Pluricomplex Green functions with colliding poles

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Joint work with Pascal Thomas (Toulouse)
1. Setting of the problem

In $\mathbb{C}$: Green function of $D \subset \mathbb{C}$ with pole at $a \in D$: the (unique) solution $G(z, a)$ of $\frac{1}{2\pi} \Delta G = \delta_a$, $G|_{\partial D} = 0$.

With several poles at $A = \{a_1, \ldots, a_N\}$: $\frac{1}{2\pi} \Delta G = \sum_k \delta_{a_k}$.

Obviously, $G(z, A) = \sum_k G(z, a_k)$.

If all $a_k \rightarrow a \in D$: $G(z, A) \rightarrow N \cdot G(z, a)$, locally uniformly outside $a$.

In $\mathbb{C}^n$, $n > 1$: $D$ - bounded hyperconvex domain (f. ex., polydisk/ball)

pluricomplex Green function: $G(z, a) \in \text{PSH}^-(D)$, $G|_{\partial D} = 0$, $(dd^c G)^n = 0$ on $D \setminus \{a\}$, $G(z, a) = \log |z - a| + O(1)$ as $z \rightarrow a$.

If $A = \{a_1, \ldots, a_N\} \subset D$: $G(z, a_j) = \log |z - a_j| + O(1)$ as $z \rightarrow a_j$, $1 \leq j \leq N$ (V. Zakhariuta)

$G(z, A) \geq \sum_k G(z, a_k)$, typically $>.$

$G(z, A)$ depends continuously on the poles, provided they do not collide.

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First examples

Example 1. \( D = \mathbb{D}^2 \subset \mathbb{C}^2, \ N = 2, \ a_1 = 0, \ a_2 \to 0. \)

If \( a_2 = (\varepsilon, 0) \), then \( G(z, A_\varepsilon) \to \max\{2 \log |z_1|, \log |z_2|\} \).

If \( a_2 = (0, \varepsilon) \), then \( G(z, A_\varepsilon) \to \max\{\log |z_1|, 2 \log |z_2|\} \).

So: no 'unrestricted' limit exists.

Can be shown: a 'restricted' limit exists iff \( \exists \lim_{\varepsilon \to 0} \frac{a_2}{|a_2|} = v \in S_1 \), and then

\[
\lim_{\varepsilon \to 0} G(z, A_\varepsilon) = \max\{2 \log |\xi_1|, \log |\xi_2|\}
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with \((\xi_1, \xi_2)\) - coordinates of \( z \) in an orthonormal basis \( \{v, w\} \).
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Example 2. $D = \mathbb{D}^2, N = 4, a_1 = 0, a_2 = (\varepsilon, 0), a_3 = (0, \varepsilon), a_4 = (\varepsilon, \varepsilon)$.

\[
G(z, A_\varepsilon) \rightarrow \max\{2 \log |z_1|, 2 \log |z_2|\}.
\]
Green functions of ideals

Given an ideal \( \mathcal{I} = \langle \psi_1, \ldots, \psi_p \rangle \subset \mathcal{O}(D) \), the Green function \( G_\mathcal{I} \) satisfies

\[
(dd^c G)^n = 0 \text{ on } D \setminus V(\mathcal{I}), \quad G = \log |\psi| + O(1), \quad G|_{\partial D} = 0,
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where \( V(\mathcal{I}) = \{z : f(z) = 0 \ \forall f \in \mathcal{I}\} \) (A.R.-R. Sigurdsson, 2005)

If \( \mathcal{I} = \mathfrak{m}_a \), the maximal ideal at \( a \in D \), then \( G_\mathcal{I}(z) = G(z, a) \).
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**Question:** What happens when \( \mathcal{I} = \mathcal{I}_\varepsilon, \varepsilon \to 0 \)?

We need a notion of convergence of ideals.
Given an ideal \( \mathcal{I} = \langle \psi_1, \ldots, \psi_p \rangle \subset \mathcal{O}(D) \), the Green function \( G_{\mathcal{I}} \) satisfies

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Convergence of ideals


$\{I_\varepsilon\}_{\varepsilon \in E}$: finite length ideals in $\mathcal{O}(D)$ (i.e., $\dim \mathcal{O}(D)/I < \infty$);
$E \subset \mathbb{C}$, $0 \in \overline{E} \setminus E$.

(i) $\liminf_{\varepsilon \to 0} I_\varepsilon$: ideal consisting of all $f$ such that $I_\varepsilon \ni f_\varepsilon \to f$ locally uniformly on $D$, as $\varepsilon \to 0$.

(ii) $\limsup_{\varepsilon \to 0} I_\varepsilon$: ideal generated by all $f$ such that $f_j \to f$ locally uniformly, as $j \to \infty$, for some sequence $\varepsilon_j \to 0$ in $E$ and $f_j \in I_{\varepsilon_j}$.

(iii) If the two limits are equal, we say that the family $I_\varepsilon$ converges and write $\lim_{\varepsilon \to 0} I_\varepsilon$ for the common values of the limits.

The convergence is equivalent to the one in the topology of the Douady space (determined by flat families.)
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Examples of convergence

Example 1, cont’d. 2-point ideals
\[ \mathcal{I}_\varepsilon = m_0 \cap m_{(\varepsilon,0)} \to \mathcal{I} = \langle z_1^2, z_2 \rangle, \] and \[ G_{\mathcal{I}_\varepsilon} \to G_{\mathcal{I}}. \]

Example 2, cont’d. 4-point ideals
\[ \mathcal{I}_\varepsilon = m_0 \cap m_{(\varepsilon,0)} \cap m_{(0,\varepsilon)} \cap m_{(\varepsilon,\varepsilon)} \to \mathcal{I} = \langle z_1^2, z_2^2 \rangle, \] and \[ G_{\mathcal{I}_\varepsilon} \to G_{\mathcal{I}}. \]

Example 3. 3-point ideals
\[ \mathcal{I}_\varepsilon = m_0 \cap m_{(\varepsilon,0)} \cap m_{(0,\varepsilon)} \to \mathcal{I} = m_0^2 = \langle z_1^2, z_1 z_2, z_2^2 \rangle. \] At the same time, \[ \lim G_{\mathcal{I}_\varepsilon} \] exists and does not equal \[ G_{\mathcal{I}} \] [MRST].

Question: What is the difference in these examples?
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**Question:** What is the difference in these examples?
Keeping the mass

In Ex. 1 and 2, the total Monge-Ampère masses of $G_{I_\varepsilon}$ and $G_I$ coincide:

$$(dd^c G_{I_\varepsilon})^2(D) = 2 = (dd^c G_{I_\varepsilon})^2(D) \quad (\text{Ex. 1})$$

$$(dd^c G_{I_\varepsilon})^2(D) = 4 = (dd^c G_{I_\varepsilon})^2(D) \quad (\text{Ex. 2})$$

In Ex. 3, $(dd^c G_{I_\varepsilon})^2(D) = 3 < (dd^c G_I)^2(D) = 4$.

Why?

Because the limit transition keeps the length of the ideals, but not the (Hilbert-Samuel) multiplicity.

Length $\ell(I) = \dim \mathcal{O}/I$,

Hilbert-Samuel multiplicity $e(I) = \lim_{k \to \infty} n! k^{-n} \ell(I^k)$.

Known: $e(I) \geq \ell(I)$, with the equality iff $I$ is a complete intersection ideal (has precisely $n$ generators).

Relation with Monge-Ampère: $e(I) = (dd^c G_I)^n(D)$. 
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In Ex. 1 and 2, the total Monge-Ampère masses of $G_{\mathcal{I}_\varepsilon}$ and $G_{\mathcal{I}}$ coincide:

$$(dd^c G_{\mathcal{I}_\varepsilon})^2(D) = 2 = (dd^c G_{\mathcal{I}_\varepsilon})^2(D) \quad \text{(Ex. 1)}$$

$$(dd^c G_{\mathcal{I}_\varepsilon})^2(D) = 4 = (dd^c G_{\mathcal{I}_\varepsilon})^2(D) \quad \text{(Ex. 2)}$$

In Ex. 3, $(dd^c G_{\mathcal{I}_\varepsilon})^2(D) = 3 < (dd^c G_{\mathcal{I}})^2(D) = 4$.

Why?

Because the limit transition keeps the length of the ideals, but not the (Hilbert-Samuel) multiplicity.

Length $\ell(\mathcal{I}) = \dim \mathcal{O}/\mathcal{I}$,

Hilbert-Samuel multiplicity $e(\mathcal{I}) = \lim_{k \to \infty} n! k^{-n} \ell(\mathcal{I}^k)$.

Known: $e(\mathcal{I}) \geq \ell(\mathcal{I})$, with the equality iff $\mathcal{I}$ is a complete intersection ideal (has precisely $n$ generators).

Relation with Monge-Ampère: $e(\mathcal{I}) = (dd^c G_{\mathcal{I}})^n(D)$.
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Complete intersection case

Proved in [MRST].

**Theorem**

Let $\mathcal{I}_\varepsilon$ be $N$-point ideals in $\mathcal{O}(D)$, converging to $\mathcal{I}$ with $V(\mathcal{I}) = \{0\}$.

(i) If $G_{\mathcal{I}_\varepsilon}$ converges to a function $g$, locally uniformly outside $0$, then $g \geq G_{\mathcal{I}}$.

(ii) If $\mathcal{I}$ is a complete intersection ideal, then $G_{\mathcal{I}_\varepsilon} \to G_{\mathcal{I}}$, locally uniformly outside $0$.

(iii) If $\mathcal{I}$ is not a complete intersection ideal, then $G_{\mathcal{I}_\varepsilon}$ does not converge to $G_{\mathcal{I}}$, even in $L^1_{loc}$.

$\mathcal{I}$ are complete intersection ideals in Ex. 1 and 2, while not in Ex. 3.

What will be in the incomplete intersection case?
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Weak vs. uniform

Weak \((L^1_{loc})\) convergence of psh functions does not imply their uniform convergence.

But: it does - in the case of \(N\)-poles Green functions!

**Theorem**

Let \(I_\varepsilon\) be \(N\)-point ideals in \(O(D)\) with \(V(I_\varepsilon) \to \{0\}\). If \(G_{I_\varepsilon} \to g\) in \(L^1_{loc}(D \setminus \{0\})\), then the convergence is actually locally uniform on \(D \setminus \{0\}\). In particular, \((dd^c g)^n = N\delta_0\).

Crucial point of the proof: the family \(G_{I_\varepsilon}\) is shown to be locally equicontinuous, so every sequence \(G_{I_{\varepsilon_k}}\) satisfies local uniform Cauchy criterion.
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Powers of ideals

Assume that not only $\mathcal{I}_\varepsilon$ converge, but also all their powers do:

$$\mathcal{I}_\varepsilon^p \to \mathcal{I}_{(p)}, \quad p = 1, 2, \ldots$$

Since $\mathcal{I}_{(p)} \cdot \mathcal{I}_{(q)} \subset \mathcal{I}_{(p+q)}$ (i.e., $\mathcal{I}_{(p)}$ form a graded family), we have

$$G_{\mathcal{I}_{(p)}} \cdot \mathcal{I}_{(q)} \leq G_{\mathcal{I}_{(p+q)}}.$$

Therefore, the scaled Green functions $\hat{G}_{\mathcal{I}_{(p)}} = p^{-1} G_{\mathcal{I}_{(p)}}$ converge:

**Lemma**

[RT] There exists the limit

$$V(z) = \lim_{p \to \infty} \hat{G}_{\mathcal{I}_{(p)}}(z) = \sup_p \hat{G}_{\mathcal{I}_{(p)}}(z)$$

whose upper regularization $G_{\mathcal{I}_{\bullet}}(z) = \lim \sup_{y \to z} V(y) \in \text{PSH}(D)$ satisfies

$$(dd^c G_{\mathcal{I}_{\bullet}})^n = N\delta_0.$$ 

Furthermore, $\hat{G}_{\mathcal{I}_{(p)}} \to G_{\mathcal{I}_{\bullet}}$ in $L^p(\Omega)$ for all $p \in [1, n]$. 
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Assume that not only $I_{\varepsilon}$ converge, but also all their powers do:

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Main result

Final ingredient:

**Domination principle [R]** If \( u, v \in \text{PSH}(D) \) solve the Dirichlet problem \((dd^c u)^n = \alpha \delta_0, u|_{\partial D} = 0, \) and \( u_1 \leq u_2 \) on \( D \), then \( u_1 \equiv u_2 \).

Combining all this, we get the main result:

**Theorem**

Let \( \{I_\varepsilon\}_{\varepsilon \in E} \) be a family of ideals of holomorphic functions vanishing at distinct points \( a_1(\varepsilon), \ldots, a_N(\varepsilon) \) of a bounded hyperconvex domain \( D \subset \mathbb{C}^n \), where \( E \subset \mathbb{C}, 0 \in \overline{E} \setminus E \). Assume that all \( a_j \to a \in D \) and \( I_\varepsilon(p) \to I(p) \) for all \( p \in \mathbb{Z}_+ \) as \( \varepsilon \to 0 \) along \( E \). Then the Green functions \( G_{I_\varepsilon} \) converge, locally uniformly on \( D \setminus \{0\} \), to the upper regularization of the upper envelope of the scaled Green functions of the limit ideals:

\[
\lim_{\varepsilon \to 0} G_{I_\varepsilon}(z) = \limsup_{y \to z} \sup_{p} p^{-1} G_{I(p)}(y).
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Examples

**Example 3, cont’d.** In 3-point model, $\mathcal{I}_2^2 \to \mathcal{I}_{(2)} = m_0^4 + \langle z_1 z_2 (z_1 + z_2) \rangle$. Since the multiplicity of $\mathcal{I}_{(2)}$ equals 12,

$$\left( dd^c \hat{G}_{\mathcal{I}_{(2)}} \right)^2 (0) = 3,$$

so $G_{\mathcal{I}_\bullet} = \hat{G}_{\mathcal{I}_{(2)}}$ and

$$\lim G_{\mathcal{I}_\varepsilon} = \hat{G}_{\mathcal{I}_{(2)}} = \max\{2 \log |z_1|, 2 \log |z_2|, \frac{1}{2} \log |z_1^2 z_2 + z_1 z_2^2|\} + O(1).$$

**Example 4.** In a similar model of $n + 1$ points in $\mathbb{C}^n$,

$$\lim G_{\mathcal{I}_\varepsilon} = \hat{G}_{\mathcal{I}_{(n)}} = \max \left\{ \frac{1}{\# J} \log \left| \left( \sum_{j \in J} z_j \right) \prod_{j \in J} z_j \right|, J \subset \{1, \ldots, n\} \right\} + O(1).$$
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Example 3, cont’d. In 3-point model, \( \mathcal{I}_ε^2 \rightarrow \mathcal{I}_{(2)} = m_0^4 + \langle z_1 z_2 (z_1 + z_2) \rangle \). Since the multiplicity of \( \mathcal{I}_{(2)} \) equals 12, 

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\]
Examples

Example 5. *Hyperplane sections of holomorphic curves*

More generally: a holomorphic curve $\Gamma \in \mathbb{C}^{n+1}$ such that 0 is its singular point.

Let $\mathcal{I}_\varepsilon \subset \mathcal{O}(\mathbb{D}^n)$ be determined by the points $a_k = a_k(\varepsilon) : (a_k, \varepsilon) \in \Gamma$.

Then the limit of the corresponding Green functions $G_{I_\varepsilon}$ exists and equals the function $G_{I_\bullet}$. 
Further developments

Mostly by Pascal Thomas and his students/collaborators:

Three- and four-point models with different rates of approaching 0: [MRST], [DT1], [DT2].
Several limit points: [NT].
Questions

1. In the $n + 1$-point example, $G_{I_e} = \hat{G}_{I(t(n))}$ with $t(n) = \text{lcm}\{1, \ldots, n\}$ (in particular, $t(n) \leq n!$). What is (the asymptotic of) the best possible index $p(n)$ such that $G_{I_e} = \hat{G}_{I_{p(n)}}$?

2. Is it always true that $G_{I_e} = \hat{G}_{I_p}$ for some $p$, at least in the setting of hyperplane sections?

3. What can be said in the case of non-radical ideals $I_e$ whose varieties tend to a single point?
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