Asymptotic properties of the Delannoy numbers and similar arrays

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The sections marked * are not needed for the main result in Sections 11 and 12.

1. Introduction

The Delannoy numbers $d(x,y)$, $(x,y) \in \mathbb{Z}^2$, are defined as 0 when $x \leqslant -1$ or when $y \leqslant -1$, as 1 when $(x,y) = (0,0)$, and for $(x,y) \in \mathbb{N}^2 \smallsetminus \{(0,0)\}$ by the recursion formula

$$d(x,y) = d(x-1,y) + d(x-1,y-1) + d(x,y-1). \quad (1)$$

They are named for Henri-Auguste Delannoy (1833–1915). He investigated the possible moves on a chessboard. The numbers under consideration here appear when one studies "la marche de la Reine." For biographies of Delannoy, see Banderier & Schwer (2005) and Schwer & Autebert (2006).

The purpose here is to determine the growth at infinity of the array of Delannoy numbers, more precisely to determine the regularized upper and lower radial indicators of the array. They are proved to coincide and to be concave. We also describe the radial indicator as an infimum of linear functions, which amounts to determining its Fenchel transform.

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The Delannoy numbers appear in many problems in mathematics; see Sulanke (2003), who lists 29 different examples. To mention just one, $d(n,r) = d(r,n)$ is the cardinality of the ball of radius r in $\mathbb{Z}^n$ equipped with the l^1 metric (also known as the hyperoctahedron),

$$\left\{x \in \mathbb{Z}^n;\ \|x\|_1 = |x_1| + \cdots + |x_n| \leqslant r\right\};$$

Vassilev & Atanasov (1987), quoted here from Sulanke (2003, note 18).

To Sulanke's examples I added a thirtieth (2008:609): for $(a,b) \in \mathbb{Z}^2$, $a+b \geqslant 0$, the number of Khalimsky-continuous functions $[0,a+b]_{\mathbb{Z}} \to \mathbb{Z}$ satisfying $f(0)=0$ and $f(a+b)=a-b$ is equal to $d(a,b)$. For a detailed proof, see Samieinia (2010: Theorem 2.2).

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And then a thirty-first: a fundamental solution

$$E(x+iy) = i^{y-x} d(x,y), \qquad x+iy \in \mathbb{Z}[i],$$

for a discrete analogue of the Cauchy–Riemann operator (2008:608).

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digital geometry, where the Khalimsky topology is a useful structure; and

discrete complex analysis.

To get a rough idea of the growth of this array we note the following easy result.

Proposition

The Delannoy numbers satisfy

$$3^{x \wedge y} \leqslant d(x,y) \leqslant \left(\sqrt{2}+1\right)^{x+y}, \qquad (x,y) \in \mathbb{N}^2.$$

In particular the diagonal numbers satisfy

$$3^{x} \leqslant d(x,x) \leqslant \left(3+\sqrt{8}\right)^{x}, \qquad x \in \mathbb{N}.$$

The Delannoy numbers are explicitly given by

$$d(x,y) = \sum_{j=0}^{x} \binom{x}{j}\binom{y}{j} 2^j = \sum_{j=0}^{x} \binom{y}{j}\binom{x+y-j}{y}, \qquad (x,y) \in \mathbb{N}^2,$$

Delannoy (1895:77), Comtet (1974:81). They have a generating function

$$G_d(z,w) = \sum_{(x,y)\in\mathbb{N}^2} d(x,y) z^x w^y = \frac{1}{1-z-w-zw},$$

Comtet (1974:81).

A lot is known about the *diagonal* (or *central*) *Delannoy numbers* $d(x,x)$; see Comtet (1974:81), Stanley (2001:185) and Sulanke (2003). Actually $d(x,x) = P_x(3)$, where $P_x(t)$ is the Legendre polynomial (Comtet 1974:81). The sequence $(d(x,x))_{x\in\mathbb{N}}$ has a generating function

$$\sum_{x\in\mathbb{N}} d(x,x)t^x = \frac{1}{\sqrt{1-6t+t^2}}$$

(Comtet 1974:81, Stanley 2001:185). We have

$$d(x,x) = 3\left(2-\tfrac{1}{x}\right) d(x-1,x-1) - \left(1-\tfrac{1}{x}\right) d(x-2,x-2).$$

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$$d(x,x) = 3\left(2-\tfrac{1}{x}\right) d(x-1,x-1) - \left(1-\tfrac{1}{x}\right) d(x-2,x-2).$$

$$d(x,x) \sim \left(\sqrt{2}+1\right)^{2x}.$$

Not so much is known about the general numbers $d(x, y)$.
I conjectured in (2008:609–610) that

$$d(x, y) \sim \left(\frac{r+y}{x}\right)^{x} \left(\frac{r+x}{y}\right)^{y},$$

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more precisely that

$$\begin{aligned}\lim_{\substack{t\in\dot{\mathbb{N}}\\ t\to+\infty}} \frac{1}{t}\log d(tx,ty) &= x\log\frac{r+y}{x} + y\log\frac{r+x}{y}\\ &= x\log\left(\frac{1+\sin\theta}{\cos\theta}\right) + y\log\left(\frac{1+\cos\theta}{\sin\theta}\right)\end{aligned}$$

for all $(x,y)\in\dot{\mathbb{N}}^2$, where (r,θ) are the usual polar coordinates in $\mathbb{R}^2$ and $\dot{\mathbb{N}} = \mathbb{N}\smallsetminus\{0\}$. The right-hand side is a concave function of $(x,y)\in\mathbb{R}^2_+$ and is positively homogeneous of degree one. Here we define $\mathbb{R}_+ = \{x\in\mathbb{R};\ x>0\}$.

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We shall prove this conjecture.

Table 1. The Delannoy numbers $d(x, y)$ with $0 \leqslant x, y \leqslant 9$.

9	1	19	181	1159	5641	22,363	75,517	224,143	598,417	1,462,563
8	1	17	145	833	3649	13,073	40,081	108,545	265,729	598,417
7	1	15	113	575	2241	7183	19,825	48,639	108,545	224,143
6	1	13	85	377	1289	3653	8989	19,825	40,081	75,517
5	1	11	61	231	681	1683	3653	7183	13,073	22,363
4	1	9	41	129	321	681	1289	2241	3649	5641
3	1	7	25	63	129	231	377	575	833	1159
2	1	5	13	25	41	61	85	113	145	181
1	1	3	5	7	9	11	13	15	17	19
0	1	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8	9

2. Convexity properties of the array of Delannoy numbers

Definition

Let G be an abelian group (in this talk only $G = \mathbb{Z}$ and $G = \mathbb{Z}^2$ will occur), and let $a \in G$. We define a difference operator D_a by

$$(D_a f)(x) = f(x+a) - f(x), \qquad x \in G, \quad f \in \mathbb{R}^G. \qquad \square$$

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Definition

We shall say that a function $f\colon \mathbb{Z}^n \to [-\infty, +\infty]$ is ***convex extensible*** if it is the restriction to $\mathbb{Z}^n$ of some convex function $F\colon \mathbb{R}^n \to [-\infty, +\infty]$. We say that f is ***concave extensible*** if $-f$ is convex extensible. $\square$

For $n = 1$ it is easy to see that $f\colon \mathbb{Z} \to \mathbb{R}$ is convex extensible if and only if $D_1^2 f \geqslant 0$. It is equivalent to require that $D_b D_a f \geqslant 0$ for all $a, b \in \mathbb{N}$.

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Proposition

For every $y \in \mathbb{Z}$, the partial function $x \mapsto d(x, y)$ is increasing and convex extensible. Moreover $x \mapsto d(x, y)$ is more strongly convex than both $x \mapsto d(x, y - 1)$ and $x \mapsto d(x - 1, y)$. By symmetry we get the corresponding results for the partial functions $y \mapsto d(x, y)$, $x \in \mathbb{Z}$.

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Proposition

The restriction of d to any bidiagonal is increasing and convex extensible, i.e., the function $\mathbb{Z} \ni x \mapsto f_m(x) = d(x, m+x)$ is increasing and convex extensible for all choices of $m \in \mathbb{Z}$.

Proposition

The restriction of d to any antidiagonal is concave on an interval close to the main diagonal; more precisely, the function

$$J_m = [s_m, m - s_m] \ni x \mapsto g_m(x) = d(x, m - x)$$

is concave, where $s_m = 0$ *for* $m = 0, 1, 2$, $s_m = \frac{1}{2}(m - 4)$ *for* m *even,* $m \geqslant 4$, *and* $s_m = \frac{1}{2}(m - 3)$ *for* m *odd,* $m \geqslant 3$.

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The interval J_m is not long; in fact $\mathbf{card}(J_m) = m - 2s_m + 1$ is 4 for odd $m \geqslant 3$ and 5 for even $m \geqslant 4$.

* 3. The logarithms of the Delannoy numbers

The logarithms of the Delannoy numbers seem to display interesting concavity properties, but are less easy to calculate with. I list a few conjectures concerning them. First a table with some of the values of the function $\log d$.

Table 2. The natural logarithms of the Delannoy numbers with $0 \leqslant x, y \leqslant 9$, rounded off to three decimal places.

9	0	2.944	5.198	7.055	8.638	10.015	11.232	12.320	13.302	14.196
8	0	2.833	4.977	6.725	8.202	9.478	10.599	11.594	12.490	13.302
7	0	2.708	4.727	6.354	7.715	8.879	9.895	10.792	11.595	12.320
6	0	2.565	4.443	5.932	7.162	8.203	9.104	9.895	10.599	11.232
5	0	2.398	4.111	5.442	6.524	7.428	8.203	8.879	9.478	10.015
4	0	2.197	3.714	4.860	5.771	6.524	7.162	7.715	8.202	8.638
3	0	1.946	3.219	4.143	4.860	5.442	5.932	6.354	6.725	7.055
2	0	1.609	2.565	3.289	3.714	4.111	4.443	4.727	4.977	5.198
1	0	1.099	1.609	1.946	2.197	2.398	2.565	2.708	2.833	2.944
0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0
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$$\frac{1}{2} \cdot 12.320 + \frac{1}{2} \cdot 14.196 = 13.258 < 13.302$$

$$\frac{1}{2} \cdot 10.792 + \frac{1}{2} \cdot 14.196 = 12.494 > 12.490$$

Conjecture

For every $y \in \mathbb{Z}$, the partial function $x \mapsto \log d(x, y)$ is concave extensible. $\square$

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The conjectured concavity of $x \mapsto \log d(x,y)$ and the proved convexity of $x \mapsto d(x,y)$ can be expressed by a double inequality

$$\sqrt{d(x-1,y)d(x+1,y)} \leqslant d(x,y) \leqslant \tfrac{1}{2}d(x-1,y) + \tfrac{1}{2}d(x+1,y),$$

$(x,y) \in \mathbb{Z}^2$.

Conjecture

The function $\mathbb{Z} \ni x \mapsto \log g_m(x) = \log d(x, m - x) \in [-\infty, +\infty[$ *is concave extensible for all choices of* $m \in \mathbb{Z}$. $\square$

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From the conjectured concavity of these functions one might guess that the function $\log d$ would be concave extensible as a function of two integer variables. However, as is evident from Table 2, this is not true. Instead I make the following conjecture.

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Conjecture

The function $\mathbb{N} \ni x \mapsto \log d(x, x) \in [0, +\infty[$ *is convex extensible.* □

The function
$[-m^-, +\infty[\ni x \mapsto \log f_m(x) = \log d(x, m+x) \in [-\infty, +\infty[$
seems to be convex extensible close to the diagonal (for small values of $|m|$) and concave extensible in certain regions away from the diagonal.

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If f is only defined on the integer points, we extend it as $+\infty$ on $\mathbb{R}^n \smallsetminus \mathbb{Z}^n$.

If f takes only the values 0 and $+\infty$, then $\tilde{f}$ is positively homogeneous of degree one as the supremum of a family of linear functions:

$$\tilde{f}(\xi) = \sup_{\substack{x \in \mathbb{R}^n \\ f(x)=0}} \xi \cdot x, \qquad \xi \in \mathbb{R}^n.$$

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Conversely, if f is positively homogeneous of degree one, then $\tilde{f}$ can take only the values 0, $+\infty$, $-\infty$.

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Conversely, if f is positively homogeneous of degree one, then $\tilde{f}$ can take only the values 0, $+\infty$, $-\infty$.

Indeed, if $f(tx) = tf(x)$ for all $t > 0$, then $t\tilde{f} = \tilde{f}$ for all $t > 0$, and this is only true for the three values 0, $+\infty$, $-\infty$. The value $-\infty$ will not occur for the functions we are studying, so then $\tilde{f}$ is an indicator function, $\tilde{f} = \mathbf{ind}_M$ for some set M, where we define generally $\mathbf{ind}_M$ to take the value 0 in M and $+\infty$ in its complement.

5. The radial indicators

Definition

Given a function $f\colon \mathbb{R}^n \to [0,+\infty[$ we define its ***upper radial indicator*** as

$$p_f(x) = \limsup_{t\to+\infty} \frac{1}{t}\log f(tx), \qquad x \in \mathbb{R}^n,$$

and its ***lower radial indicator*** as

$$q^f(x) = \liminf_{t\to+\infty} \frac{1}{t}\log f(tx), \qquad x \in \mathbb{R}^n.$$

Definition

We also define the ***regularized upper radial indicator*** as

$$p_f^*(x) = \limsup_{y \to x} p_f(y), \qquad x \in \mathbb{R}^n,$$

the smallest upper semicontinuous majorant of p_f, and the ***regularized lower radial indicator*** as

$$q_*^f(x) = \liminf_{y \to x} q_f(y), \qquad x \in \mathbb{R}^n,$$

the largest lower semicontinuous minorant of q^f. $\square$

If f is only defined in $\mathbb{Z}^n$, we have to restrict t in the definition of p_f and q^f to those t for which tx belongs to $\mathbb{Z}^n$. This means that p_f and q^f will be defined only in $\mathbb{R}\mathbb{Z}^n$.

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Similarly, in the definitions of the regularized indicators, we have to restrict y to $\mathbb{R}\mathbb{Z}^n$. But then p_f^* and q_*^f will be well defined everywhere, since $\mathbb{R}\mathbb{Z}^n \supset \mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Proposition

Let $f\colon \mathbb{N}^n \to \mathbb{R}_+$ *be any function and let* $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$*. Then the following four properties are equivalent.*

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(A). *For every positive* ε *there exists a constant* C_ε *such that*

$$f(x) \leqslant C_\varepsilon e^{-(\sigma - \varepsilon \mathbf{1}) \cdot x}, \qquad x \in \mathbb{N}^n,$$

where $\mathbf{1} = (1, 1, \ldots, 1)$*.*

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(D). $\sigma \in M_f$*, where* M_f *is the set such that* $(-p_f^*)^{\sim} = \mathbf{ind}_{M_f}$*.*

Conjecture

For all $(x, y) \in \mathbb{N}^2$*, the function* $\dot{\mathbb{N}} \ni t \mapsto t^{-1} \log d(tx, ty)$ *is increasing.* $\square$

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For all $(x,y) \in \mathbb{N}^2$, the function $\dot{\mathbb{N}} \ni t \mapsto t^{-1} \log d(tx, ty)$ is increasing. $\square$

If this is true, the upper radial indicator satisfies

$$p_d(x,y) = \lim_{\substack{t \in \dot{\mathbb{N}} \\ t \to +\infty}} \frac{1}{t} \log d(tx, ty) = \sup_{t \in \dot{\mathbb{N}}} \frac{1}{t} \log d(tx, ty), \qquad (x,y) \in \mathbb{N}^2.$$

However, we shall prove that the first equality holds, i.e., that the limit superior is actually a limit.

6. Convolution

We define the convolution product $h = f * g$ of two functions $f, g\colon \mathbb{Z}^n \to \mathbb{R}$ by

$$h(x) = \sum_{y \in \mathbb{Z}^n} f(x-y)g(y), \qquad x \in \mathbb{Z}^n,$$

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(1) The first is the algebra of all functions with finite support. (The support of a function is here just the set where it is nonzero.)

(2) Given a vector $\alpha \neq 0$ we consider the algebra of all functions with support contained in a translate of the cone $K_\alpha = \{x \in \mathbb{R}^n;\ \alpha \cdot x \geqslant \|x\|\}$.

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In these cases, the associative law holds.

Example

That some care is needed is shown by a simple example: take $f(x) = 1$ for all $x \in \mathbb{Z}$; $g = \delta_{-1} - \delta_0$ (a difference operator); and $h(x) = 1$ for all $x \in \mathbb{N}$, $h(x) = 0$ for $x \leqslant -1$.

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Then $f * g = 0$ (case (3)) and $(f * g) * h = 0$, while $g * h = \delta_{-1}$ (case (3)) and $f * (g * h) = 1 \neq 0$.

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Then $f * g = 0$ (case (3)) and $(f * g) * h = 0$, while $g * h = \delta_{-1}$ (case (3)) and $f * (g * h) = 1 \neq 0$.

Note that neither $f * h$ nor $f * g * h$ can be defined in accordance with any of the cases (1)–(4). $\square$

The recursion formula (1) can be written

$$d * (\delta - \delta_{(1,0)} - \delta_{(1,1)} - \delta_{(0,1)}) = \delta,$$

where δ_a is the Kronecker delta placed at a and $\delta = \delta_0$.

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For the binomial coefficients we have a similar formula,

$$b * (\delta - \delta_{(1,0)} - \delta_{(1,0)}) = \delta.$$

Both equations are thus of the form

$$f * (\delta - \mu) = \delta.$$

7. The Fourier transformation

We define the Fourier transform $\hat{f}$ of a function $f\colon \mathbb{Z}^n \to \mathbb{C}$ by

$$\hat{f}(\zeta) = \sum_{x\in\mathbb{Z}^n} f(x)e^{i\zeta\cdot x}, \qquad \zeta \in \mathbb{C}^n,$$

for those ζ for which the sum has a good sense. Since $d(x,y) \leqslant (\sqrt{2}+1)^{x+y}$, we see that $\hat{d}$ is well defined for $\operatorname{Im}\zeta_1,\ \operatorname{Im}\zeta_2 > \log(\sqrt{2}+1)$.

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If f satisfies condition (A), then for every positive ε, $\hat{f}$ is holomorphic for $\operatorname{Im}\zeta_j > -\sigma_j + \varepsilon$, $j = 1, \ldots, n$, and satisfies

$$|\hat{f}(\zeta)| \leqslant \prod_{j=1}^{n} \left(1 - e^{-\sigma_j - \operatorname{Im}\zeta_j + \varepsilon}\right)^{-1} \tag{2}$$

there. Letting ε tend to zero we see that $\hat{f}$ is holomorphic for $\operatorname{Im}\zeta_j > -\sigma_j$, $j = 1, \ldots, n$.

We have adapted the signs to the usual conventions concerning Fourier series. The Fourier inversion formula therefore takes the form

$$f(x) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{f}(\xi) e^{-i\xi \cdot x}\, d\xi_1 \cdots d\xi_n,$$

for $x \in \mathbb{Z}^n$. Here $\xi = (\xi_1, \ldots, \xi_n)$ are n real variables.

The Fourier transform of a convolution product is given by

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Since d grows fast, we cannot apply the inversion formula to $\hat{d}$, but to $\hat{d}_a$, the Fourier transform of $d_a(x,y) = d(x,y)a^{x+y}$, for any positive constant $a < \sqrt{2} - 1$. We obtain

$$d(x,y)a^{x+y} = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \hat{d}_a(\xi) e^{-i(\xi_1 x + \xi_2 y)}\, d\xi_1\, d\xi_2,$$

$(x,y) \in \mathbb{Z}^2$, where $\hat{d}_a(\zeta) = \hat{d}(\zeta_1 - i\log a, \zeta_2 - i\log a)$, which means that for $\hat{d}$, the integral goes over a square in $\mathbb{R}^2$ translated in $\mathbb{C}^2$ by the imaginary vector $-i(\log a, \log a)$.

The convolution formula yields

$$\hat{f}(\zeta) = \frac{1}{1 - \hat{\mu}(\zeta)}, \qquad \zeta \in \mathbb{C}^n, \quad \operatorname{Im} \zeta_j \text{ large.}$$

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In case $f = d$ the formula reads

$$\hat{d}(\zeta) = \frac{1}{1 - e^{i\zeta_1} - e^{i\zeta_2} - e^{i(\zeta_1+\zeta_2)}},$$

$$\zeta \in \mathbb{C}^2, \quad \operatorname{Im} \zeta_1, \operatorname{Im} \zeta_2 > \log(\sqrt{2}+1).$$

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$$\zeta \in \mathbb{C}^2, \quad \operatorname{Im}\zeta_1, \operatorname{Im}\zeta_2 > \log(\sqrt{2}+1).$$

This is of course equivalent to the formula for the generating function. In fact, $\hat{d}(\zeta) = G_d(e^{i\zeta_1}, e^{i\zeta_2})$.

With this expression for $\hat{d}$ we obtain

$$d(x,y) = \frac{1}{4\pi^2 a^{x+y}} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{-i(\xi_1 x + \xi_2 y)}}{1 - ae^{i\xi_1} - ae^{i\xi_2} - a^2 e^{i(\xi_1+\xi_2)}} \, d\xi_1 \, d\xi_2,$$

$(x,y) \in \mathbb{Z}^2$, where a is any constant satisfying $0 < a < \sqrt{2} - 1$.

8. Fundamental solutions for convolution operators

Both the Delannoy numbers and the binomial coefficients satisfy convolution equations of the form $f * (\delta - \mu) = \delta$. In the first case, $\mu = \delta_{(1,0)} + \delta_{(1,1)} + \delta_{(0,1)}$; in the second, $\mu = \delta_{(1,0)} + \delta_{(0,1)}$. In this section we shall prove some results on such convolution equations in a slightly more general setting.

8. Fundamental solutions for convolution operators

Both the Delannoy numbers and the binomial coefficients satisfy convolution equations of the form $f*(\delta-\mu)=\delta$. In the first case, $\mu=\delta_{(1,0)}+\delta_{(1,1)}+\delta_{(0,1)}$; in the second, $\mu=\delta_{(1,0)}+\delta_{(0,1)}$. In this section we shall prove some results on such convolution equations in a slightly more general setting.

So let $\mu\colon \mathbb{Z}^n\to[0,+\infty[$ be a function which is nonzero only at finitely many points $x\in\mathbb{N}^n$, $x\neq 0$. This condition will enable us to make inductions on $\mathbf{1}\cdot x=\sum x_j$.

Then $\delta - \mu$ is a fundamental solution to the operator $u \mapsto u * f$, more precisely, if we define F_θ for a fixed vector θ with positive components as the set of all functions u such that $u(x)$ is nonzero only if $\theta \cdot x \geqslant -C_u$, then with

$$P(u) = u * f, \qquad u \in F_\theta,$$

(case (4)) and

$$Q(v) = v * (\delta - \mu), \qquad v \in F_\theta,$$

(case (3)) we have

$$Q(P(u)) = (u * f) * (\delta - \mu) = u * (f * (\delta - \mu)) = u * \delta = u, \qquad u \in F_\theta,$$

and

$$P(Q(v)) = (v * (\delta - \mu)) * f = v * ((\delta - \mu) * f) = v * \delta = v, \qquad v \in F_\theta.$$

Proposition

Let a function $\mu\colon \mathbb{Z}^n \to [0, +\infty[$ *which is nonzero only at finitely many points* x *such that* $x \neq 0$ *and* $x_j \geqslant 0$*, and let a real vector* $\sigma = (\sigma_1, \dots, \sigma_n)$ *be given. Assume that*

$$\hat{\mu}(-i\sigma) = \sum_y \mu(y) e^{\sigma \cdot y} \leqslant 1. \tag{3}$$

Then the unique function $f\colon \mathbb{Z}^n \to \mathbb{R}$ *which is zero outside* $\mathbb{N}^n$ *and solves* $f * (\delta - \mu) = \delta$ *can be estimated as*

$$f(x) \leqslant e^{-\sigma \cdot x}, \qquad x \in \mathbb{Z}^n.$$

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$$f(x) \leqslant e^{-\sigma\cdot x}, \qquad x \in \mathbb{Z}^n.$$

Conversely, if for any positive ε *an estimate*

$$f(x) \leqslant C_\varepsilon e^{-(\sigma-\varepsilon\mathbf{1})\cdot x}, \qquad x \in \mathbb{Z}^n,$$

holds for some constant C_ε*, then the condition* (3) *holds.*

Theorem

Given a function $\mu\colon \mathbb{Z}^n \to [0, +\infty[$ *which is nonzero only at finitely many points* x *such that* $x \neq 0$ *and* $x_j \geqslant 0$*, the function* $f\colon \mathbb{Z}^n \to \mathbb{R}$ *which is nonzero only in* $\mathbb{N}^n$ *and solves the equation* $f * (\delta - \mu) = \delta$ *satisfies* $(-p_f^*)^{\sim} = \mathbf{ind}_{M_f}$*, where*

$$M_f = \{\sigma \in \mathbb{R}^n;\ \hat{\mu}(-i\sigma) \leqslant 1\}.$$

$\square$

* 9. The binomial coefficients

Let

$$b(x,y) = \binom{x+y}{x} = \frac{(x+y)!}{x!\,y!}, \qquad (x,y) \in \mathbb{N}^2,$$

be the binomial coefficients. We define them also when $x \leqslant -1$ or $y \leqslant -1$ by taking them equal to zero then. Using Stirling's formula we see that the regularized radial indicators are

$$q_*^b(x,y) = p_b^*(x,y) = x\log(1+y/x) + y\log(1+x/y), \quad (x,y) \in \mathbb{R}_+^2.$$

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$$q_*^b(x,y) = p_b^*(x,y) = x\log(1+y/x) + y\log(1+x/y), \quad (x,y) \in \mathbb{R}_+^2.$$

The function p_b^* is positively homogeneous of order 1 and concave. To prove concavity it is enough to note that $x \mapsto \log(1+x)$ is concave. This implies that the homogeneous function $(x,y) \mapsto y\log(1+x/y)$ is concave; by symmetry also $(x,y) \mapsto x\log(1+y/x)$ is concave.

Proposition

The Fenchel transform $(-p_b^*)^\sim$ *of* $-p_b^*$ *is equal to* $\mathbf{ind}_{M_b}$*, the indicator function of the set*

$$M_b = \{\sigma \in \mathbb{R}^2;\ e^{\sigma_1} + e^{\sigma_2} \leqslant 1\}.$$

Since $-p_b^*$ *is convex, lower semicontinuous, and does not take the value* $-\infty$*, we have*

$$q_*^b = p_b^* = -(-p_b^*)^{\tilde{\tilde{}}} = -(\mathbf{ind}_{M_b})^\sim.$$

The array of binomial coefficients satisfies the assertions on convexity that we stated for d, and also for concavity but with a different interval: for b, the interval of concavity is $[t_m, m - t_m]$, where

$$t_m = \lceil \tfrac{1}{2}\left(m - 2 - \lfloor \sqrt{m+2} \rfloor\right) \rceil.$$

10. An array for comparison

In this section we shall study another array, denoted by c, which will be used to describe the asymptotic properties of the array of Delannoy numbers.

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We define

$$c(x,y) = \left(\frac{r+y}{x}\right)^x \left(\frac{r+x}{y}\right)^y, \qquad (x,y) \in \mathbb{R}_+^2,$$

where $r = \sqrt{x^2+y^2}$. We put $c(x,0) = c(0,y) = 1$ for $x, y \geqslant 0$, and $c(x,y) = 0$ when $x < 0$ or $y < 0$.

Its regularized upper and lower radial indicators coincide:

$$q_*^c(x,y) = q^c(x,y) = p_c^*(x,y) = p_c(x,y) = \log c(x,y)$$

$$= x \log \frac{r+y}{x} + y \log \frac{r+x}{y}, \qquad (x,y) \in \mathbb{R}_+^2.$$

At $(x,0)$ and $(0,y)$, $x, y \geqslant 0$, the indicators take the value 0, and they take the value minus infinity when $x < 0$ or $y < 0$. Hence the asymptotic properties of c can be read off directly. By way of contrast, some work will be needed for setting up an approximate recursion formula similar to (1). Note the similarity between the formula for p_c^* and that for p_b^*.

The function $\log c$ is concave. To verify that this is so, it is enough to verify that $x \mapsto \log(\sqrt{x^2+1}+x) = \psi_1(x)$ is concave. And this is indeed the case: $\psi_1'(x) = 1/r$, a decreasing function. This implies that also $(x,y) \mapsto y\psi_1(x/y) = y\log((r+x)/y)$ is concave; by symmetry, also
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$(x,y) \mapsto x\psi_1(y/x) = x\log((r+y)/x)$ is concave.

Proposition

Let $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}_+^2$. The function $\log c$ has a linear majorant

$$\log c(x,y) \leqslant -\sigma_1 x - \sigma_2 y, \qquad (x,y) \in \mathbb{R}^2,$$

if and only if $e^{\sigma_1} + e^{\sigma_2} + e^{\sigma_1+\sigma_2} \leqslant 1$.

Corollary

The Fenchel transform $(-p_c)^{\sim}$ *of* $-p_c$ *is equal to* $\mathbf{ind}_{M_c}$*, where*

$$M_c = \{\sigma \in \mathbb{R}^2;\ e^{\sigma_1} + e^{\sigma_2} + e^{\sigma_1+\sigma_2} \leqslant 1\}.$$

Since $-p_c$ *is convex, lower semicontinuous, and does not take the value* $-\infty$*, we also get* $q^c = p_c = -(-p_c)^{\sim\sim} = -(\mathbf{ind}_{M_c})^{\sim}$*.*

$\square$

11. Estimates from above for the Delannoy numbers

We note the following special case of an earlier theorem.

Proposition

The Fenchel transform $(-p_d^*)^{\sim}$ *of the function* $-p_d^*$ *is equal to* $\mathbf{ind}_{M_d}$*, where*

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$$M_d = \{\sigma \in \mathbb{R}^2;\ e^{\sigma_1} + e^{\sigma_2} + e^{\sigma_1+\sigma_2} \leqslant 1\}.$$

Theorem

The regularized upper radial indicator p_d^* *of the array of Delannoy numbers is*

$$p_d^*(x,y) = p_c(x,y) = \log c(x,y) = x \log \frac{r+y}{x} + y \log \frac{r+x}{y},$$

for all $(x,y) \in \mathbb{R}_+^2$*, where* $r = \sqrt{x^2+y^2}$*.*

Proof. We observe that the sets M_d and M_c coincide, and find that

$$(-p_d^*)^{\sim} = (-p_c)^{\sim} = \mathbf{ind}_{M_c}.$$

This implies that

$$p_d^* \leqslant -(-p_d^*)^{\widetilde{\sim}} = -(-p_c)^{\widetilde{\sim}} = p_c,$$

where the last equality holds because $-p_c$ is convex, lower semicontinuous and does not take the value $-\infty$.

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$$(-p_d^*)^{\sim} = (-p_c)^{\sim} = \mathbf{ind}_{M_c}.$$

This implies that

$$p_d^* \leqslant -(-p_d^*)^{\approx} = -(-p_c)^{\approx} = p_c,$$

where the last equality holds because $-p_c$ is convex, lower semicontinuous and does not take the value $-\infty$.

We do not know yet whether $-p_d^*$ is convex, so we cannot assert right now that the first inequality is an equality. However, from the properties of the function $p_c = \log c$ we shall exclude that the inequality $p_d^* \leqslant p_c$ can be strict at any point. This depends on the strict concavity of the function $\omega(x) = p_c(x, 1) = \log c(x, 1)$.

If there exists a point $(x_0, y_0) \in \mathbb{R}^2$ with $p_d^*(x_0, y_0) < p_c(x_0, y_0)$, we must have $x_0, y_0 > 0$, and in view of the homogeneity we may assume that $y_0 = 1$. The tangent to the graph of ω at the point $(x_0, \omega(x_0))$ has the form $L(x) = -\xi x - \eta$, where $-\xi = \omega'(x_0)$ and $-\eta = \omega(x_0) - \omega'(x_0)x_0$. Hence (ξ, η) belongs to the boundary of M_c: we have $e^{\xi} + e^{\eta} + e^{\xi+\eta} = 1$.

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Now if $p_d^*(x_0, 1)$ is smaller than $p_c(x_0, 1)$, this must be true also in a neighborhood of x_0, since p_d^* is upper semicontinuous and p_c is continuous. In view of the strict concavity of ω, this implies that we can lower the line defined by L and still keep it as a majorant of p_d^*, so that $p_d^*(x, 1) \leqslant -\xi x - \eta - \varepsilon$ for some positive number ε and all $x > 0$. This implies that $(-p_d^*)^{\sim}(\xi, \eta + \varepsilon) \leqslant 0$, which means that $(\xi, \eta + \varepsilon)$ belongs to M_d. But $e^\xi + e^{\eta+\varepsilon} + e^{\xi+\eta+\varepsilon} > e^\xi + e^\eta + e^{\xi+\eta} = 1$, contradicting the result on M_d. This contradiction proves that the inequality $p_d^* \leqslant p_c$ cannot be strict anywhere; we must have $p_d^* = p_c$ everywhere. $\square$

12. An approximate recursion formula

While d is defined by a recursion formula, the array c is defined explicitly without recursion. We shall now prove an approximate recursion formula for c.

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Lemma

For every positive ε there exist a number N_ε such that the quotient

$$\frac{c(x-1,y)+c(x-1,y-1)+c(x,y-1)}{c(x,y)} \tag{4}$$

is at least $e^{-\varepsilon}$ and at most e^{ε} when $(x,y)\in\mathbb{R}^2$, $x,y\geqslant 0$, $x+y\geqslant N_\varepsilon$.

Proof. Let us introduce

$$(x_j, y_j) = \begin{cases} (x, y), & j = 0, \\ (x-1, y), & j = 1, \\ (x-1, y-1), & j = 2, \\ (x, y-1), & j = 3, \end{cases}$$

$r_j = \sqrt{x_j^2 + y_j^2}$, $j = 0, 1, 2, 3$, and $r = r_0$. Then $r_1 \approx r - x/r$, $r_2 \approx r - (x+y)/r$, and $r_3 \approx r - y/r$. In the approximations we do, it is not enough that $r_1/(r - x/r)$ tend to one; we must prove that even raised to the power x this expression tends to one. However, these approximations are quite good as shown by the next lemma.

Lemma

We have

$$1 \leqslant \frac{r_1}{r - x/r} \leqslant 1 + \frac{1}{2r^2}, \qquad x \geqslant 2, \quad y \geqslant 0;$$

$$1 \leqslant \frac{r_2}{r - (x+y)/r} \leqslant 1 + \frac{1}{r^2}, \qquad x \geqslant 0, \quad y \geqslant 0, \quad x + y \geqslant 4;$$

$$1 \leqslant \frac{r_3}{r - y/r} \leqslant 1 + \frac{1}{2r^2}, \qquad x \geqslant 0, \quad y \geqslant 2.$$

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$$1 \leqslant \frac{r_2}{r - (x+y)/r} \leqslant 1 + \frac{1}{r^2}, \qquad x \geqslant 0, \ \ y \geqslant 0, \ \ x+y \geqslant 4;$$

$$1 \leqslant \frac{r_3}{r - y/r} \leqslant 1 + \frac{1}{2r^2}, \qquad x \geqslant 0, \ \ y \geqslant 2.$$

It follows that

$$1 \leqslant \left(\frac{r_1}{r - x/r}\right)^x \leqslant \left(\frac{r_1}{r - x/r}\right)^r \leqslant \left(1 + \frac{1}{2r^2}\right)^r \leqslant e^{1/(2r)} \to 1.$$

The approximation $r_1 \approx r - 1$ is not sharp enough: if we have an inequality

$$\frac{r_1}{r-1} \leqslant 1 + h(r),$$

then $h(r) \geqslant (1 + o(r))/r$.

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And $(1+1/r)^r$ tends to $e \neq 1$ while $(1+1/r)^x$ behaves like $e^{x/r}$, which does not converge.

Proof of the lemma, cont'd. The quotient (4) can be written as the sum of three terms,

$$\frac{c(x_j, y_j)}{c(x, y)} = \left(\frac{r_j + y_j}{r + y} \cdot \frac{x}{x_j}\right)^{x_j} \left(\frac{x}{r + y}\right)^{x - x_j} \left(\frac{r_j + x_j}{r + x} \cdot \frac{y}{y_j}\right)^{y_j} \left(\frac{y}{r + x}\right)^{y - y_j}.$$

The quotient $c(x_1, y_1)/c(x, y)$

The quotient $c(x_1, y_1)/c(x, y) = c(x-1, y)/c(x, y)$ can be written

$$\frac{c(x-1,y)}{c(x,y)} = \left(\frac{r_1+y}{r+y} \cdot \frac{x}{x-1}\right)^{x-1} \frac{x}{r+y} \left(\frac{r_1+x-1}{r+x}\right)^{y},$$

for $x \geqslant 1$, $y \geqslant 0$.

Using the formula $r_1 \approx r - x/r$, we find this to be approximately equal to

$$\left(\frac{r+y-x/r}{r+y} \cdot \frac{x}{x-1}\right)^{x-1} \frac{x}{r+y} \left(\frac{r+x-1-x/r}{r+x}\right)^{y}$$

$$= \left(1 - \frac{x/r}{r+y}\right)^{x-1} \left(\frac{x}{x-1}\right)^{x-1} \frac{x}{r+y} \left(1 - \frac{1+x/r}{r+x}\right)^{y}$$

$$\approx \exp\left(-\frac{x^2}{r(r+y)} - \frac{y}{r}\right) \left(\frac{x}{x-1}\right)^{x-1} \frac{x}{r+y} = e^{-1} \left(\frac{x}{x-1}\right)^{x-1} \frac{x}{r+y} = \varphi(x,y),$$

where the last inequality is a definition of φ for $(x,y) \in \mathbb{R}^2$, $x \geqslant 1$, $y \geqslant 0$. We define $(x/(x-1))^{x-1} = 1$ for $x = 1$ to make it continuous and increasing on $[1, +\infty[$. Thus

$$\varphi(1,y) = \frac{1}{e\left(\sqrt{1+y^2}+y\right)}.$$

Here an approximate equality $A \approx B$ means that the quotient A/B tends to 1 as $r \to +\infty$.

It will be convenient to have $\varphi(x,y)$ defined for all nonnegative real numbers x, y. For $x = 0$ the quotient is $c(-1,y)/c(0,y) = 0$. We define $\varphi(x,y)$ for $0 \leqslant x < 1$ by linear interpolation between the values $x = 0$ and $x = 1$, thus

$$\varphi(x,y) = x\varphi(1,y) = \frac{x}{e\left(\sqrt{1+y^2}+y\right)}, \qquad 0 \leqslant x \leqslant 1, \quad y \geqslant 0.$$

Then $[0,+\infty[\ \ni x \mapsto \varphi(x,y)$ will be increasing.

The quotient $c(x_2, y_2)/c(x, y)$

The quotient $c(x_2, y_2)/c(x, y) = c(x-1, y-1)/c(x, y)$ can be written

$$\frac{c(x-1, y-1)}{c(x, y)} = \left(\frac{r_2 + y - 1}{r + y} \cdot \frac{x}{x-1}\right)^{x-1} \left(\frac{r_2 + x - 1}{r + x} \cdot \frac{y}{y-1}\right)^{y-1} \frac{xy}{(r+x)(r+y)}.$$

We find that this is approximately equal to

$$\left(1 - \frac{r+x+y}{r(r+y)}\right)^{x-1} \left(1 - \frac{r+x+y}{r(r+x)}\right)^{y-1} \left(\frac{x}{x-1}\right)^{x-1} \left(\frac{y}{y-1}\right)^{y-1} \frac{xy}{(r+y)(r+x)}$$

$$\approx e^{-2} \left(\frac{x}{x-1}\right)^{x-1} \left(\frac{y}{y-1}\right)^{y-1} \frac{xy}{(r+x)(r+y)} = \varphi(x, y)\varphi(y, x)$$

for $x, y \geqslant 1$.

The quotient $c(x_3, y_3)/c(x, y)$

We need not do any calculations here: we just interchange x and y in the quotient $c(x-1, y)/c(x, y)$ and obtain that the quotient is approximately equal to $\varphi(y, x)$.

Adding the three terms

We have found that the quotient (4), which is the sum of the three quotients $c(x_j, y_j)/c(x, y)$, is approximately equal to

$$\varphi(x,y)+\varphi(x,y)\varphi(y,x)+\varphi(y,x)=(1+\varphi(x,y))(1+\varphi(y,x))-1, \tag{5}$$

where

$$\varphi(x,y)=e^{-1}\left(\frac{x}{x-1}\right)^{x-1}\frac{x}{r+y}, \qquad x\geqslant 1,\;\; y\geqslant 0,$$

and $\varphi(x,y)$ is defined also for $0\leqslant x<1$ by a special formula.

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We shall study the expression (5) as $r\to+\infty$, equivalently, as $x+y\to+\infty$. It is easy to do so if y/r tends to a limit, say $\sin\theta$; let us do this first. (Actually this result is enough for our conclusion concerning the regularized indicators.)

It is clear that if $0 < \theta < \frac{1}{2}\pi$, then as $r \to +\infty$, both x and y tend to infinity, so that the factors $(x/(x-1))^{x-1}$ and $(y/(y-1))^{y-1}$ tend to e, and, if y/r tends to $\sin\theta$, the expression (5) tends to

$$\left(1+\frac{\cos\theta}{1+\sin\theta}\right)\left(1+\frac{\sin\theta}{1+\cos\theta}\right)-1=1.$$

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If $\theta = 0$, then $\varphi(x,y)$ tends to 1 while $\varphi(y,x)$ tends to 0 (this is because of the bound $x/(x-1)^{x-1} \leqslant e$). Hence (5) still tends to 1.

And if $\theta = \frac{1}{2}\pi$, then $\varphi(x,y)$ tends to 0 while $\varphi(y,x)$ tends to 1, so that also in this case (5) tends to 1.

However, we claim more: we claim not just that (5) tends to 1 as y/r tends to $\sin\theta$ for each θ but that the convergence is uniform in $\theta \in [0, \frac{1}{2}\pi]$, in other words that $|(1+\varphi(x,y))(1+\varphi(y,x))-2| \leqslant \varepsilon$ when $r \geqslant R_\varepsilon$. Dini's theorem will save us from explicit estimates, which are, however, not difficult.

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$$\varphi_r(\theta) = \varphi(r\cos\theta, r\sin\theta), \qquad \theta \in [0, \tfrac{1}{2}\pi], \quad r > 0.$$

These functions form an increasing family. As we saw, φ_r tends to

$$\varphi_\infty(\theta) = \frac{\cos\theta}{1+\sin\theta}.$$

All functions φ_r as well as φ_∞ are continuous, and the convergence $\varphi_r \nearrow \varphi_\infty$ is monotone on the compact set $[0, \frac{1}{2}\pi]$.

However, we claim more: we claim not just that (5) tends to 1 as y/r tends to $\sin\theta$ for each θ but that the convergence is uniform in $\theta \in [0, \frac{1}{2}\pi]$, in other words that $|(1+\varphi(x,y))(1+\varphi(y,x))-2| \leqslant \varepsilon$ when $r \geqslant R_\varepsilon$. Dini's theorem will save us from explicit estimates, which are, however, not difficult. Indeed, let us define

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All functions φ_r as well as φ_∞ are continuous, and the convergence $\varphi_r \nearrow \varphi_\infty$ is monotone on the compact set $[0, \frac{1}{2}\pi]$. Dini's theorem says that the convergence is uniform. Then $(1+\varphi(x,y))(1+\varphi(y,x))$ tends uniformly to 2. $\square$

We can now compare d and c.

Proposition

For every positive ε there exist positive constants λ_ε and Λ_ε such that

$$\lambda_\varepsilon e^{-\varepsilon(x+y)} c(x,y) \leqslant d(x,y) \leqslant \Lambda_\varepsilon e^{\varepsilon(x+y)} c(x,y), \qquad (x,y) \in \dot{\mathbb{N}}^2.$$

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Proof. An induction on the sum $x+y$ is possible for large values of $x+y$:

$$\begin{aligned} d(x,y) &= d(x-1,y) + d(x-1,y-1) + d(x,y-1) \\ &\leqslant \Lambda_\varepsilon e^{\varepsilon(x+y-1)} c(x-1,y) + \Lambda_\varepsilon e^{\varepsilon(x+y-2)} c(x-1,y-1) \\ &\quad + \Lambda_\varepsilon e^{\varepsilon(x+y-1)} c(x,y-1) \\ &\leqslant \Lambda_\varepsilon e^{\varepsilon(x+y-1)} [c(x-1,y) + c(x-1,y-1) + c(x,y-1)] \\ &\leqslant \Lambda_\varepsilon e^{\varepsilon(x+y)} c(x,y). \end{aligned}$$

The estimate from below is done similarly. For smaller values of $x+y$ we obtain the result at the price of taking a smaller constant λ_ε and a larger constant Λ_ε.

Main result

We can now formulate our main result.

Theorem

The regularized radial indicators of the array of Delannoy numbers are

$$p_d^*(x,y) = q_*^d(x,y) = p_c(x,y) = \log c(x,y) = x \log \frac{r+y}{x} + y \log \frac{r+x}{y}$$

for all $(x,y) \in \mathbb{R}_+^2$*. The limit superior and limit inferior in the definitions of the upper and lower radial indicators* p_d *and* q^d *are actually limits.*

Proof.

From Proposition 17 we know that

$$\lambda_\varepsilon e^{-\varepsilon(x+y)}c(x,y) \leqslant d(x,y) \leqslant \Lambda_\varepsilon e^{\varepsilon(x+y)}c(x,y).$$

From this it follows that the radial indicators satisfy

$$p_c(x,y) - \varepsilon(x+y) \leqslant p_d(x,y) \leqslant \varepsilon(x+y) + p_c(x,y).$$

Since ε is arbitrary, we find that $p_d = p_c$. Similarly $q^d = q^c$ and for the regularized indicators. $\square$

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Thank you!

Thank you!

Teşekkür ederim!