Lineal convexity

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linealistanbul

Abstract
A bounded open set with boundary of class C^1 which is locally weakly lineally convex is weakly lineally convex, but, as shown by Yuriĭ Zelinskiĭ, this is not true for unbounded domains. The purpose here is to construct explicit examples, Hartogs domains, showing this. Their boundary can have regularity $C^{1,1}$ or C^∞. Obstructions to constructing certain smooth domains will be discussed.

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2004 September 20 at Sabancı: *Holomorphic functions on discrete sets* (see my papers 05-1 and 08-3).

2007 September 19 at Karaköy: *Vyacheslav Zakharyuta's complex analysis* (see my paper 09-1).

2007 September 20 at Sabancı: *Three problems in discrete convexity: local minima, marginal functions, and separating hyperplanes* (see my papers 08-1, 10-3).

2012 April 06 at Karaköy: *Asymptotic properties of the Delannoy numbers and similar arrays* (this paper is now generalized but not yet published).

1. Introduction

In my paper (1998) I claimed that a differential condition which I called the Behnke–Peschl condition implies that a connected open subset of $\mathbb{C}^n$ with boundary of class C^2 is weakly lineally convex. The proof in the case of bounded domains relied on a result by Južakov and Krivokolesko (1971), proved also in Hörmander (1994: Proposition 4.6.4), but in the case of unbounded domains, the proof of their result breaks down.

1. Introduction

In my paper (1998) I claimed that a differential condition which I called the Behnke–Peschl condition implies that a connected open subset of $\mathbb{C}^n$ with boundary of class C^2 is weakly lineally convex. The proof in the case of bounded domains relied on a result by Yužakov and Krivokolesko (1971), proved also in Hörmander (1994: Proposition 4.6.4), but in the case of unbounded domains, the proof of their result breaks down.

Yuriĭ Zelinskiĭ (2002a, 2002b) published a counterexample in the case of an unbounded set. His example is not very explicit. We shall construct here an explicit example—actually a Hartogs domain, which has the advantage of being easily visualized in three real dimensions.

2. Lineal convexity

Definition

A subset of $\mathbb{C}^n$ is said to be ***lineally convex*** if its complement is a union of complex affine hyperplanes. $\square$

To every set A there exists a smallest lineally convex subset $\mu(A)$ which contains A. Clearly the mapping $\mu\colon \mathscr{P}(\mathbb{C}^n) \to \mathscr{P}(\mathbb{C}^n)$, where $\mathscr{P}(\mathbb{C}^n)$ denotes the family of all subsets of $\mathbb{C}^n$ (the power set), is increasing and idempotent, in other words an **ethmomorphism** (morphological filter). It is also larger than the identity, so that μ is a **cleistomorphism** (closure operator) in the ordered set $\mathscr{P}(\mathbb{C}^n)$.

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This kind of complex convexity was introduced by Heinrich Behnke (1898–1997) and Ernst Ferdinand Peschl (1906–1986). I learnt about it from André Martineau (1930–1972) when I was in Nice (1967/68). See Martineau (1966, 1967, 1968).

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Are there lineally convex sets which are not convex? This is obvious in one complex variable, and from there we can easily construct, by taking Cartesian products, lineally convex sets in any dimension which are not convex. But these sets do not have smooth boundaries. Hörmander (1994:293, Remark 3) constructs open connected sets in $\mathbb{C}^n$ with boundary of class C^2 as perturbations of a convex set. These sets are lineally convex and close to a convex set in the C^2 topology, and therefore starshaped with respect to some point if the perturbation is small. Also the symmetrized bidisk $\{(z_1 + z_2, z_1 z_2) \in \mathbb{C}^2; |z_1|, |z_2| < 1\}$ studied by Agler & Young (2004) and Pflug & Zwonek (2012) is not convex, but it is starshaped. So we may ask:

Question

Does there exist a lineally convex set in $\mathbb{C}^n$, $n \geqslant 2$, with smooth boundary which is not starshaped with respect to any point? □

3. Weak lineal convexity

Definition

An open subset Ω of $\mathbb{C}^n$ is said to be ***weakly lineally convex*** if there passes, through every point on the boundary of Ω, a complex affine hyperplane which does not cut Ω. $\square$

From this definition it is clear that every lineally convex open set is weakly lineally convex. The converse does not hold. This is not difficult to see if we allow sets that are not connected:

Example

Given a number c with $0 < c < 1$, define an open set Ω_c in $\mathbb{C}^2$ as the union of the set

$$\{z \in \mathbb{C}^2;\ |y_1| < 1,\ c < |x_1| < 1,\ |x_2| < c,\ |y_2| < c\}$$

with the two sets obtained by permuting x_1, x_2 and y_2. Thus Ω_c consists of six boxes. It is easy to see that it is weakly lineally convex, but there are many points in its complement such that every complex line passing through that point hits Ω_c.

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Any complex line intersects the real hyperplane defined by $y_1 = 0$ in the empty set or in a real line or in a real two-dimensional plane, and the three-dimensional set $\{z;\ y_1 = 0\} \cap \Omega_c$ is easy to visualize. $\square$

It is less easy to construct a connected set with these properties, but this has been done by Yužakov & Krivokolesko (1971:325, Example 2). See also an example due to Hörmander in the book by Andersson, Passare & Sigurdsson (2004:20–21, Example 2.1.7).

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However, the boundary of the constructed set is not of class C^1, and this is essential. Indeed, Yužakov & Krivokolesko (1971:323, Theorem 1) proved that a connected bounded open set with "smooth" boundary is locally weakly lineally convex if and only if it is lineally convex. It is then even $\mathbb{C}$-convex (1971:324, Assertion). See also Corollary 4.6.9 in Hörmander (1994), which states that a connected bounded open set with boundary of class C^1 is locally weakly lineally convex if and only if it is $\mathbb{C}$-convex (and every $\mathbb{C}$-convex open set is lineally convex).

There cannot be any cleistomorphism connected with the notion of weak lineal convexity for the simple reason that the property is defined only for open sets. We might therefore want to define weak lineal convexity for arbitrary sets. We may ask:

Question

Is there a reasonable definition of weak lineal convexity for all sets which keeps the definition for open sets and is such that there is a cleistomorphism associating to any $A \subset \mathbb{C}^n$ the smallest set which contains A and is weakly lineally convex?

□

For an open set Ω with boundary of class C^1, there is, at any given boundary point p, only one possible complex hyperplane that might be in the complement of the set: The complex tangent plane at p, $Y = p + T_{\mathbb{C}}(p)$. That Y does not meet Ω is equivalent to $L \subset \complement\Omega$ for all complex lines L contained in Y. This is convenient, because it allows us to work with 1-dimensional complex subspaces rather than $(n-1)$-dimensional complex subspaces.

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Question

Is this worth studying?

4. Local weak lineal convexity

Definition

We shall say that an open set $\Omega \subset \mathbb{C}^n$ is ***locally weakly lineally convex*** if for every point p there exists a neighborhood V of p such that $\Omega \cap V$ is weakly lineally convex. $\square$

Obviously, a weakly lineally convex open set has this property, but the converse does not hold, which is obvious for sets which are not connected: Take the union of two open balls whose closures are disjoint. Also for connected sets the converse does not hold:

Example (Kiselman 1996, Example 3.1.)
Define first

$$\Omega_+ = \{(z,t);\ |z| < 1 \text{ and } |t| < |z-2|\};$$

$$\Omega_- = \{(z,t);\ |z| < 1 \text{ and } |t| < |z+2|\},$$

and then

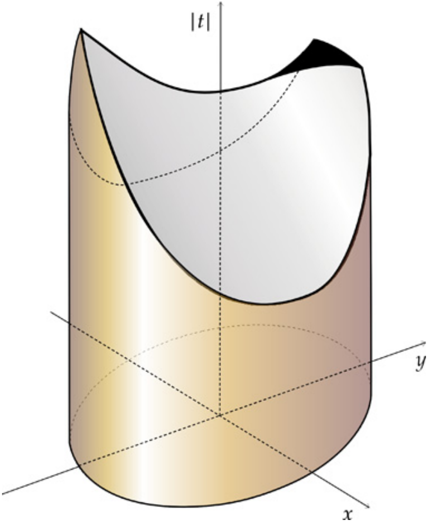
$$\Omega_0 = \Omega_+ \cap \Omega_-; \qquad \Omega_0^r = \{(z,t) \in \Omega_0;\ |t| < r\},$$

where r is a constant with $2 < r < \sqrt{5}$. All these sets are lineally convex. The two points $(\pm i, \sqrt{5})$ belong to the boundary of Ω_0; in the three-dimensional space of the variables $(\operatorname{Re} z, \operatorname{Im} z, |t|)$, the set representing Ω_0 has two peaks, which have been truncated in Ω_0^r.

We now define Ω^r by glueing together Ω_0 and Ω_0^r: Define Ω^r as the subset of Ω_0 such that $(z,t) \in \Omega_0^r$ if $\operatorname{Im} z > 0$; we truncate only one of the peaks of Ω_0.

The point $(i - \varepsilon, r)$ for a small positive ε belongs to the boundary of Ω^r and the tangent plane at that point has the equation $t = r$ and so must cut Ω^r at the point $(-i + \varepsilon, r)$. Therefore Ω^r is not lineally convex, but it agrees with the lineally convex sets Ω_0 and Ω_0^r when $\operatorname{Im} z < \delta$ and $\operatorname{Im} z > -\delta$, respectively, for a small positive δ. The set has Lipschitz boundary; in particular it is equal to the interior of its closure.

$\square$



An open connected Hartogs set in $\mathbb{C}^2$ which is locally weakly lineally convex but not weakly lineally convex. Coordinates $(z, t) \in \mathbb{C}^2$; $(x, y, |t|) \in \mathbb{R}^3$. (Graphics by Erik Melin.)

In this example it is essential that the boundary is not smooth.

Zelinskiĭ (1993:118, Example 13.1) constructs an open set which is locally weakly lineally convex but not weakly lineally convex. The set is not equal to the interior of its closure.

Definition

Let us say that an open set Ω is ***locally weakly lineally convex in the sense of Yužakov and Krivokolesko*** (1971:323) if for every boundary point p there exists a complex hyperplane Y passing through p and a neighborhood V of p such that Y does not meet $V \cap \Omega$. □

Zelinskiĭ (1993:118, Definition 13.1) uses this definition and calls the property локальая линейная выпуклость, thus a property which is strictly weaker than the local weak lineal convexity defined here as we shall see.

Hörmander (1994: Proposition 4.6.4) and Andersson et al. (2004: Proposition 2.5.8) use this property only for open sets with boundary of class C^1. Then the hyperplane Y is unique.

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For all open sets, local weak lineal convexity obviously implies local weak lineal convexity in the sense of Yužakov and Krivokolesko. In the other direction, Hörmander's Proposition 4.6.4 shows that for bounded open sets with boundary of class C^1, local weak lineal convexity in the sense of Yužakov and Krivokolesko implies local weak lineal convexity (even weak lineal convexity if the set is connected).

Nikolov (2012: Proposition 3.7.1) and Nikolov et al. (2010: Proposition 3.3) have a local result in the same direction: If Ω has a boundary of class C^k, $2 \leqslant k \leqslant \infty$, and $\Omega \cap B_<(p, r)$, where p is a given point, is locally weakly lineally convex in the sense of Yužakov and Krivokolesko at all points near p, then there exists a $\mathbb{C}$-convex open set ω (hence lineally convex) with boundary of class C^k such that $\omega \cap B_<(p, r') = \Omega \cap B_<(p, r')$ for some positive r'.

However, in general the two properties are not equivalent:

Example

There exists a bounded connected open set in $\mathbb{C}^2$ with Lipschitz boundary which is locally weakly lineally convex in the sense of Yužakov and Krivokolesko but not locally weakly lineally convex. While Ω^r is locally weakly lineally convex for $2 < r < \sqrt{5}$, the set Ω^2 for $r = 2$ is not locally weakly lineally convex: The point $(0, 2)$ does not have a neighborhood with the desired property. But it does satisfy the property of Yužakov and Krivokolesko. $\square$

5. The Behnke–Peschl and Levi conditions

The ***real Hessian*** of a C^2 function f is

$$H_f^{\mathbb{R}}(p;s) = \sum f_{x_j x_k}(p) s_j s_k, \qquad p \in \mathbb{R}^m, \quad s \in \mathbb{R}^m.$$

The ***complex Hessian*** is

$$H_f^{\mathbb{C}}(p;t) = \sum f_{z_j z_k}(p) t_j t_k, \qquad p \in \mathbb{C}^n, \quad t \in \mathbb{C}^n.$$

The ***Levi form*** is

$$L_f(p;t) = \sum f_{z_j \bar{z}_k}(p) t_j \bar{t}_k, \qquad p \in \mathbb{C}^n, \quad t \in \mathbb{C}^n.$$

If we let the relation between the real s and the complex t be the usual one:

$$t_j = s_{2j-1} + i s_{2j}, \qquad j = 1, \dots, n,\ s \in \mathbb{R}^n,\ t \in \mathbb{C}^n,$$

we get

$$\tfrac{1}{2} H_f^{\mathbb{R}}(p;s) = \operatorname{Re} H_f^{\mathbb{C}}(p;t) + L_f(p;t), \qquad p \in \mathbb{C}^n,\ s \in \mathbb{R}^{2n},\ t \in \mathbb{C}^n.$$

Let now Ω_f be the set of all points where f is negative. We should assume that $\|grad\, f\| + |f| > 0$ everywhere, so that the boundary of Ω_f is of class C^2.

The complex tangent space $T_{\mathbb{C}}(p)$ at a point $p \in \partial\Omega_f$ is defined by $\sum f_{z_j}(p)t_j = 0$; the real tangent space $T_{\mathbb{R}}(p)$ by $\operatorname{Re} \sum f_{z_j}(p)t_j = 0$.

Definition

An open set Ω with boundary of class C^2 is said to satisfy the ***Behnke–Peschl condition*** if, for every point $p \in \partial\Omega_f$, we have

$$\tfrac{1}{2} H_f^{\mathbb{R}}(p; s) = \operatorname{Re} H_f^{\mathbb{C}}(p; t) + L_f(p; t) \geqslant 0 \qquad \text{when } t \in T_{\mathbb{C}}(p),$$

i.e., when $\sum f_{z_j}(p)t_j = 0$. We say that it satisfies the ***strong Behnke–Peschl condition*** if, for $t \in T_{\mathbb{C}}(p) \smallsetminus \{0\}$, we have strict inequality. $\square$

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That this condition is necessary for weak lineal convexity was known since Behnke and Peschl (1935); the sufficiency was unknown.

I stated the result also for unbounded connected open sets with C^2 boundary. The proof relied on Proposition 4.6.4 in Hörmander (1994), which is stated there for bounded connected open sets with boundary of class C^1; I wrote (1998:4) that the assumption that the domain be bounded is not needed for the conclusion.

We also recall the following classical definition.

Definition

An open set Ω with boundary of class C^2 is said to satisfy the ***Levi condition*** if, for every point $p \in \partial\Omega_f$, we have

$$L_f(p;t) \geqslant 0 \qquad \text{when } t \in T_{\mathbb{C}}(p),$$

i.e., when $\sum f_{z_j}(p)t_j = 0$. We say that it satisfies the ***strong Levi condition*** if, for $t \in T_{\mathbb{C}}(p) \smallsetminus \{0\}$, we have strict inequality. □

The inequality $L_f \geqslant |H^{\mathbb{C}}| \geqslant 0$ shows that the Behnke–Peschl condition implies the Levi condition.

6. Južakov and Krivokolesko: Passage from local to global

Let us quote the part of Proposition 4.6.4 in Hörmander (1994) which is important for us:

Proposition

Let $\Omega \subset \mathbb{C}^n$ be a bounded connected open set with boundary of class C^1 and assume that Ω is locally weakly lineally convex in the sense of Južakov and Krivokolesko. Then Ω is weakly lineally convex.

The result was proved by Južakov & Krivokolelsko (1971) under the condition that the boundary is "smooth." In my paper (1998) I needed this result for boundaries of class C^2, and carelessly claimed that it is true then also if the set is unbounded.

7. Zelinskiĭ's example of 2002

In a paper published in 2002, Zelinskiĭ (2002a, 2002b) gave a counterexample to the result I used for unbounded domains: There exists an unbounded open connected set in $\mathbb{C}^2$ which is locally weakly lineally convex but not weakly lineally convex.

7. Zelinskiĭ's example of 2002

In a paper published in 2002, Zelinskiĭ (2002a, 2002b) gave a counterexample to the result I used for unbounded domains: There exists an unbounded open connected set in $\mathbb{C}^2$ which is locally weakly lineally convex but not weakly lineally convex.

In this paper, Zelinskiĭ first defines a domain $D_1 \subset \mathbb{C}$ as follows. D_1 is the set of all $z \in \mathbb{C}$ such that

$$\left[(|z|^2 < 2) \wedge (x < 0)\right] \vee \left[x^2 + (y-1)^2 < 1\right] \vee \left[x^2 + (y+1)^2 < 1\right],$$

where $z = x + iy$, $(x, y) \in \mathbb{R}^2$. He then says (2002b:346): "It is obvious that this domain is locally convex and smooth at all points of the boundary, except the origin $(0, 0)$." This is not true, since the points $(-1, 1)$ and $(-1, -1)$ are also points where the boundary is not smooth and not convex.

However, I guess that this is due to a typo, and that the author means instead D_1 as the set of all $z \in \mathbb{C}$ such that

$$[(|z| < 2) \wedge (x < 0)] \vee \left[x^2 + (y-1)^2 < 1\right] \vee \left[x^2 + (y+1)^2 < 1\right].$$

In the sequel I shall use this set.

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In the sequel I shall use this set.

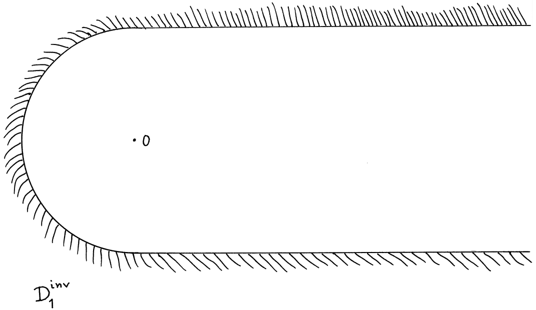
Zelinskiĭ considers the set $G = D_1 \times D_<(0,1)$, which is lineally convex since it is the Cartesian product of two lineally convex sets. It is, however, not $\mathbb{C}$-convex, since the intersection with the complex line of equation $2z_1 - 3z_2 - 4 = 0$ is not connected.

Under inversion of $z = x + iy$ we get a set ${D_1}^{\text{inv}}$ which has boundary of class $C^{1,1}$, since the only point where the boundary is not of class $C^{1,1}$ is thrown to infinity:

$$ {D_1}^{\text{inv}} = \{z;\ 1/z \in D_1\} $$

$$ = \left\{z \in \mathbb{C};\ |\mathrm{Im}\, z| > \tfrac{1}{2} \text{ or } \left[\mathrm{Re}\, z < 0 \text{ and } |z| > \tfrac{1}{2}\right]\right\}. $$

The complement of this set is convex.



The set $D_1{}^{\mathrm{inv}}$.

Zelinskiĭ considers the set $G = D_1 \times D_<(0,1)$ and approximations G_ε of G from the inside. Their boundaries are of class $C^{1,1}$ at all points of the boundary except at $D_0 = \{(0,0)\} \times D_<(0,1)$ and are locally convex at these points. Then he applies a projective mapping (not specified) so that the straight line $\{0\} \times \mathbb{C}$ is mapped to infinity. The resulting domains W_ε then have boundaries of class $C^{1,1}$. For small ε (how small is not specified) they cannot be $\mathbb{C}$-convex, nor weakly lineally convex, while they are locally weakly lineally convex.

8. A new example

We shall construct explicit Hartogs domains here with the properties mentioned in Zelinskiĭ's example.

Example

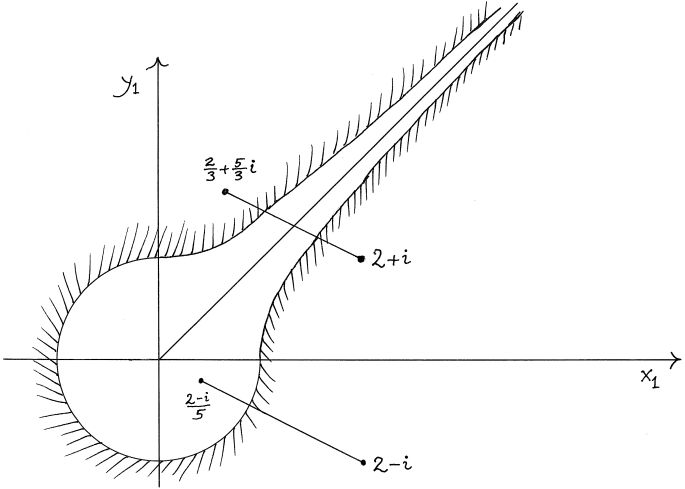
Define a function $\varphi^\diamond\colon \mathbb{C} \to \mathbb{R}$ by

$$\varphi^\diamond(z_1) = \begin{cases} -x_1^2 - y_1^2, & x_1 \leqslant 0 \text{ or } y_1 \leqslant 0; \\ -x_1^2 + y_1^2, & 0 \leqslant y_1 \leqslant x_1; \\ x_1^2 - y_1^2, & 0 \leqslant x_1 \leqslant y_1. \end{cases}$$

Then

$$\Omega_{\varphi^\diamond} = \{z \in \mathbb{C}^2;\ 1 + \varphi^\diamond(z_1) + |z_2|^2 < 0\}$$

has boundary of class $C^{1,1}$ and is locally weakly lineally convex but not weakly lineally convex, since the tangent plane at the boundary point $p = (2+i, \sqrt{2})$ passes through $q = (q_1, 0) = ((2+5i)/3, 0) \in \Omega_{\varphi^\diamond}$. $\square$



The set $\Omega_{\varphi^\diamond} \cap \{z \in \mathbb{C}^2;\ z_2 = 0\}$.

The set in this example is not of class C^2 at the points where $y_1 = 0$, $x_1 > 0$ or $x_1 = 0$, $y_1 > 0$, but we can make it smoother.

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We note that the function $\varphi^\diamond$ is homogeneous of degree two:

$$\varphi^\diamond(z_1) = \varphi^\diamond(|z_1|e^{it}) = |z_1|^2\psi(t), \qquad z_1 \in \mathbb{C},\ t \in \mathbb{R},$$

and it is therefore natural to ask whether there are smooth homogeneous functions with the good properties—or even an analytic homogeneous function.

Four questions

We ask for functions $\varphi\colon \mathbb{C} \to \mathbb{R}$ which yield a locally weakly lineally convex domain which is not weakly lineally convex in four different cases.

1.1. *Is there a C^∞ function φ with these properties?*

1.2. *Is there a homogeneous C^∞ function φ with these properties?*

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As we shall see, the answer to the first question is in the affirmative. But the answer to Question 1.2 is in the negative.

Example. Now define $\varphi^\star : \mathbb{C} \to \mathbb{R}$ by

$$\varphi^\star(z_1) = \begin{cases} -x_1^2 + \chi(y_1), & x_1 \geqslant y_1; \\ -y_1^2 + \chi(x_1), & x_1 \leqslant y_1, \end{cases}$$

where $\chi \in C^\infty(\mathbb{R})$ is a function of one real variable such that χ' is convex and which satisfies

$$\chi(y_1) = \begin{cases} -y_1^2 + \rho, & y_1 \leqslant -\frac{1}{2}; \\ y_1^2 + \sigma, & y_1 \geqslant \frac{1}{2}, \end{cases}$$

where $\sigma - \rho + \frac{1}{2} = \chi(\frac{1}{2}) - \chi(-\frac{1}{2})$.

The convexity of χ' implies that $2|y_1| \leqslant \chi'(y_1) \leqslant \max(2|y_1|, 1)$ with equality to the left for $|y_1| \geqslant \frac{1}{2}$. This implies that we must have $\frac{1}{2} < \chi(\frac{1}{2}) - \chi(-\frac{1}{2}) < 1$, and we can actually choose χ so that $\chi(\frac{1}{2}) - \chi(-\frac{1}{2})$ is any given number in that interval. For definiteness we can choose $\rho = -\frac{1}{4}$, $\sigma = 0$, $\chi(\frac{1}{2}) - \chi(-\frac{1}{2}) = \frac{3}{4}$, which implies that the part where z_1 is in the first quadrant is unchanged compared to $\Omega_{\varphi^\diamond}$. We can for example choose χ as a suitable third primitive of

$$\chi'''(y_1) = C \exp(1/(y_1 - c) - 1/(y_1 + c)), \qquad -c < y_1 < c,$$

for a number c, $0 < c \leqslant \frac{1}{2}$ and a positive constant C, taking $\chi'''(y_1)$ equal to zero when $|y_1| \geqslant c$.

Then

$$\Omega_{\varphi^\star} = \{z \in \mathbb{C}^2;\ 1 + \varphi^\star(z_1) + |z_2|^2 < 0\}$$

has boundary of class C^∞ and is locally weakly lineally convex but not lineally convex, since the tangent plane at the boundary point $p = (2 + i, \sqrt{2})$ passes through
$q = (q_1, 0) = ((2 + 5i)/3, 0) \in \Omega_{\varphi^\star}$. $\square$

The properties mentioned in these examples will follow from the next proposition.

Proposition

Let $\varphi\colon \mathbb{C} \to \mathbb{R}$ *be a function of class* C^k*,* $k = 2, 3, \ldots, \infty, \omega$*, and define an open set in* $\mathbb{C}^2$ *as*

$$\Omega_\varphi = \{z \in \mathbb{C}^2;\ 1 + \varphi(z_1) + |z_2|^2 < 0\}.$$

We assume that $\varphi_{z_1} \neq 0$ *wherever* $\varphi = -1$*, and that*

$$(-\varphi - 1)\,(|\varphi_{z_1 z_1}| - \varphi_{z_1 \bar{z}_1}) \leqslant |\varphi_{z_1}|^2 \text{ in the set where } -\varphi - 1 \geqslant 0.$$

Then Ω_φ *has boundary of class* C^k *and satisfies the Behnke–Peschl condition at every boundary point. If the inequality is strict at a certain point, we get the strong Behnke–Peschl condition at that point.*

Proof. With $f(z) = 1 + \varphi(z_1) + |z_2|^2$ we get

$$f_{z_1}(z) = \varphi_{z_1}(z_1), \quad f_{z_2}(z) = \bar{z}_2.$$

In view of our assumption, the gradient of f cannot vanish at a boundary point, so smoothness is inherited by the boundary.

The tangent plane at a boundary point p has the equation

$$\varphi_{z_1}(p_1)(z_1 - p_1) + \bar{p}_2(z_2 - p_2) = 0.$$

It hits the plane $z_2 = 0$ at the point

$$q = (q_1, 0) = (p_1 + |p_2|^2/\varphi_{z_1}(p_1), 0).$$

We get $f_{z_1 z_1} = \varphi_{z_1 z_1}$, $f_{z_1 z_2} = f_{z_2 z_2} = 0$; thus
$H_f^{\mathbb{C}}(p; t) = \varphi_{z_1 z_1}(p_1) t_1^2$.

Similarly we get $f_{z_1 \bar{z}_1} = \varphi_{z_1 \bar{z}_1}$, $f_{z_1 \bar{z}_2} = 0$, $f_{z_2 \bar{z}_2} = 1$; thus

$$L_f(p; t) = \varphi_{z_1 \bar{z}_1}(p_1)|t_1|^2 + |t_2|^2.$$

In the tangent space we have $t_1 = -t_2\bar{p}_2/\varphi_{z_1}(p_1)$, which inserted into $\frac{1}{2}H_f^{\mathbb{R}} = \operatorname{Re} H_f^{\mathbb{C}} + L_f$ yields

$$\operatorname{Re}\frac{\bar{p}_2^2\varphi_{z_1z_1}(p_1)}{\varphi_{z_1}(p_1)^2}t_2^2 + \frac{|p_2|^2\varphi_{z_1\bar{z}_1}(p_1)}{|\varphi_{z_1}(p_1)|^2}|t_2|^2 + |t_2|^2.$$

This expression majorizes

$$\frac{|p_2|^2}{|\varphi_{z_1}(p_1)|^2}\big(\varphi_{z_1\bar{z}_1}(p_1) - |\varphi_{z_1z_1}(p_1)|\big)|t_2|^2 + |t_2|^2.$$

Here, as we know, $\varphi_{z_1}(p_1) \neq 0$ and $|p_2|^2 = -\varphi(p_1) - 1 \geqslant 0$. We see that the condition on φ implies that this quantity is nonnegative; if the condition holds with strict inequality we conclude as indicated. $\square$

Corollary

Let φ *have the form* $\varphi(z_1) = -x_1^2 + \chi(y_1)$ *for* $x_1 \geqslant y_1$ *and* $\varphi(z_1) = -y_1^2 + \chi(x_1)$ *for* $y_1 \geqslant x_1$*. We assume that* $\chi(y_1) = -y_1^2$ *for* $y_1 \leqslant -\frac{1}{2}$*. We also assume that* $\chi \in C^k(\mathbb{R})$*,* $k \geqslant 2$*, with* $-2 \leqslant \chi''$ *and such that* $\chi(y_1) > -1$ *when* $\chi'(y_1) = 0$*. Then the conclusion of the proposition holds under the assumption*

$$\frac{1}{4}\chi'(y_1)^2 + \chi(y_1) + 1 \geqslant 0, \qquad y_1 \in \mathbb{R}.$$

The function $\varphi^\diamond$ is not of class $C^{1,1}$ at the points where $x_1 = y_1$, $x_1 > 0$, but this is of no consequence, since these points do not belong to the closure of the set it defines.

We note that the tangent plane at a boundary point p with $\operatorname{Re} p_1 \leqslant 0$ or $\operatorname{Im} p_1 \leqslant 0$ is contained in the complement of $\Omega_{\varphi^\diamond}$; in particular, it hits the plane $z_2 = 0$ at the point $q = (p_1/|p_1|^2, 0) \notin \Omega_{\varphi^\diamond}$. We also note that the part of $\Omega_{\varphi^\diamond}$ where $0 < x_1 < y_1$ is convex, so any tangent plane does not cut this part. Similarly, the part where $0 < y_1 < x_1$ is convex. Therefore $\Omega_\varphi^\diamond$ is the union of three lineally convex sets—actually two, since we can take one open set to be given by $y_1 < x_1$ or $y_1 < 0$, and the other by $x_1 < y_1$ or $x_1 < 0$.

When $x_1 < 0$ or $y_1 < 0$ we get $\varphi^\diamond_{z_1}(z_1) = -\bar{z}_1$, $\varphi^\diamond_{z_1 z_1}(z_1) = 0$, $\varphi^\diamond_{z_1 \bar{z}_1}(z_1) = -1$; when $0 < y_1 \leqslant x_1$ we have $\varphi^\diamond_{z_1}(z_1) = -z_1$, $\varphi^\diamond_{z_1 z_1}(z_1) = -1$ and $\varphi^\diamond_{z_1 \bar{z}_1}(z_1) = 0$; when $0 < x_1 \leqslant y_1$ we have $\varphi^\diamond_{z_1}(z_1) = z_1$, $\varphi^\diamond_{z_1 z_1}(z_1) = 1$ and $\varphi^\diamond_{z_1 \bar{z}_1}(z_1) = 0$. In all three cases $|\varphi_{z_1 z_1}| - \varphi_{z_1 \bar{z}_1} = 1$. An application of the proposition now gives the result, except that it does not give anything at the exceptional points, where the function is not of class C^∞, i.e., those with $y_1 = 0$, $x_1 > 0$ or $x_1 = 0$, $y_1 > 0$. However, we have already seen that at these points, the tangent plane does not cut $\Omega_{\varphi^\diamond}$.

The defining function $1-x_1^2+\chi(y_1)+|z_2|^2$ has nonvanishing gradient everywhere since $\chi' > 0$ everywhere. Smoothness follows.

The function $\varphi^\star$ is not of class C^∞ in the set where $x_1 = y_1$, $x_1 > 0$, but again this is unimportant since these points do not belong to the closure of $\Omega_{\varphi^\star}$. An application of the corollary now gives the result. In fact, with the choice of $\rho = -\frac{1}{4}$, $\sigma = 0$, we need only note that $\chi(y_1) \geqslant -y_1^2 - \frac{1}{4}$ everywhere, and that $\chi'(y_1) \geqslant 2|y_1|$, so that

$$\tfrac{1}{4}\chi'(y_1)^2+\chi(y_1)+1 \geqslant \tfrac{3}{4} > 0,$$

thus with strict inequality.

Now for an impossibility result! (Found in Singapore in December 2013.)

Proposition

Let

$$\Omega_\varphi = \{z \in \mathbb{C}^2;\ 1 + \varphi(z_1) + |z_2|^2 < 0\},$$

where $\varphi \leqslant 0$ *is homogeneous of degree two and of class* C^2 *where it is negative. Then either* φ *is constant and* Ω_φ *is lineally convex; or* φ *is not constant and* Ω_φ *is not connected.*

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A striking contrast between $C^{1,1}$ and C^2.

Proof. For functions $\varphi\colon \mathbb{C} \to \mathbb{R}$ which are homogeneous of degree two, i.e., of the form $\varphi(z_1) = |z_1|^2 g(t)$, $z_1 = |z_1|e^{it} \in \mathbb{C}$, $t \in \mathbb{R}$, the condition on φ takes the form

$$(-r^2 g - 1)\left[-g + \sqrt{(\tfrac{1}{4}g'')^2 + (\tfrac{1}{2}g')^2} - \tfrac{1}{4}g''\right] \leqslant r^2(g^2 + \tfrac{1}{4}g'^2),$$

where $-r^2 g - 1 \geqslant 0$; equivalently

$$4g + (-r^2 g - 1)\left[\sqrt{g''^2 + 4g'^2} - g''\right] \leqslant r^2 g'^2.$$

From this we obtain, if we divide by r^2 and let r tend to $+\infty$,

$$(-g)\left[\sqrt{g''^2 + 4g'^2} - g''\right] \leqslant g'^2.$$

But this condition is also sufficient, which follows on multiplication by r^2 and adding the trivial inequality $4g - \left[\sqrt{g''^2 + 4g'^2} - g''\right] \leqslant 0.$

To get rid of the square root we rewrite the condition as

$$g'^2\left(g'^2+2(-g)g''-4g^2\right)\geqslant 0,$$

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Now if we take $g=-h^2$, where $h\geqslant 0$ and h is of class C^2 where it is positive, we get an inequality of degree eight but which is easy to analyze:

$$h^5h'^2\left(h+h''\right)\leqslant 0.$$

Thus, for each t such that $h(t)>0$, either $h'(t)=0$ or $h(t)+h''(t)\leqslant 0$. If h' is zero everywhere, it is known that Ω_φ is lineally convex, in particular weakly lineally convex. Wherever h is positive and h' is nonzero we get $h+h''\leqslant 0$. This implies that any local maximum of h is isolated and that there can only be one point where the maximum is attained. Hence, unless h' vanishes everywhere, $h+h''\leqslant 0$ everywhere. We define $k=h+h''\leqslant 0$ and obtain for any $a\in\mathbb{R}$

$$h(t)=h(a)\cos(t-a)+h(a)\int_a^t\sin(t-s)k(s)ds,\qquad t\in\mathbb{R}.$$

The function h attains its maximum at some point which we may call a and the formula then shows that $h(t) \leqslant h(a)\cos(t-a)$ for all t with $a \leqslant t \leqslant t+\pi/2$. In particular, h must have a zero t_0 in the interval $]a, a+\pi/2]$. By symmetry, h has a zero t_1 also in the interval $[a-\pi/2, a[$, hence at least two zeros in a period. This means that Ω is not connected, since the union of the rays $\arg z_1 = t_0$ and $\arg z_1 = t_1$ divides the z_1-plane. $\square$

It is of interest to understand in what way the proof of Hörmander's Proposition 4.6.4 quoted above breaks down in the unbounded case. An important step in the proof is to see that, if we have a continuous family $(L_t)_{t\in[0,1]}$ of complex lines, the set T of parameter values t such that $L_t \cap \Omega$ is connected is both open and closed. Thus, if $0 \in T$, then also $1 \in T$. We shall see that closedness is no longer true for the sets in the two examples.

Example

Define complex lines

$$L_t = \{z \in \mathbb{C}^2;\ z_2 = t(z_1 - 1 - i)\}, \qquad t \in [0,1],$$

which all pass through $(1+i, 0) \notin \Omega_{\varphi^\star}$. Then $L_t \cap \Omega_{\varphi^\star}$ is connected for $0 \leqslant t < 1$ while $L_1 \cap \Omega_{\varphi^\star}$ is not.

We shall first see that the real hyperplane given by $y_1 = x_1$ divides $L_1 \cap \Omega_{\varphi^\star}$ into two parts. If $y_1 = x_1$ and $x_1 > 0$, then z does not belong to $\Omega_{\varphi^\star}$. If $y_1 = x_1$, $x_1 < 0$, and $z \in L_t \cap \Omega_{\varphi^\star}$, we shall reach a contradiction. In fact, if $z_2 = z_1 - 1 - i$ and $z \in \Omega_{\varphi^\star}$, then

$$|z_1 - 1 - i|^2 = |x_1 - 1|^2|1 + i|^2 < -\varphi^\star(z_1) - 1 = 2x_1^2 - \rho - 1,$$

which implies that $4x_1 > 3 + \rho = 2\frac{3}{4} > 0$, contradicting the assumption $x_1 < 0$. We also need to know that the two parts are not empty. This is clear, since $(2, 1 - i)$ and $(2i, i - 1)$ both belong to $L_1 \cap \Omega_{\varphi^\star}$, the first with $y_1 < x_1$, the second with $y_1 > x_1$.

Next, we shall see that $L_t \cap \Omega_{\varphi^\star}$ is connected when $0 \leqslant t < 1$. Given t such that $0 \leqslant t < 1$, we fix an r such that the point z with $z_1 = -r - ir$ and $z_2 = t(z_1 - 1 - i) = -t(1+i)(1+r)$ belongs to $L_t \cap \Omega_{\varphi^\star}$. This is possible for a large r. But then also all points $z \in L_t$ with $|z_1| = \sqrt{2}\,r$ and $x_1 \leqslant 0$ or $y_1 \leqslant 0$ belong to $\Omega_{\varphi^\star}$. Thus the circular arc Γ so described lies in $L_t \cap \Omega_{\varphi^\star}$. Now a arbitrary point $a \in L_t \cap \Omega_{\varphi^\star}$ can be joined to a point $a' \in \Gamma$ contained in $L_t \cap \Omega_{\varphi^\star}$ by a straight line segment. We take $a' = \sqrt{2}ra/|a_1|$ if $\operatorname{Re} a_1 \leqslant 0$ or $\operatorname{Im} a_1 \leqslant 0$; $a' = \sqrt{2}\,r$ if $a \in \Omega_{\varphi^\star}$ and $0 < \operatorname{Im} a_1 < \operatorname{Re} a_1$; and $a' = \sqrt{2}\,ir$ if $a \in \Omega_{\varphi^\star}$ and $0 < \operatorname{Re} a_1 < \operatorname{Im} a_1$. $\square$

Another method is to construct a set from its complement. We know that the complement of any union of complex hyperplanes is lineally convex. We define

$$L_{c,\beta} = \{z \in \mathbb{C}^2;\ z_1 - c + \beta z_2 = 0\},$$

a complex line which passes through $(c, 0)$, and Ω as the complement of the union

$$\bigcup_{\substack{c \notin \omega \\ |\beta| \leqslant \psi(c)}} L_{c,\beta}.$$

Here $\psi\colon \mathbb{C} \to \mathbb{R}$ is a function like $\psi(c) = \min(d(c, \omega), 1)$, where $d(c, \omega) = \inf_{z \in \omega}(\|c - z\|_2)$ is the distance from c to ω.

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We now take ω as the set of all z_1 such that $y_1 \geqslant x_1$ and $(z_1, 0)$ belongs to $\Omega_{\varphi^\star}$. The set so constructed is lineally convex. We then take the union of this set and the set where x_1 and y_1 have been permuted. The new set is locally weakly lineally convex but not lineally convex.

A suitable modification of this set has a smooth boundary.

9. A set which is not starshaped

In answer to an earlier question we mention a modification of the set $\Omega_{\varphi^\diamond}$ which is not starhaped.

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Example. (Found on 2014 February 15.) Define $\varphi^\sharp \colon \mathbb{C} \to \mathbb{R}$ by

$$\varphi^\sharp(z_1) = \begin{cases} -x_1^2 - y_1^2, & x_1 + y_1 \leqslant 0; \\ -\frac{1}{2}(x_1 - y_1)^2, & x_1 + y_1 \geqslant 0. \end{cases}$$

Then

$$\Omega_{\varphi^\sharp} = \{z \in \mathbb{C}^2;\ 1 + \varphi^\sharp(z_1) + |z_2|^2 < 0\}$$

has boundary of class $C^{1,1}$ and is lineally convex, but it is not starshaped with respect to any point. $\square$

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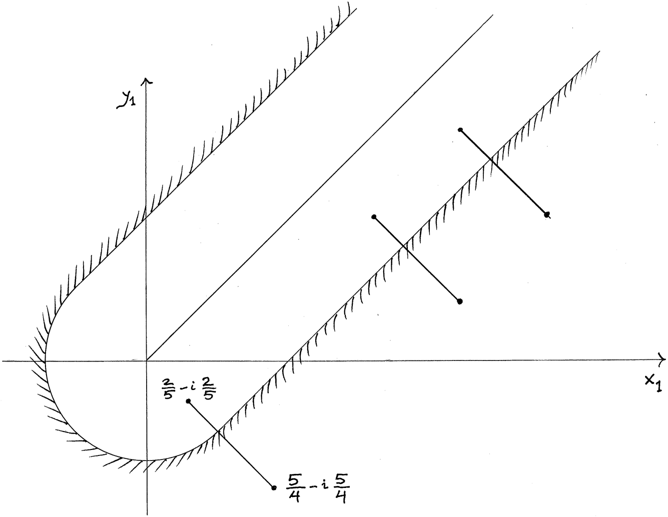
Unbounded! What about bounded?



The set $\Omega_{\varphi^\sharp} \cap \{z \in \mathbb{C}^2;\ z_2 = 0\}$.

10. Cutting off domains is sometimes possible

A convex unbounded domain can always be cut off to yield a bounded convex domain, and the smoothness of the boundary can be preserved. However, it seems to be possible that there is a larger class of unbounded domains which satisfy the Behnke–Peschl condition and allows cutting off so that the Behnke–Peschl condition as well as the smoothness are preserved. …

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ÇOOOK TEŞEKKÜRLER!

(Spelling courtesy of Aişe Özyılmazel)