Pluricomplex Green functions with colliding poles

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Joint work with Pascal Thomas (Toulouse)

1. Setting of the problem

In $\mathbb{C}$: Green function of $D \subset \mathbb{C}$ with pole at $a \in D$:
the (unique) solution $G(z,a)$ of $\frac{1}{2\pi}\Delta G = \delta_a$, $G|_{\partial D} = 0$.

With several poles at $A = \{a_1, \dots, a_N\}$: $\frac{1}{2\pi}\Delta G = \sum_k \delta_{a_k}$.

Obviously, $G(z,A) = \sum_k G(z,a_k)$.

If all $a_k \to a \in D$: $G(z,A) \to N \cdot G(z,a)$, locally uniformly outside a.

In $\mathbb{C}^n$, $n > 1$: D - bounded hyperconvex domain (f. ex., polydisk/ball)

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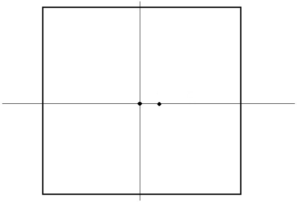
First examples

Example 1. $D = \mathbb{D}^2 \subset \mathbb{C}^2$, $N = 2$, $a_1 = 0$, $a_2 \to 0$.



If $a_2 = (\varepsilon, 0)$, then $G(z, A_\varepsilon) \to \max\{2\log|z_1|, \log|z_2|\}$.

If $a_2 = (0, \varepsilon)$, then $G(z, A_\varepsilon) \to \max\{\log|z_1|, 2\log|z_2|\}$.
So: no 'unrestricted' limit exists.

Can be shown: a 'restricted' limit exists iff $\exists \lim \frac{a_2}{|a_2|} = v \in S_1$, and then

$$\lim_{\varepsilon \to 0} G(z, A_\varepsilon) = \max\{2\log|\xi_1|, \log|\xi_2|\}$$

with (ξ_1, ξ_2) - coordinates of z in an orthonormal basis $\{v, w\}$.

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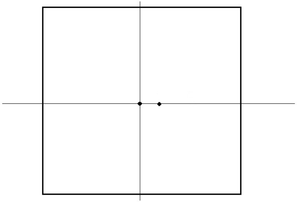

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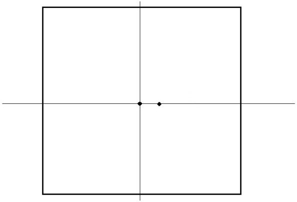

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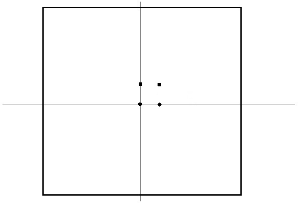
First examples (cont.)

Example 2. $D = \mathbb{D}^2$, $N = 4$, $a_1 = 0$, $a_2 = (\varepsilon, 0), a_3 = (0, \varepsilon), a_4 = (\varepsilon, \varepsilon)$.



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Green functions of ideals

Given an ideal $\mathcal{I} = \langle \psi_1, \dots \psi_p \rangle \subset \mathcal{O}(D)$, the *Green function* $G_{\mathcal{I}}$ satisfies

$(dd^c G)^n = 0$ on $D \setminus V(\mathcal{I})$, $G = \log|\psi| + O(1)$, $G|_{\partial D} = 0$,
where $V(\mathcal{I}) = \{z : f(z) = 0 \ \forall f \in \mathcal{I}\}$ (A.R.- R. Sigurdsson, 2005)

If $\mathcal{I} = \mathfrak{m}_a$, the maximal ideal at $a \in D$, then $G_{\mathcal{I}}(z) = G(z, a)$.
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Question: What happens when $\mathcal{I} = \mathcal{I}_\varepsilon$, $\varepsilon \to 0$?

We need a notion of convergence of ideals.

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$(\mathcal{I}_\varepsilon)_{\varepsilon\in E}$: finite length ideals in $\mathcal{O}(D)$ (i.e., $\dim \mathcal{O}(D)/\mathcal{I} < \infty$);
$E \subset \mathbb{C}$, $0 \in \overline{E} \setminus E$.

(i) $\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon$: ideal consisting of all f such that $\mathcal{I}_\varepsilon \ni f_\varepsilon \to f$ locally uniformly on D, as $\varepsilon \to 0$.

(ii) $\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon$: ideal generated by all f such that $f_j \to f$ locally uniformly, as $j \to \infty$, for some sequence $\varepsilon_j \to 0$ in E and $f_j \in \mathcal{I}_{\varepsilon_j}$.

(iii) If the two limits are equal, we say that the family $\mathcal{I}_\varepsilon$ *converges* and write $\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon$ for the common values of the limits.

The convergence is equivalent to the one in the topology of the Douady space (determined by *flat families*.)

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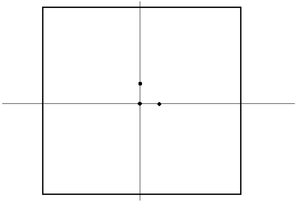
Examples of convergence

Example 1, cont'd. 2-point ideals
$\mathcal{I}_\varepsilon = \mathfrak{m}_0 \cap \mathfrak{m}_{(\varepsilon,0)} \to \mathcal{I} = \langle z_1^2, z_2 \rangle$, and $G_{\mathcal{I}_\varepsilon} \to G_{\mathcal{I}}$.

Example 2, cont'd. 4-point ideals
$\mathcal{I}_\varepsilon = \mathfrak{m}_0 \cap \mathfrak{m}_{(\varepsilon,0)} \cap \mathfrak{m}_{(0,\varepsilon)} \cap \mathfrak{m}_{(\varepsilon,\varepsilon)} \to \mathcal{I} = \langle z_1^2, z_2^2 \rangle$, and $G_{\mathcal{I}_\varepsilon} \to G_{\mathcal{I}}$.

Example 3. 3-point ideals



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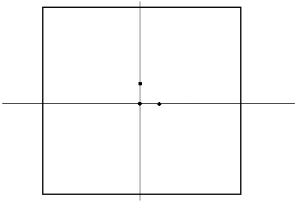

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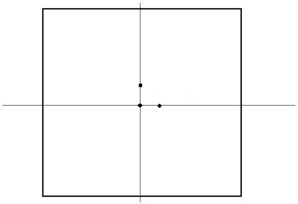

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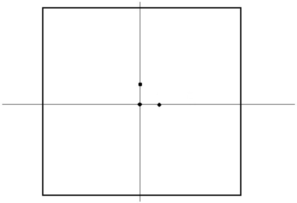

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Keeping the mass

In Ex. 1 and 2, *the total Monge-Ampère masses of $G_{\mathcal{I}_\varepsilon}$ and $G_{\mathcal{I}}$ coincide*:

$$(dd^c G_{\mathcal{I}_\varepsilon})^2(D) = 2 = (dd^c G_{\mathcal{I}_\varepsilon})^2(D) \quad \text{(Ex. 1)}$$

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In Ex. 3, $(dd^c G_{\mathcal{I}_\varepsilon})^2(D) = 3 < (dd^c G_{\mathcal{I}})^2(D) = 4$.

Why?

Because the limit transition keeps the *length* of the ideals, but not the *(Hilbert-Samuel) multiplicity*.

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Hilbert-Samuel multiplicity $e(\mathcal{I}) = \lim_{k\to\infty} n! k^{-n} \ell(\mathcal{I}^k)$.
Known: $e(\mathcal{I}) \geq \ell(\mathcal{I})$, with the equality iff $\mathcal{I}$ is a *complete intersection ideal* (has precisely n generators).

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Complete intersection case

Proved in [MRST].

Theorem

Let $\mathcal{I}_\varepsilon$ be N-point ideals in $\mathcal{O}(D)$, converging to $\mathcal{I}$ with $V(\mathcal{I}) = \{0\}$.

(i) *If $G_{\mathcal{I}_\varepsilon}$ converges to a function g, locally uniformly outside 0, then $g \geq G_{\mathcal{I}}$.*

(ii) *If $\mathcal{I}$ is a complete intersection ideal, then $G_{\mathcal{I}_\varepsilon} \to G_{\mathcal{I}}$, locally uniformly outside 0.*

(iii) *If $\mathcal{I}$ is not a complete intersection ideal, then $G_{\mathcal{I}_\varepsilon}$ does not converge to $G_{\mathcal{I}}$, even in L^1_{loc}.*

$\mathcal{I}$ are complete intersection ideals in Ex. 1 and 2, while not in Ex. 3.

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Weak vs. uniform

Weak (L^1_{loc}) convergence of psh functions does not imply their uniform convergence.

But: it does - in the case of N-poles Green functions!

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Let $\mathcal{I}_\varepsilon$ be N-point ideals in $\mathcal{O}(D)$ with $V(\mathcal{I}_\varepsilon) \to \{0\}$. If $G_{\mathcal{I}_\varepsilon} \to g$ in $L^1_{loc}(D \setminus \{0\})$, then the convergence is actually locally uniform on $D \setminus \{0\}$. In particular, $(dd^c g)^n = N\delta_0$.

Crucial point of the proof: the family $G_{\mathcal{I}_\varepsilon}$ is shown to be locally equicontinuous, so every sequence $G_{\mathcal{I}_{\varepsilon_k}}$ satisfies local uniform Cauchy criterion.

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Powers of ideals

Assume that not only $\mathcal{I}_\varepsilon$ converge, but also all their powers do:

$$\mathcal{I}_\varepsilon^p \to \mathcal{I}_{(p)}, \quad p = 1, 2, \ldots$$

Since $\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)} \subset \mathcal{I}_{(p+q)}$ (i.e., $\mathcal{I}_{(p)}$ form a *graded family*), we have $G_{\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)}} \leq G_{\mathcal{I}_{(p+q)}}$.

Therefore, the scaled Green functions $\widehat{G}_{\mathcal{I}_{(p)}} = p^{-1} G_{\mathcal{I}_{(p)}}$ converge:

Lemma

[RT] *There exists the limit*

$$V(z) = \lim_{p \to \infty} \widehat{G}_{\mathcal{I}_{(p)}}(z) = \sup_p \widehat{G}_{\mathcal{I}_{(p)}}(z)$$

whose upper regularization $G_{\mathcal{I}_\bullet}(z) = \limsup_{y \to z} V(y) \in \mathrm{PSH}(D)$ *satisfies*

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Main result

Final ingredient:
Domination principle [R] *If $u, v \in \mathrm{PSH}(D)$ solve the Dirichlet problem $(dd^c u)^n = \alpha\,\delta_0$, $u|_{\partial D} = 0$, and $u_1 \le u_2$ on D, then $u_1 \equiv u_2$.*

Combining all this, we get the main result:

Theorem

Let $\{\mathcal{I}_\varepsilon\}_{\varepsilon\in E}$ be a family of ideals of holomorphic functions vanishing at distinct points $a_1(\varepsilon),\ldots,a_N(\varepsilon)$ of a bounded hyperconvex domain $D \subset \mathbb{C}^n$, where $E \subset \mathbb{C}$, $0 \in \overline{E}\setminus E$. Assume that all $a_j \to a \in D$ and $\mathcal{I}_\varepsilon^p \to \mathcal{I}_{(p)}$ for all $p \in \mathbb{Z}_+$ as $\varepsilon \to 0$ along E. Then the Green functions $G_{\mathcal{I}_\varepsilon}$ converge, locally uniformly on $D\setminus\{0\}$, to the upper regularization of the upper envelope of the scaled Green functions of the limit ideals:

$$\lim_{\varepsilon\to 0} G_{\mathcal{I}_\varepsilon}(z) = \limsup_{y\to z}\,\sup_p\; p^{-1} G_{\mathcal{I}_{(p)}}(y).$$

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Examples

Example 3, cont'd. In 3-point model, $\mathcal{I}_\varepsilon^2 \to \mathcal{I}_{(2)} = \mathfrak{m}_0^4 + \langle z_1 z_2(z_1+z_2)\rangle$. Since the multiplicity of $\mathcal{I}_{(2)}$ equals 12,
$\left(dd^c \widehat{G}_{\mathcal{I}_{(2)}}\right)^2(0) = 3$, so $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{(2)}}$ and

$$\lim G_{\mathcal{I}_\varepsilon} = \widehat{G}_{\mathcal{I}_{(2)}} = \max\{2\log|z_1|, 2\log|z_2|, \frac{1}{2}\log|z_1^2 z_2 + z_1 z_2^2|\} + O(1).$$

Example 4. In a similar model of $n+1$ points in $\mathbb{C}^n$,

$$\lim G_{\mathcal{I}_\varepsilon} = \widehat{G}_{\mathcal{I}_{(n)}} = \max\left\{\frac{1}{\#J}\log\left|(\sum_{j\in J} z_j)\prod_{j\in J} z_j\right|, J\subset\{1,\dots,n\}\right\} + O(1).$$

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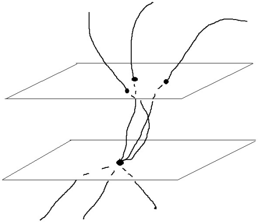
Examples

Example 5. *Hyperplane sections of holomorphic curves*



More generally: a holomorphic curve $\Gamma \in \mathbb{C}^{n+1}$ such that 0 is its singular point.
Let $\mathcal{I}_\varepsilon \subset \mathcal{O}(\mathbb{D}^n)$ be determined by the points $a_k = a_k(\varepsilon): \ (a_k, \varepsilon) \in \Gamma$.
Then the limit of the corresponding Green functions $G_{\mathcal{I}_\varepsilon}$ exists and equals the function $G_{\mathcal{I}_\bullet}$.

Further developments

Mostly by Pascal Thomas and his students/collaborators:

Three- and four-point models with different rates of approaching 0: [MRST], [DT1], [DT2].

Several limit points: [NT].

Questions

1. In the $n+1$-point example, $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{(t(n))}}$ with $t(n) = lcm\{1,\dots,n\}$ (in particular, $t(n) \le n!$). *What is (the asymptotic of) the best possible index $p(n)$ such that $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{(p(n))}}$?*

2. *Is it always true that $G_{\mathcal{I}_\bullet} = \widehat{G}_{\mathcal{I}_{(p)}}$ for some p, at least in the setting of hyperplane sections?*

3. *What can be said in the case of non-radical ideals $\mathcal{I}_\varepsilon$ whose varieties tend to a single point?*

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