On the Reflexivity, Hyperreflexivity and Transitivity of Toeplitz operators

by MAREK PTAK

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$\operatorname{Lat}\mathcal{W} = \{\mathcal{L} \subset \mathcal{H} : A\mathcal{L} \subset \mathcal{L} \text{ for all } A \in \mathcal{W}\}$
$\operatorname{Alg}\operatorname{Lat}\mathcal{W} = \{B \in L(\mathcal{H}) : \operatorname{Lat}\mathcal{W} \subset \operatorname{Lat} B\}$

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Subspaces of $L(\mathcal{H})$

$\mathcal{M} \subset L(\mathcal{H})$ – subspace
$\mathrm{ref}\mathcal{M} = \{B \in L(\mathcal{H}) : Bx \in \overline{\mathcal{M}x},\ \text{for all } x \in \mathcal{H}\}$
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Toeplitz operator in finite dimensional space

$$\mathcal{J}_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

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$$\mathcal{W}(\mathcal{J}_n) = \mathcal{A}_n = \left\{ \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_0 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

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$\mathcal{A}_n \subsetneq \operatorname{Alg}\operatorname{Lat}\mathcal{A}_n \quad \Longrightarrow \mathcal{A}_n$ is not reflexive

$$\mathcal{T}_n = \left\{ \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-n} \\ a_1 & a_0 & \cdots & a_{-n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_0 \end{bmatrix}, : a_i \in \mathbb{C} \right\}$$

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ref $\mathcal{T}_n = L(\mathbb{C}^n)$

$\mathcal{T}_n$ is transitive thus not reflexive

Example

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$\operatorname{Lat} \mathcal{W}_2 \ni \mathbb{C}^2 \oplus \{0\}, \{0\} \oplus \{0\} \oplus \mathbb{C}, \mathbb{C} \oplus \{0\} \oplus \{0\},$
$\{(x, 0, x) : x \in \mathbb{C}\}, \; \{(x, x, 0) : x \in \mathbb{C}\}.$

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$\mathcal{M} \subset L(\mathcal{H})$ – w^*-closed subspace
$\mathcal{M}$ is reflexive $\iff F_1(\mathcal{H}) \cap {}_\perp\mathcal{M}$ spans ${}_\perp\mathcal{M}$
$\mathcal{M}$ is transitive $\iff F_1(\mathcal{H}) \cap {}_\perp\mathcal{M} = \{0\}$

Toeplitz Operators on the unit circle

H^2 – Hardy space $\quad P_{H^2}\colon L^2 \longrightarrow H^2$

$S \in L(H^2) \quad (Sf)(z) = z\,f(z) \quad f \in H^2$

$\varphi \in L^\infty \quad T_\varphi f = P_{H^2}(\varphi f) \quad f \in H^2$

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$$\operatorname{span}(S^*, \mathcal{W}(S)) \sim \begin{bmatrix} \alpha_0 & \alpha_{-1} & 0 & 0 & \cdots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & 0 & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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Proof:

$k_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots)$, $\{k_\lambda : \lambda \in \mathbb{D}\}$ dense in H^2
$S^* k_\lambda = \lambda k_\lambda$
$A \in \text{AlgLat } S \Longrightarrow A^* \in \text{Alg Lat } S^* \Longrightarrow \mathbb{C}k_\lambda \in \text{Lat } A^* \Longrightarrow$
$S^* A^* = A^* S \Longrightarrow SA = AS \Longrightarrow A \in \mathcal{W}(S)$

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$(\Leftrightarrow \forall \varphi \colon \mathcal{M} \to \mathbb{C}$ cont.
$\exists x, y \in H \quad \varphi(A) = < A, x \otimes y > = (A\,x, y), \quad A \in \mathcal{M})$

Proposition

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Proof:

$f, g \in H^2, \; f \otimes g \in {}_\perp\mathcal{T}(\mathbb{D})$
$\forall_{\varphi \in L^\infty} \quad 0 = \langle T_\varphi, f \otimes g \rangle = \langle T_\varphi f, g \rangle = \int \varphi f \overline{g} \, d\, m$
$f\overline{g} = 0 \Longrightarrow f = 0 \text{ or } g = 0$

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$\mathcal{T}(\mathbb{D})$ *is 2–reflexive*

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A subspace $\mathcal{M} \subset L(\mathcal{H})$ is called *k-reflexive* if $\mathcal{M}^{(k)} = \{S^{(k)} : S \in \mathcal{M}\}$ is reflexive in $L(\mathcal{H}^{(k)})$, where $S^{(k)} = S \oplus \cdots \oplus S$ and $\mathcal{H}^{(k)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$.

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$X \in \mathcal{T}(\mathbb{D})$ iff $\forall x, y \in H \quad \langle X, x \otimes y - T_z x \otimes T_z y \rangle = 0$
${}_\perp\mathcal{T}(\mathbb{D}) = [{}_\perp\mathcal{T}(\mathbb{D}) \cap F_2]$

Theorem (Azoff, MP)

$\mathcal{B} \subset \mathcal{T}(\mathbb{D})$ *w^*-closed subspace*

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Theorem (Bourgain)

$f \in L^1 \implies f \in H^2\overline{H^2} \iff \log|f| \in L^1$

Proposition (Azoff, MP)

$g \in H^2\overline{H^2}$, $g \neq 0 \Longrightarrow L^1 = H^2\overline{H^2} + \mathbb{C}g$

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Every intransitive hyperplane of $\mathcal{T}(\mathbb{D})$ is reflexive.
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Theorem (Azoff, MP)

$\mathcal{A}(\mathbb{D}) \subset \mathcal{B} \subsetneq \mathcal{T}(\mathbb{D})$
$\mathcal{B}$ *w^*-closed* $\Longrightarrow \mathcal{B}$ *is reflexive*

Example

$$\mathcal{B} = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ \alpha_{-1} & \alpha_0 & \alpha_1 & \cdots \\ 0 & \alpha_{-1} & \alpha_0 & \cdots \\ \vdots & 0 & \alpha_{-1} & \cdots \\ \alpha_{-n} & \vdots & 0 & \ddots \\ \vdots & \alpha_{-n} & \ddots & \ddots \\ \vdots & \vdots & \ddots & \end{bmatrix} \textit{ is reflexive.}$$

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- $T_z^{*n}\mathcal{A}(\mathbb{D}) = \mathrm{span}\{S^{*n}, S^{*n-1}, \ldots, S^*, I, S, S^2, \ldots\}$ *is reflexive.*

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- $T_z^{*n}\mathcal{A}(\mathbb{D}) = \text{span}\{S^{*n}, S^{*n-1}, \dots, S^*, I, S, S^2, \dots\}$ *is reflexive.*
- $\log|f| \notin L^1 \quad T_{\overline{f}}\mathcal{A}(\mathbb{D})$ *is transitive*

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- $\operatorname{span}(S^*, \mathcal{W}(S))$ *is reflexive*
- $\operatorname{span}(S^{*n}, \ldots, S^*, \mathcal{W}(S))$ *is reflexive)*
- $T_z^{*n}\mathcal{A}(\mathbb{D}) = \operatorname{span}\{S^{*n}, S^{*n-1}, \ldots, S^*, I, S, S^2, \ldots\}$ *is reflexive.*
- $\log|f| \notin L^1$ $T_{\overline{f}}\mathcal{A}(\mathbb{D})$ *is transitive*
- *(eg.* $f = \chi_E, E \subset \mathbb{T},\ m(E) > 0,\ m(E) < 1)$ $T_{\chi_E}\mathcal{A}(\mathbb{D})$ *is transitive.*

Toeplitz Operators on the upper-half plane

Definition

$H^p(\mathbb{C}_+)$ *(*$1 \leqslant p < \infty$*)*
the space of analytic functions $F\colon \mathbb{C}_+ \to \mathbb{C}$ *such that*

$$\|F\|_{H^p(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{\mathbb{R}} |F(x+iy)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

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$\gamma\colon \mathbb{C}_+ \to \mathbb{D}$, $\gamma(z) = \frac{z-i}{z+i}$, conformal mapping

$\gamma\colon \mathbb{R} \to \mathbb{T} \setminus \{1\}$, $\gamma(t) = \frac{t-i}{t+i}$, one-to-one correspondence

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Theorem

Let U_∞ and U_1 be as above then
(a) $\langle \varphi, f\rangle = \langle U_\infty \varphi, U_1 f\rangle$, *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$,
(b) $U_1(H^\infty(\mathbb{D})_\perp) = H^\infty(\mathbb{C}_+)_\perp$,
(c) $U_\infty = (U_1^{-1})^*$,
(d) U_∞ *is a weak* homeomorphism.*

Definition

$\Phi \in L^\infty(\mathbb{R})$, $P_{H^2(\mathbb{C}_+)} : L^2(\mathbb{R}) \longrightarrow H^2(\mathbb{C}_+)$

$$T_\Phi F = P_{H^2(\mathbb{C}_+)}(\Phi F), \quad F \in H^2(\mathbb{C}_+),$$

$\Phi \in L^\infty(\mathbb{R})$, T_Φ *Toeplitz operator*
$\Phi \in H^\infty(\mathbb{C}_+)$, T_Φ *analytic Toeplitz operator.*

$\mathcal{T}(\mathbb{C}_+) = \{T_\Phi : \Phi \in L^\infty(\mathbb{R})\}$

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$\xi\colon L^\infty(\mathbb{T}) \to \mathcal{T}(\mathbb{D}) \subset \mathcal{B}(H^2(\mathbb{D}))$, $\xi(\varphi) = T_\varphi$,

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the symbol maps

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$\langle T_\varphi, \xi_*^{-1}(f)\rangle = \langle \varphi, f\rangle$ for $\varphi \in L^\infty(\mathbb{T}),\ f \in L^1(\mathbb{T})$
$\langle T_\Phi, \eta_*^{-1}(F)\rangle = \langle \Phi, F\rangle$ for $\Phi \in L^\infty(\mathbb{R}),\ F \in L^1(\mathbb{R}).$

Theorem

If $\widetilde{U}_2 : \mathcal{B}(H^2(\mathbb{D})) \to \mathcal{B}(H^2(\mathbb{C}_+))$ *is given by* $\widetilde{U}_2(A) = U_2 A U_2^{-1},\ A \in \mathcal{B}(H^2(\mathbb{D}))$*, then*

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(d) *the following diagram commutes*

$$\begin{array}{ccc} L^\infty(\mathbb{T}) & \xrightarrow{\ \xi\ } & \mathcal{T}(\mathbb{D}) \\ {\scriptstyle U_\infty}\downarrow & & \downarrow{\scriptstyle \widetilde{U}_2} \\ L^\infty(\mathbb{R}) & \xrightarrow[\ \eta\ ]{} & \mathcal{T}(\mathbb{C}_+) \end{array}$$

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$\xi\colon L^\infty(\mathbb{T}) \to \mathcal{T}(\mathbb{D}) \subset \mathcal{B}(H^2(\mathbb{D})),\ \xi(\varphi) = T_\varphi$
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$$\begin{array}{ccc}
\mathcal{T}(\mathbb{C}_+)_* & \xrightarrow{\ \eta_*\ } & L^1(\mathbb{R}) \\
{\scriptstyle \widetilde{U}_{2*}}\big\downarrow & & \big\downarrow{\scriptstyle U_1^{-1}} \\
\mathcal{T}(\mathbb{D})_* & \xrightarrow[\ \xi_*\ ]{} & L^1(\mathbb{T})
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(c) $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ *is reflexive (respectively transitive) if and only if* $\widetilde{U}(\mathcal{S})$ *is reflexive (respectively transitive).*

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Theorem (Młocek, MP'2012)

Suppose that $\mathcal{F} \subset \mathcal{T}(\mathbb{C}_+)$ is a weak* closed subspace. Then the following statements are equivalent

(1) *$\mathcal{F}$ is not transitive,*

(2) *$\mathcal{F}$ is reflexive,*

(3) *there is a function $F\colon \mathbb{R} \to \mathbb{C}$ such that $F \in L^1(\mathbb{R})$, $\log|F| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ and $\int_{\mathbb{R}} \Phi F dt = 0$ for all $T_\Phi \in \mathcal{F}$.*

Example

If $G \in L^\infty(\mathbb{R})$ *and* $\int_{\mathbb{R}} \Big| \log |G(t)| \Big| \frac{dt}{1+t^2} = \infty$, *then* $T_G\mathcal{A}(\mathbb{C}_+)$ *is transitive.*

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Taking a appropriate function G *we get in particular;*

(a) *if* $G(t) = \exp(-|t|)$ *or* $G(t) = \exp(-t^2/2)$, *then* $T_G\mathcal{A}(\mathbb{C}_+)$ *is transitive,*

(b) *if* G *is the characteristic function of* $E \subset \mathbb{R}$ *with* E *having finite non–zero Lebesgue measure, then* $T_G\mathcal{A}(\mathbb{C}_+)$ *is transitive.*

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Taking a appropriate function G *we get in particular;*

(a) *the subspace* $T_{e^{i\lambda t}} \mathcal{A}(\mathbb{C}_+)$ *is reflexive for any* $\lambda > 0$*,*
(b) *if* $\overline{G}$ *is an inner function on* $\mathbb{C}_+$ *(i.e.* $\overline{G} \in H^\infty(\mathbb{C}_+)$ *and* $|\overline{G}(t)| = 1$ *a.e.), then* $T_G \mathcal{A}(\mathbb{C}_+)$ *is reflexive,*
(c) *if* $G(t) = \frac{1}{1+t^2}$*, then* $T_G \mathcal{A}(\mathbb{C}_+)$ *is reflexive.*

Example

Let $G \in L^1(\mathbb{R})$ *and* $\mathcal{B}_G := \{T_\Phi \in \mathcal{T}(\mathbb{C}_+) : \int_{\mathbb{R}} G\Phi \, dt = 0\}$. *Let* $F \in L^1(\mathbb{R})$ *then* $\int_{\mathbb{R}} F\Phi \, dt = 0$ *for all* Φ *such that* $T_\Phi \in \mathcal{B}_G$ *iff* $F \in \mathrm{span}\{G\}$. *Hence the following holds:*

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(c) *if* $G(t) = \frac{1}{1+t^2}$ *(or more generally* $G(t) = (1+t^2)^\alpha, \alpha < -\frac{1}{2}$*), then* $\mathcal{B}_G$ *is reflexive.*

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Arveson

$\mathcal{W}$ is called hyperreflexive iff there is k such that

$$\operatorname{dist}(A, \mathcal{W}) \leqslant k\, \alpha(A, \mathcal{W})$$

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Proof:

$A \in \mathrm{ref}\mathcal{M} \Longrightarrow \alpha(A,\mathcal{M}) = 0 \Longrightarrow \mathrm{dist}(A,\mathcal{M}) = 0 \Longrightarrow A \in \mathcal{M}$

Hyperreflexivity of analytic Toeplitz operators

Theorem (Davidson)

$\mathcal{W}(S)$ is hyperrreflexive, $\kappa(\mathcal{W}(S)) < 19$
(Kliś, MP) $\kappa(\mathcal{W}(S)) < 13$

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<u>Tool: Nehari Theorem</u>
$\|H_\varphi\| = \mathrm{dist}(T_\varphi, \mathcal{A}) = d(f, H^\infty)$

H_φ – Hankel operator

$$\alpha_k(A,\mathcal{M}) = \sup\{|tr(Af)| : f \in F_k(\mathcal{H}) \cap \mathrm{ball}_{\perp}\mathcal{M}\}$$

A subspace $\mathcal{M}$ is *k–hyperreflexive*
if there is a such that for any $A \in L(\mathcal{H})$:

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Theorem (Kliś)

$\mathcal{M}^{(k)} = \{S^{(k)} : S \in \mathcal{M}\} \quad S^{(k)} = S \oplus \cdots \oplus S$
$\mathcal{M}^{(k)}$ *hyperreflexive* $\Longrightarrow$ $\mathcal{M}$ *k–hyperreflexive*
$\not\Longleftarrow$

$\mathcal{T}(\mathbb{T})$ is not hyperreflexive

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Theorem (Kliś, MP)

$\mathcal{T}(\mathbb{T})$ is 2–hyperreflexive and $\kappa_2(\mathcal{T}(\mathbb{T})) \leqslant 2$.

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Theorem (Arveson)

There is a positive linear projection $\pi\colon L(H^2) \to \mathcal{T}(\mathbb{T})$ such that

1. $\pi(I) = I$, $\|\pi\| = 1$,
2. $\pi(T) = T$ *for* $T \in \mathcal{T}(\mathbb{T})$,
3. $\pi(A)$ *belongs to the weakly–closed convex hull of* $\{T_z^{k*} A T_z^k : k \in \mathbb{N}\}$ *for* $A \in L(H^2)$,

Proof:

$A \in L(H^2(\mathbb{T}))$
$\pi(A) \in$ weakly–closed convex hull $\{T^*_{z^k} A T_{z^k} : k \in \mathbb{N}\}$

$$d(A, \mathcal{T}(\mathbb{T})) \leqslant \|A - \pi(A)\| \leqslant \sup_{k\in\mathbb{N}} \|A - T^*_{z^k} A T_{z^k}\|$$
$$\leqslant \sup_{k\in\mathbb{N}} \sup\{|((A - T^*_{z^k} A T_{z^k})x, y)| : x, y \in H^2(\mathbb{T}), \|x \otimes y\| = 1\}$$
$$\leqslant \sup_{k\in\mathbb{N}} \sup\{|(Ax, y) - (A\, z^k x, z^k y)| : x, y \in H^2(\mathbb{T}), \|x \otimes y\| = 1\}$$
$$\leqslant \sup_{k\in\mathbb{N}} \sup\{|tr(A(x \otimes y - z^k x \otimes z^k y))| : x, y \in H^2(\mathbb{T}), \|x \otimes y\| = 1\}$$

$\mathrm{rank}(x \otimes y - z^k x \otimes z^k y) \leqslant 2$,
$\|x \otimes y - z^k x \otimes z^k y\| \leqslant 2$ if $\|x \otimes y\| = 1$,
$d(A, \mathcal{T}(\mathbb{T})) \leqslant 2\, \alpha_2(A, \mathcal{T}(\mathbb{T}))$.

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Example

$T_z^{*n}\mathcal{A}(\mathbb{T}) = \operatorname{span}\{S^{*n}, S^{*n-1}, \ldots, S^*, I, S, S^2, \ldots\}$ *is hyperreflexive.*

Toeplitz Operators in Bergman space

$L_a^2 \subset L^2(\mathbb{D})$ – Bergman space
$P_{L_a^2} : L^2(\mathbb{D}) \rightarrow L_a^2$

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Theorem (Bercovici)

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Theorem (Conway, MP)

$\mathcal{T}(B) = \overline{\{p(B) + q(B)^* : p, q \text{ polynomials}\}}^*$ *is hyperreflexive with constant at most 3*

$\mathcal{B} \subset \mathcal{T}(B)$ w^**-closed subspace* $\Longrightarrow$ *hyperreflexive.*