obtain

$$d(x, y) = \frac{1}{4\pi^2 a^{x+y}} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{-i(\xi_1 x + \xi_2 y)}}{1 - ae^{i\xi_1} - ae^{i\xi_2} - a^2 e^{i(\xi_1+\xi_2)}}\, d\xi_1\, d\xi_2, \qquad (x, y) \in \mathbf{Z}^2,$$

where a is any constant satisfying $0 < a < \sqrt{2} - 1$.

The simple estimate $\sup \leqslant \sum$ for nonnegative functions defined on $\mathbf{Z}^n$ gives, for $g = e^{-f}$,

$$e^{\tilde{f}(\xi)} = \sup_{x\in\mathbf{Z}^n} e^{\xi\cdot x} e^{-f(x)} \leqslant \sum_{x\in\mathbf{Z}^n} e^{\xi\cdot x} e^{-f(x)} = \hat{g}(-i\xi), \qquad \xi \in \mathbf{R}^n.$$

8. Fundamental solutions for convolution operators

Both the Delannoy numbers and the binomial coefficients satisfy convolution equations of the form $f * (\delta - \mu) = \delta$. In the first case, $\mu = \delta_{(1,0)} + \delta_{(1,1)} + \delta_{(0,1)}$; in the second, $\mu = \delta_{(1,0)} + \delta_{(0,1)}$. In this section we shall prove some results on such convolution equations in a slightly more general setting.

So let $\mu\colon \mathbf{Z}^n \to [0, +\infty[$ be a function which is nonzero only at finitely many points $x \in \mathbf{N}^n$, $x \neq 0$. This condition will enable us to make inductions on $\mathbf{1} \cdot x = \sum x_j$. Then $\delta - \mu$ is a fundamental solution to the operator $u \mapsto u * f$, more precisely, if we define F_θ for a fixed vector θ with positive components as the set of all functions u such that $u(x)$ is nonzero only if $\theta \cdot x \geqslant -C_u$, then with

$$P(u) = u * f, \qquad u \in F_\theta,$$

(case (4) in Section 6) and

$$Q(v) = v * (\delta - \mu), \qquad v \in F_\theta,$$

(case (3) in Section 6) we have

$$Q(P(u)) = (u * f) * (\delta - \mu) = u * (f * (\delta - \mu)) = u * \delta = u, \qquad u \in F_\theta,$$

and

$$P(Q(v)) = (v * (\delta - \mu)) * f = v * ((\delta - \mu) * f) = v * \delta = v, \qquad v \in F_\theta.$$

The associative law holds under the conditions mentioned.

Proposition 8.1. *Let a function $\mu\colon \mathbf{Z}^n \to [0, +\infty[$ which is nonzero only at finitely many points x such that $x \neq 0$ and $x_j \geqslant 0$ and a real vector $\sigma = (\sigma_1, \dots, \sigma_n)$ be given. Assume that*

$$\hat{\mu}(-i\sigma) = \sum_y \mu(y) e^{\sigma \cdot y} \leqslant 1. \tag{8.1}$$

Then the unique function $f\colon \mathbf{Z}^n \to \mathbf{R}$ which is zero outside $\mathbf{N}^n$ and solves $f*(\delta-\mu) = \delta$ can be estimated as

$$f(x) \leqslant e^{-\sigma \cdot x}, \qquad x \in \mathbf{Z}^n. \tag{8.2}$$

Conversely, if for any positive ε an estimate

$$f(x) \leqslant C_\varepsilon e^{-(\sigma - \varepsilon \mathbf{1}) \cdot x}, \qquad x \in \mathbf{Z}^n, \tag{8.3}$$

holds for some constant C_ε, then the condition (8.1) *holds.*

Proof. For $x \neq 0$ we have

$$f(x) = (f * \mu)(x) = \sum_y \mu(y) f(x - y).$$

Now the values of y for which $\mu(y) \neq 0$ must satisfy $\mathbf{1} \cdot y \geqslant 1$, so that $\mathbf{1} \cdot (x - y) \leqslant \mathbf{1} \cdot x - 1$. By induction we may therefore assume that all the $f(x - y)$ that occur satisfy the estimate. We get

$$f(x) \leqslant \sum \mu(y) e^{-\sigma \cdot (x-y)} \leqslant e^{-\sigma \cdot x},$$

provided the inequality (8.1) holds.

Conversely, assume that the estimate (8.3) is valid, but that condition (8.1) does not hold:

$$\hat{\mu}(-i\sigma) = \sum_y \mu(y) e^{\sigma \cdot y} > 1.$$

We note that $\mathbf{1} \cdot y = \sum y_j \geqslant 1$ in the support of μ, which means that $\hat{\mu}(-i(\sigma - t\mathbf{1})) = \sum_y \mu(y) e^{(\sigma - t\mathbf{1}) \cdot y}$ takes a value larger than 1 for $t = 0$ and tends to zero when t tends to $+\infty$ since $\mathbf{1} \cdot y = \sum_j y_j > 0$. We first determine a positive number s such that $\sum_y \mu(y) e^{(\sigma - s\mathbf{1}) \cdot y} = 1$. Hence $\hat{\mu}(-i(\sigma - t\mathbf{1}))$ is smaller than 1 for $t > s$ and equal to 1 when $t = s$. This implies that

$$\hat{f}(-i(\sigma - t\mathbf{1})) = \frac{1}{1 - \hat{\mu}(-i(\sigma - t\mathbf{1}))}, \qquad t > s,$$

is finite for $t > s$ and tends to $+\infty$ as $t \searrow s$.

On the other hand, we have the estimate (7.1), which shows that $\hat{f}(\zeta)$ is bounded when $\operatorname{Im} \zeta_j \geqslant -\sigma_j + \varepsilon$; hence $\hat{f}(-i(\sigma - t\mathbf{1}))$ is bounded when $t \geqslant \varepsilon$. If we choose $\varepsilon = s > 0$ we obtain a contradiction. Hence we cannot have $\hat{\mu}(-i\sigma) > 1$. ☐

By combining Propositions 5.2 and 8.1 we obtain the following result.

Theorem 8.2. *Given a function $\mu \colon \mathbf{Z}^n \to [0, +\infty[$ which is nonzero only at finitely many points x such that $x \neq 0$ and $x_j \geqslant 0$, the function $f \colon \mathbf{Z}^n \to \mathbf{R}$ which is nonzero only in $\mathbf{N}^n$ and solves the equation $f * (\delta - \mu) = \delta$ satisfies $(-p_f^*)^{\sim} = \mathbf{ind}_{M_f}$, where*

$$M_f = \{\sigma \in \mathbf{R}^n;\ \hat{\mu}(-i\sigma) \leqslant 1\}.$$ ☐

9. The binomial coefficients

Let

$$b(x, y) = \binom{x+y}{x} = \frac{(x+y)!}{x!\, y!}, \qquad (x, y) \in \mathbf{N}^2,$$

be the binomial coefficients. We define them also when $x \leqslant -1$ or $y \leqslant -1$ by taking them equal to zero then. Using Stirling's formula in the simple form

$$\log x! = x \log x - x + O(\log x), \qquad x \to +\infty,$$

we see that the regularized radial indicators are

$$q_*^b(x, y) = p_b^*(x, y) = x \log(1 + y/x) + y \log(1 + x/y), \qquad (x, y) \in \mathbf{R}_+^2. \tag{9.1}$$

In particular, $p_b^*(x,x) = 2x\log 2 \approx 1.3863x$.

The function p_b^* is positively homogeneous of order 1 and concave. To prove concavity it is enough to note that $x \mapsto \log(1+x)$ is concave. This implies that the homogeneous function $(x,y) \mapsto y\log(1+x/y)$ is concave; by symmetry also $(x,y) \mapsto x\log(1+y/x)$ is concave.

The gradient of p_b^* is

$$\operatorname{grad} p_b^*(x,y) = (\log(1+y/x), \log(1+x/y)), \qquad x,y>0.$$

We note the following special case of Theorem 8.2.

Proposition 9.1. *The Fenchel transform* $(-p_b^*)^\sim$ *of the function* $-p_b^*$ *is equal to* $\mathbf{ind}_{M_b}$, *where*

$$M_b = \{\sigma \in \mathbf{R}^2;\ e^{\sigma_1} + e^{\sigma_2} \leqslant 1\}.$$

Since $-p_b$ *is convex, lower semicontinuous, and does not take the value* $-\infty$, *we also get* $q^b = p_b = -(-p_b)^{\sim\sim} = -(\mathbf{ind}_{M_b})^\sim$.

The array of binomial coefficients satisfies the assertions in Propositions 2.3 and 2.4, and also that in Proposition 2.5, but with a different interval: for b, the interval of concavity is $[t_m, m-t_m]$, where

$$t_m = \left\lceil \tfrac{1}{2}\left(m - 2 - \left\lfloor \sqrt{m+2} \right\rfloor\right)\right\rceil.$$

10. An array for comparison

In this section we shall study another array, denoted by c, which will be used to describe the asymptotic properties of the array of Delannoy numbers.

We define

$$c(x,y) = \left(\frac{r+y}{x}\right)^x \left(\frac{r+x}{y}\right)^y, \qquad (x,y) \in \mathbf{R}^2_+,$$

where $r = \sqrt{x^2+y^2}$. We put $c(x,0) = c(0,y) = 1$ for $x,y \geqslant 0$, and $c(x,y) = 0$ when $x<0$ or $y<0$.

Its regularized upper and lower radial indicators coindide:

$$q_*^c(x,y) = q^c(x,y) = p_c^*(x,y) = p_c(x,y) = \log c(x,y) = x\log\frac{r+y}{x} + y\log\frac{r+x}{y}$$

for all $(x,y) \in \mathbf{R}^2_+$. At $(x,0)$ and $(0,y)$, $x,y \geqslant 0$, the indicators take the value 0, and they take the value minus infinity when $x<0$ or $y<0$. Hence the asymptotic properties of c can be read off directly. By way of contrast, some work will be needed for setting up an approximate recursion formula similar to (1.1). Note the similarity between the formula for p_c^* above and that for p_b^* in (9.1).

The function $\log c$ is concave. To verify that this is so, it is enough to verify that $x \mapsto \log(\sqrt{x^2+1}+x) = \psi_1(x)$ is concave. And this is indeed the case: $\psi_1'(x) = 1/r$, a decreasing function. This implies that also $(x,y) \mapsto y\psi_1(x/y) = y\log((r+x)/y)$ is concave; by symmetry, also $(x,y) \mapsto x\psi_1(y/x) = x\log((r+y)/x)$ is concave.

Proposition 10.1. *Let* $\sigma = (\sigma_1, \sigma_2) \in \mathbf{R}^2_+$. *The function* $\log c$ *has a linear majorant*

$$\log c(x,y) \leqslant -\sigma_1 x - \sigma_2 y, \qquad (x,y) \in \mathbf{R}^2,$$

if and only if $e^{\sigma_1} + e^{\sigma_2} + e^{\sigma_1+\sigma_2} \leqslant 1$.

Proof. In view of the homogeneity it is enough to consider the restriction of the functions to $y = 1$. Thus the question is when we have

$$\omega(x) = x \log \frac{r+1}{x} + \log(r+x) \leqslant -\sigma_1 x - \sigma_2, \qquad x > 0, \tag{10.1}$$

where $r = \sqrt{x^2+1}$. Now ω is strictly concave: its derivative is

$$\omega'(x) = \log(r+1) - \log x$$

with second derivative

$$\omega''(x) = -\frac{1}{rx} < 0.$$

So the inequality holds for all positive x if and only if it holds at the point x for which the derivatives of the two sides are equal, thus for x such that $\omega'(x) = -\sigma_1$, i.e., $r+1 = e^{-\sigma_1}x$. This is the point $x = 2e^{-\sigma_1}/(e^{-2\sigma_1} - 1)$. For this x the inequality reads $1 + e^{-\sigma_1} + e^{-\sigma_2} \leqslant e^{-\sigma_1-\sigma_2}$. We are done. ☐

Corollary 10.2. *The Fenchel transform* $(-p_c)^{\sim}$ *of* $-p_c$ *is equal to* $\mathbf{ind}_{M_c}$, *where*

$$M_c = \{\sigma \in \mathbf{R}^2;\ e^{\sigma_1} + e^{\sigma_2} + e^{\sigma_1+\sigma_2} \leqslant 1\}.$$

Since $-p_c$ *is convex, lower semicontinuous, and does not take the value* $-\infty$, *we also get* $q^c = p_c = -(-p_c)^{\approx} = -(\mathbf{ind}_{M_c})^{\sim}$. ☐

11. Estimates from above for the Delannoy numbers

We note the following special case of Theorem 8.2.

Proposition 11.1. *The Fenchel transform* $(-p_d^*)^{\sim}$ *of the function* $-p_d^*$ *is equal to* $\mathbf{ind}_{M_d}$, *where*

$$M_d = \{\sigma \in \mathbf{R}^2;\ e^{\sigma_1} + e^{\sigma_2} + e^{\sigma_1+\sigma_2} \leqslant 1\}.$$ ☐

Theorem 11.2. *The regularized upper radial indicator* p_d^* *of the array of Delannoy numbers is*

$$p_d^*(x,y) = p_c(x,y) = \log c(x,y) = x \log \frac{r+y}{x} + y \log \frac{r+x}{y}, \qquad (x,y) \in \mathbf{R}^2_+,$$

where $r = \sqrt{x^2+y^2}$.

Proof. We compare Corollary 10.2 and Proposition 11.1, observe that the sets M_d and M_c coincide, and find that

$$(-p_d^*)^{\tilde{}} = (-p_c)^{\tilde{}} = \mathbf{ind}_{M_c}.$$

This implies that

$$p_d^* \leqslant -(-p_d^*)^{\tilde{}\tilde{}} = -(-p_c)^{\tilde{}\tilde{}} = p_c,$$

where the last equality holds because $-p_c$ is convex, lower semicontinuous and does not take the value $-\infty$. We do not know yet whether $-p_d^*$ is convex, so we cannot assert right now that the first inequality is an equality. However, from the properties of the function $p_c = \log c$ we shall exclude that the inequality $p_d^* \leqslant p_c$ can be strict at any point. This depends on the strict concavity of the function $\omega(x) = p_c(x, 1) = \log c(x, 1)$; see (10.1).

If there exists a point $(x_0, y_0) \in \mathbf{R}^2$ with $p_d^*(x_0, y_0) < p_c(x_0, y_0)$, we must have $x_0, y_0 > 0$, and in view of the homogeneity we may assume that $y_0 = 1$. The tangent to the graph of ω at the point $(x_0, \omega(x_0))$ has the form $L(x) = -\xi x - \eta$, where $-\xi = \omega'(x_0)$ and $-\eta = \omega(x_0) - \omega'(x_0)x_0$. Hence (ξ, η) belongs to the boundary of M_c: we have $e^\xi + e^\eta + e^{\xi+\eta} = 1$.

Now if $p_d^*(x_0, 1)$ is smaller than $p_c(x_0, 1)$, this must be true also in a neighborhood of x_0, since p_d^* is upper semicontinuous and p_c is continuous. In view of the strict concavity of ω, this implies that we can lower the line defined by L and still keep it as a majorant of p_d^*, so that $p_d^*(x, 1) \leqslant -\xi x - \eta - \varepsilon$ for some positive number ε and all $x > 0$. This implies that $(-p_d^*)^{\tilde{}}(\xi, \eta+\varepsilon) \leqslant 0$, which means that $(\xi, \eta+\varepsilon)$ belongs to M_d. But $e^\xi + e^{\eta+\varepsilon} + e^{\xi+\eta+\varepsilon} > e^\xi + e^\eta + e^{\xi+\eta} = 1$, contradicting the result on M_d. This contradiction proves that the inequality $p_d^* \leqslant p_c$ cannot be strict anywhere; we must have $p_d^* = p_c$ everywhere. □

If Conjecture 3.1 were true, the concavity of p_d^* would follow rather directly. Here I have obtained the concavity of p_d^* indirectly, by proving that it equals p_c, which is known to be concave.

12. An approximate recursion formula

While d is defined by a recursion formula, the array c is defined explicitly without recursion. We shall now prove an approximate recursion formula for c.

Lemma 12.1. *For every positive ε there exist a number N_ε such that the quotient*

$$\frac{c(x-1, y) + c(x-1, y-1) + c(x, y-1)}{c(x, y)} \tag{12.1}$$

is at least $e^{-\varepsilon}$ and at most e^ε when $(x, y) \in \mathbf{R}^2$, $x, y \geqslant 0$, $x + y \geqslant N_\varepsilon$.

Proof. Let us introduce

$$(x_j, y_j) = \begin{cases} (x, y), & j = 0, \\ (x-1, y), & j = 1, \\ (x-1, y-1), & j = 2, \\ (x, y-1), & j = 3, \end{cases}$$

$r_j = \sqrt{x_j^2 + y_j^2}$, $j = 0, 1, 2, 3$, and $r = r_0$. Then $r_1 \approx r - x/r$, $r_2 \approx r - (x+y)/r$, and $r_3 \approx r - y/r$. In the approximations we do, it is not enough that $r_1/(r - x/r)$ tend to one; we must prove that even raised to the power x this expression tends to one. However, these approximations are quite good as shown by the next lemma.

Lemma 12.2. *We have*

$$\begin{aligned} &1 \leqslant \frac{r_1}{r - x/r} \leqslant 1 + \frac{1}{2r^2}, && x \geqslant 2, \quad y \geqslant 0; \\ &1 \leqslant \frac{r_2}{r - (x+y)/r} \leqslant 1 + \frac{1}{r^2}, && x \geqslant 0, \quad y \geqslant 0, \quad x + y \geqslant 4; \\ &1 \leqslant \frac{r_3}{r - y/r} \leqslant 1 + \frac{1}{2r^2}, && x \geqslant 0, \quad y \geqslant 2. \end{aligned}$$

It follows that

$$1 \leqslant \left(\frac{r_1}{r - x/r}\right)^x \leqslant \left(\frac{r_1}{r - x/r}\right)^r \leqslant \left(1 + \frac{1}{2r^2}\right)^r \leqslant e^{1/(2r)} \to 1.$$

Remark 12.3. The weaker approximation $r_1 \approx r - 1$ is not sharp enough: if we have an inequality

$$\frac{r_1}{r-1} \leqslant 1 + h(r),$$

then $h(r) \geqslant (1 + o(r))/r$. And $(1 + 1/r)^r$ tends to $e \neq 1$ while $(1 + 1/r)^x$ behaves like $e^{x/r}$, which does not converge.

Proof of Lemma 12.2. We shall prove that

$$\left(r - \frac{x}{r}\right)^2 \leqslant r_1^2 \leqslant \left(r - \frac{x}{r}\right)^2 \left(1 + \frac{1}{2r^2}\right)^2$$

i.e.,

$$r^2 - 2x + \frac{x^2}{r^2} \leqslant r^2 - 2x + 1 \leqslant r^2 + 1 + \frac{1}{4r^2} - 2x - \frac{2x}{r^2} - \frac{x}{2r^4} + \frac{x^2}{r^2} + \frac{x^2}{r^4} + \frac{x^2}{4r^6},$$

equivalently

$$\frac{x^2}{r^2} - 1 \leqslant 0 \leqslant \frac{1}{4r^2} - \frac{2x}{r^2} - \frac{x}{2r^4} + \frac{x^2}{r^2} + \frac{x^2}{r^4} + \frac{x^2}{4r^6}.$$

The first inequality is clear. For the second we regroup the terms a little:

$$0 \leqslant \frac{1}{4r^2}\left(1 - \frac{2x}{r^2}\right) + \frac{x}{r^2}(x-2) + \frac{x^2}{6r^4} + \frac{x^2}{r^6}.$$

It is now obvious that the inequality holds when $x \geqslant 2$. The inequalities for r_2 and r_3 can be proved similarly. This finishes the proof of Lemma 12.2. ☐

Proof of Lemma 12.1, cont'd. The quotient (12.1) can be written as the sum of three terms,

$$\frac{c(x_j,y_j)}{c(x,y)} = \left(\frac{r_j+y_j}{r+y}\cdot\frac{x}{x_j}\right)^{x_j}\left(\frac{x}{r+y}\right)^{x-x_j}\left(\frac{r_j+x_j}{r+x}\cdot\frac{y}{y_j}\right)^{y_j}\left(\frac{y}{r+x}\right)^{y-y_j}.$$

The quotient $c(x_1,y_1)/c(x,y)$

The quotient $c(x_1,y_1)/c(x,y) = c(x-1,y)/c(x,y)$ can be written

$$\frac{c(x-1,y)}{c(x,y)} = \left(\frac{r_1+y}{r+y}\cdot\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y}\left(\frac{r_1+x-1}{r+x}\right)^{y}, \qquad x\geqslant 1,\quad y\geqslant 0.$$

Using the formula $r_1 \approx r - x/r$, we find this to be approximately equal to

$$\begin{aligned}
&\left(\frac{r+y-x/r}{r+y}\cdot\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y}\left(\frac{r+x-1-x/r}{r+x}\right)^{y}\\
&= \left(1-\frac{x/r}{r+y}\right)^{x-1}\left(\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y}\left(1-\frac{1+x/r}{r+x}\right)^{y}\\
\approx \exp&\left(-\frac{x^2}{r(r+y)}-\frac{y}{r}\right)\left(\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y} = e^{-1}\left(\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y} = \varphi(x,y),
\end{aligned}$$

where the last inequality is a definition of φ for $(x,y)\in\mathbf{R}^2$, $x\geqslant 1$, $y\geqslant 0$. We define $(x/(x-1))^{x-1}=1$ for $x=1$ to make it continuous and increasing on $[1,+\infty[$. Thus

$$\varphi(1,y) = \frac{1}{e\left(\sqrt{1+y^2}+y\right)}.$$

Here an approximate equality $A\approx B$ means that the quotient A/B tends to 1 as $r\to+\infty$.

It will be convenient to have $\varphi(x,y)$ defined for all nonnegative real numbers x, y. For $x=0$ the quotient is $c(-1,y)/c(0,y)=0$. We define $\varphi(x,y)$ for $0\leqslant x<1$ by linear interpolation between the values $x=0$ and $x=1$, thus

$$\varphi(x,y) = x\varphi(1,y) = \frac{x}{e\left(\sqrt{1+y^2}+y\right)}, \qquad 0\leqslant x\leqslant 1,\quad y\geqslant 0. \tag{12.2}$$

Then $[0,+\infty[\ \ni x\mapsto\varphi(x,y)$ will be increasing.

The quotient $c(x_2,y_2)/c(x,y)$

The quotient $c(x_2,y_2)/c(x,y) = c(x-1,y-1)/c(x,y)$ can be written

$$\frac{c(x-1,y-1)}{c(x,y)} = \left(\frac{r_2+y-1}{r+y}\cdot\frac{x}{x-1}\right)^{x-1}\left(\frac{r_2+x-1}{r+x}\cdot\frac{y}{y-1}\right)^{y-1}\frac{xy}{(r+x)(r+y)}.$$

We find that this is approximately equal to

$$\left(1-\frac{r+x+y}{r(r+y)}\right)^{x-1}\left(1-\frac{r+x+y}{r(r+x)}\right)^{y-1}\left(\frac{x}{x-1}\right)^{x-1}\left(\frac{y}{y-1}\right)^{y-1}\frac{xy}{(r+y)(r+x)}$$

$$\approx e^{-2}\left(\frac{x}{x-1}\right)^{x-1}\left(\frac{y}{y-1}\right)^{y-1}\frac{xy}{(r+x)(r+y)}=\varphi(x,y)\varphi(y,x)$$

for $x, y \geqslant 1$.

The quotient $c(x_3, y_3)/c(x, y)$

We need not do any calculations here: we just interchange x and y in the quotient $c(x-1, y)/c(x, y)$ and obtain that the quotient is approximately equal to $\varphi(y, x)$.

Adding the three terms

We have found that the quotient (12.1), which is the sum of the three quotients $c(x_j, y_j)/c(x, y)$, is approximately equal to

$$\varphi(x,y)+\varphi(x,y)\varphi(y,x)+\varphi(y,x)=(1+\varphi(x,y))(1+\varphi(y,x))-1, \tag{12.3}$$

where

$$\varphi(x,y)=e^{-1}\left(\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y}, \qquad x \geqslant 1, \quad y \geqslant 0,$$

and $\varphi(x, y)$ is defined also for $0 \leqslant x < 1$ by (12.2).

We shall study the expression (12.3) as $r \to +\infty$, equivalently, as $x + y \to +\infty$. It is easy to do so if y/r tends to a limit, say $\sin\theta$; let us do this first. (Actually this result is enough for our conclusion concerning the regularized indicators.)

It is clear that if $0 < \theta < \frac{1}{2}\pi$, then as $r \to +\infty$, both x and y tend to infinity, so that the factors $(x/(x-1))^{x-1}$ and $(y/(y-1))^{y-1}$ tend to e, and, if y/r tends to $\sin\theta$, the expression (12.3) tends to

$$\left(1+\frac{\cos\theta}{1+\sin\theta}\right)\left(1+\frac{\sin\theta}{1+\cos\theta}\right)-1=1.$$

If $\theta = 0$, then $\varphi(x, y)$ tends to 1 while $\varphi(y, x)$ tends to 0 (this is because of the bound $x/(x-1)^{x-1} \leqslant e$). Hence (12.3) still tends to 1.

And if $\theta = \frac{1}{2}\pi$, then $\varphi(x, y)$ tends to 0 while $\varphi(y, x)$ tends to 1, so that also in this case (12.3) tends to 1.

However, we claim more: we claim not just that (12.3) tends to 1 as y/r tends to $\sin\theta$ for each θ but that the convergence is uniform in $\theta \in [0, \frac{1}{2}\pi]$, in other words that $|(1 + \varphi(x, y))(1 + \varphi(y, x)) - 2| \leqslant \varepsilon$ when $r \geqslant R_\varepsilon$. Dini's theorem will save us from explicit estimates, which are, however, not difficult. Indeed, let us define

$$\varphi_r(\theta)=\varphi(r\cos\theta, r\sin\theta), \qquad \theta\in[0,\tfrac{1}{2}\pi], \quad r>0.$$

These functions form an increasing family. As we saw, φ_r tends to

$$\varphi_\infty(\theta) = \frac{\cos\theta}{1+\sin\theta}.$$

All functions φ_r as well as φ_∞ are continuous, and the convergence $\varphi_r \nearrow \varphi_\infty$ is monotone on the compact set $[0, \frac{1}{2}\pi]$. Dini's theorem says that the convergence is uniform. Then $(1+\varphi(x,y))(1+\varphi(y,x))$ tends uniformly to 2.

This finishes the proof of Lemma 12.1. □

We can now compare d and c.

Proposition 12.4. *For every positive ε there exist positive constants λ_ε and Λ_ε such that*

$$\lambda_\varepsilon e^{-\varepsilon(x+y)} c(x,y) \leqslant d(x,y) \leqslant \Lambda_\varepsilon e^{\varepsilon(x+y)} c(x,y), \qquad (x,y) \in \dot{\mathbf{N}}^2.$$

Proof. With the quotient as in Lemma 12.1, an induction on the sum $x+y$ is possible for large values of $x+y$:

$$\begin{aligned} d(x,y) &= d(x-1,y) + d(x-1,y-1) + d(x,y-1) \\ &\leqslant \Lambda_\varepsilon e^{\varepsilon(x+y-1)} c(x-1,y) + \Lambda_\varepsilon e^{\varepsilon(x+y-2)} c(x-1,y-1) \\ &\quad + \Lambda_\varepsilon e^{\varepsilon(x+y-1)} c(x,y-1) \\ &\leqslant \Lambda_\varepsilon e^{\varepsilon(x+y-1)} [c(x-1,y) + c(x-1,y-1) + c(x,y-1)] \\ &\leqslant \Lambda_\varepsilon e^{\varepsilon(x+y)} c(x,y). \end{aligned}$$

The estimate from below is done similarly. For smaller values of $x+y$ we obtain the result at the price of taking a smaller constant λ_ε and a larger constant Λ_ε. □

We can now formulate our main result.

Theorem 12.5. *The regularized radial indicators of the array of Delannoy numbers are*

$$p_d^*(x,y) = q_*^d(x,y) = p_c(x,y) = \log c(x,y) = x \log \frac{r+y}{x} + y \log \frac{r+x}{y}$$

for all $(x,y) \in \mathbf{R}_+^2$. The limit superior and limit inferior in the definitions of the upper and lower radial indicators p_d and q^d are actually limits.

Proof. From Proposition 12.4 we know that

$$\lambda_\varepsilon e^{-\varepsilon(x+y)} c(x,y) \leqslant d(x,y) \leqslant \Lambda_\varepsilon e^{\varepsilon(x+y)} c(x,y).$$

From this it follows that the radial indicators satisfy

$$p_c(x,y) - \varepsilon(x+y) \leqslant p_d(x,y) \leqslant \varepsilon(x+y) + p_c(x,y).$$

Since ε is arbitrary, we find that $p_d = p_c$. Similarly $q^d = q^c$ and for the regularized indicators. □

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Author's address: Uppsala University, Department of Mathematics
P. O. Box 480, SE-751 06 Uppsala, Sweden
E-mail address: kiselman@math.uu.se, christer@kiselman.eu