Coherent state transforms attached to generalized Bargmann spaces on the complex plane

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Abstract

We construct a family of coherent states transforms attached to generalized Bargmann spaces \[ C.R. Acad.Sci.Paris, t.325,1997 \] in the complex plane. This constitutes another way of obtaining the kernel of an isometric operator linking the space of square integrable functions on the real line with the true-poly-Fock spaces \[ Oper.Theory. Adv.Appl.,v.117,2000 \].

1 Introduction

The Bargmann transform, was originally introduced in 1961 by V. Bargmann [1] and was closely connected to the Heisenberg group. It has found many applications in quantum optics. Another interest on this transform lies in that it is a windowed Fourier transform [2] and as such it plays an important role in signal processing and harmonic analysis on phase space [3].

This transform can be defined as

\[ B[f](z) := \pi^{-\frac{1}{4}} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2} \xi^2 + \sqrt{2} \xi \mathfrak{z} - \frac{1}{2} \mathfrak{z}^2} d\xi, z \in \mathbb{C}. \]  

(1.1)

It maps isometrically the space \( L^2(\mathbb{R}, d\xi) \) of square integrable functions \( f \) on the real line onto the Fock space \( \mathcal{F}(\mathbb{C}) \) of entire complex-valued functions which are \( e^{-|z|^2} d\lambda \) -square integrable, \( d\lambda \) denotes the ordinary planar Lebesgue measure.

Note also that the Fock space \( \mathcal{F}(\mathbb{C}) \) coincides with the null space

\[ \mathcal{A}_0(\mathbb{C}) := \left\{ \varphi \in L^2(\mathbb{C}, e^{-|z|^2} d\lambda), \widetilde{\Delta} \varphi = 0 \right\} \]  

(1.2)

of the second order differential operator [4]:

\[ \widetilde{\Delta} := -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z} \frac{\partial}{\partial z}. \]  

(1.3)

The latter constitutes (in suitable units and up to additive constant) a realization in \( L^2(\mathbb{C}, e^{-|z|^2} d\lambda) \) of the Schrödinger operator describing the motion of a charged particle evolving in the complex plane \( \mathbb{C} \) under influence of a normal uniform magnetic field. Its spectrum consists of eigenvalues of infinite multiplicity (Landau levels) of the form :

\[ \epsilon_m = m, m = 0, 1, 2, ... \]
The corresponding eigenspaces

\[ A_m(\mathbb{C}) := \{ \varphi \in L^2(\mathbb{C}, e^{-|z|^2} d\lambda), \bar{\Delta} \varphi = \epsilon_m \varphi \} \]  

are pairwise orthogonal in the Hilbert space \( L^2(\mathbb{C}, e^{-|z|^2} d\lambda) \) which decomposes as

\[ L^2(\mathbb{C}, e^{-|z|^2} d\lambda) = \bigoplus_{m \geq 0} A_m(\mathbb{C}). \]

In this Note, the main objective is to construct for each Hilbert space \( A_m(\mathbb{C}) \), \( m = 0, 1, 2, \ldots \) a unitary transformation, \( B_m : L^2(\mathbb{R}) \rightarrow A_m(\mathbb{C}) \) in such a way that for the first Hilbert space \( A_0(\mathbb{C}) \), which is the Fock space, the constructed transform \( B_0 \) coincides with the classical Bargmann transform \( B \). This will be achieved by adopting a coherent states analysis. Precisely, the constructed transforms are of the form

\[ B_m[f](z) = (-1)^m (2^m m! \sqrt{\pi})^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2} \xi^2 + \sqrt{2} \xi z - \frac{1}{2} z^2} H_m\left(\xi - \frac{z + \bar{z}}{2}\right) d\xi, \]

where \( H_m(\xi) = (-1)^m e^{\xi^2} \left(\frac{d}{d\xi}\right)^m e^{-\xi^2} \) is the \( m \)th Hermite polynomial.

We should note that the expression of the transforms \( B_m \) coincides with the expression of a family of isometric operators linking the space \( L^2(\mathbb{R}) \) with the true-poly-Fock spaces introduced by N. L. Vasilevski [5]. Thereby, the present work constitutes another way to arrive at the result of theorem 2.5 in [5], by using a coherent states method exploiting tools of the \( L^2 \)-spectral theory of the Schrödinger operator given in (1.3).

In the next section, we review briefly the coherent states formalism we will be using. Section 3 deals with some needed facts on the \( L^2 \)-spectral theory of the Schrödinger operator \( \bar{\Delta} \). In section 4 we define a family of coherent state transforms attached to the generalized Bargmann spaces \( A_m(\mathbb{C}) \).

## 2 Coherent states formalism

Here, we follow the generalization of the canonical coherent states according to the procedure in [6].

Let \((X, \mu)\) be a measure space and let \( \mathcal{H} \subset L^2(X, \mu) \) be a closed subspace of infinite dimension. Let \( \{ \Phi_n \}_{n=0}^\infty \) be an orthogonal basis of \( \mathcal{H} \) satisfying, for arbitrary \( x \in X \),

\[ \omega(x) := \sum_{n=0}^\infty \rho_n^{-1} |\Phi_n(x)|^2 < +\infty, \]

where \( \rho_n := ||\Phi_n||_{L^2(X)}^2 \). Define

\[ \mathcal{R}(x, y) := \sum_{n=0}^\infty \rho_n^{-1} \Phi_n(x) \overline{\Phi_n(y)}, \quad x, y \in X. \]

Then, \( \mathcal{R}(x, y) \) is a reproducing kernel, \( \mathcal{H} \) is the corresponding reproducing kernel Hilbert space and \( \omega(x) := \mathcal{R}(x, x), \ x \in X. \)
Definition 2.1. Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} = \infty$ and $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. The coherent states labeled by points $x \in X$ are defined as the ket-vectors $\vartheta_x \equiv |x\rangle \in \mathcal{H}$:

$$\vartheta_x \equiv |x\rangle : (\omega(x))^{\frac{1}{2}} \sum_{n=0}^{\infty} \Phi_n(x) \frac{\phi_n}{\sqrt{p_n}}.$$

(2.1)

By definition, it is straightforward to show that $\langle \vartheta_x, \vartheta_x \rangle_\mathcal{H} = 1$.

Definition 2.2. The coherent state transform associated to the set of coherent states $(\vartheta_x)_{x \in X}$ is the isometric map

$$W : \mathcal{H} \rightarrow \mathcal{H}^2 \subset L^2(X, \mu)$$

(2.2)

defined for every $x \in X$ by

$$W[\phi](x) := (\omega(x))^{\frac{1}{2}} < \phi, \vartheta_x >_\mathcal{H}.$$

Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$< \phi, \psi >_\mathcal{H} = < W[\phi], W[\psi] >_{L^2(X)} = \int_X d\mu(x) \omega(x) < \phi, \vartheta_x > < \vartheta_x, \psi >.$$

Thereby, we have a resolution of the identity of $\mathcal{H}$ which can be expressed in Dirac’s bra-ket notation as:

$$1_\mathcal{H} = \int_X d\mu(x) \omega(x) |x\rangle < x |,$$

and where $\omega(x)$ appears as a weight function.

Remark 2.1. Note that formula (2.1) can be considered as a generalization of the series expansion of the canonical coherent states

$$\vartheta_\zeta \equiv |\zeta\rangle := e^{-\frac{1}{2} |\zeta|^2} \sum_{k=0}^{+\infty} \frac{\zeta^k}{\sqrt{k!}} \phi_k, \zeta \in \mathbb{C}$$

with $\{\phi_k\}_{k=0}^{+\infty}$ being an orthonormal basis of eigenstates of the quantum harmonic oscillator. Here, the space $\mathcal{H}^2$ is the Fock space $\mathfrak{F}(\mathbb{C})$ and $\omega(\zeta) = \pi^{-1} e^{|\zeta|^2}, \zeta \in \mathbb{C}$.

3 The generalized Fock spaces $\mathcal{A}_m(\mathbb{C})$

As the Fock space $\mathfrak{F}(\mathbb{C})$ has $K_0(z, w) := \pi^{-1} e^{z \bar{w}}$ as reproducing kernel, we have shown [4] that the Hilbert spaces $\mathcal{A}_m(\mathbb{C})$ also have explicit reproducing kernel of the form

$$K_m(z, w) := \pi^{-1} e^{(z, w)} L_m^{(0)} (\overline{|z - w|^2}), z, w \in \mathbb{C},$$

(3.1)

where $L_m^{(\alpha)} (t)$ is the Laguerre polynomial defined by the Rodriguez formula as

$$L_m^{(\alpha)} (t) = \frac{1}{m!} t^{-\alpha} e^t \left( \frac{d}{dt} \right)^m (e^{\alpha + m} e^{-t}), t \in \mathbb{R}$$

In particular, if we set $\omega_m(z) := K_m(z, z)$, then $\omega_m(z) = \pi^{-1} e^{|z|^2}, z \in \mathbb{C}$.
The spaces $A_m(C)$ have been also used to study the spectral properties of the Cauchy transform on $L^2(C,e^{-|z|^2}d\lambda)$; see [7] where the authors exhibited for each fixed $m = 0,1,2,...$ an orthogonal basis denoted $\{h_{m,p}\}_{p=0}^{+\infty}$ and defined by

$$h_{m,p}(z) := \gamma_{m,p} \Gamma_1 \left( -\min(m,p), |m-p|+1, |z|^2 \right) |z|^{m-p} e^{-i(m-p)\arg z}$$  \hspace{1cm} (3.2)

where

$$\gamma_{m,p} := \frac{(-1)^{\min(m,p)} (\max(m,p))!}{(|m-p)|!},$$

and $\Gamma_1$ is the confluent hypergeometric function given by [8]:

$$\Gamma_1(a,b;u) = \frac{\Gamma (b)}{\Gamma (a)} \sum_{j=0}^{+\infty} \frac{\Gamma (a+j) \ u^j}{\Gamma (b+j) \ j!}, \ |u| < +\infty, b \neq 0, -1, -2, ... .$$

Here $\Gamma (a)$ is the Euler’s Gamma function such that $\Gamma (j+1) = j!$ if $j = 0,1,2,...$.

Note that for $a = -n$ with $n$ being a positive integer, the hypergeometric function $\Gamma_1$ becomes a polynomial and can be expressible in term of Laguerre polynomial according to [8]:

$$\Gamma_1 (-n, \alpha+1; u) = \frac{n! \Gamma (\alpha+1)}{\Gamma (n+\alpha+1)} L_n^{(\alpha)} (u).$$

For our purpose we shall consider the orthogonal basis of $A_m(C)$ in the following form

$$h_{m,p}(z) = (-1)^{\min(m,p)} (\min(m,p))! |z|^{m-p} e^{-i(m-p)\arg z} \Gamma_1^{(\min(m,p))} \left( |z|^2 \right), z \in C,$$  \hspace{1cm} (3.3)

with the square norm in $L^2(C,e^{-|z|^2}d\lambda)$ given by

$$\rho_{m,p} := \|h_{m,p}\|^2 = \pi m! .$$

**Remark 3.1.** In [7] p. 404] the elements of the orthogonal basis given in (3.2) have been also expressed as

$$h_{m,p}(z) = \sum_{j=0}^{\min(m,p)} (-1)^j \frac{m!}{j! (m-j)! (p-j)!} z^{m-j} z^{p-j}.$$  \hspace{1cm} (3.4)

We should note these complex polynomials in (3.4) were considered also by Itô [9] in the context of complex Markov process.

4 Coherent states attached to $A_m(C)$

In this section, we shall attach to each space $A_m(C)$ a set coherent states via series expansion according to the procedure presented in section 2. We will also give expressions of these coherent states in a closed form by using direct calculations.

**Definition 4.1.** For $m = 0,1,2,...$, the coherent states associated with the space $A_m(C)$ and labelled by points $z \in C$ are defined formally according to formula (2.1) as

$$\theta_{z,m} \equiv |z,m> := (\omega_m(z))^{-\frac{1}{2}} \sum_{p=0}^{+\infty} \frac{h_{m,p}(z)}{\sqrt{\rho_{m,p}}} \psi_p$$
where \( \psi_p \) are elements of a total orthonormal system of \( L^2(\mathbb{R}, \, d\xi) \) given

\[
\psi_p(\xi) := (\sqrt{\pi}2^p p!)^{-\frac{1}{2}} e^{-\frac{1}{4}\xi^2} H_p(\xi), \quad p = 0, 1, 2, \ldots, \quad \xi \in \mathbb{R},
\]

and \( H_p(\xi) \) is the \( p \)th Hermite polynomial.

**Proposition 4.1.** The wave functions of these coherent states are expressed as

\[
\vartheta_{z,m}(\xi) = (-1)^m (2^m m! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{1}{4}\xi^2 + \sqrt{\pi} \xi - \frac{1}{2} \xi^2} H_m \left( \xi - \frac{z + \xi}{2} \right), \quad \xi \in \mathbb{R}.
\]

**Proof.** According to Definition 4.1, we start by writing

\[
\vartheta_{z,m}(\xi) = \left( \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}|z|^2} \right)^{-\frac{1}{2}} \sum_{p=0}^{+\infty} \frac{h_{m,p}(z)}{\sqrt{\pi m!} p!} \psi_p(\xi).
\]

Recalling the expression of \( h_{m,p}(z) \) in (3.3), then these wave functions can be rewritten as

\[
\vartheta_{z,m}(\xi) = \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}} \sum_{p=0}^{+\infty} \frac{(1)^{\min(m,p)}}{\sqrt{p!}} (\min(m, p))! |z|^{m-p} e^{-i(m-p) \arg z} L^{(m-p)}_{\min(m, p)} \left( |z|^2 \right) \psi_p(\xi).
\]

The integer \( m \) being fixed, we denote by \( S_m(z, \xi) \) the following series:

\[
S_m(z, \xi) := \sum_{p=0}^{+\infty} \frac{(1)^{\min(m,p)}}{\sqrt{p!}} (\min(m, p))! |z|^{m-p} e^{-i(m-p) \arg z} L^{(m-p)}_{\min(m, p)} \left( |z|^2 \right) \psi_p(\xi)
\]

and we split it into two part as

\[
S_m(z, \xi) = \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^p p! |z|^{m-p} e^{-i(m-p) \arg z} L^{(m-p)}_{p} \left( |z|^2 \right) \psi_p(\xi)
\]

\[
+ \sum_{p=m}^{+\infty} \frac{1}{\sqrt{p!}} (-1)^m m! |z|^{p-m} e^{-i(m-p) \arg z} L^{(p-m)}_{m} \left( |z|^2 \right) \psi_p(\xi)
\]

This can also be written as

\[
S_m(z, \xi) = S_{(<\infty)}(m, z, \xi) + S_{(\infty)}(m, z, \xi)
\]

with

\[
S_{(<\infty)}(m, z, \xi) = \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^p p! |z|^{m-p} e^{-i(m-p) \arg z} L^{(m-p)}_{p} \left( |z|^2 \right) \psi_p(\xi)
\]

\[
- \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^m m! |z|^{p-m} e^{-i(m-p) \arg z} L^{(p-m)}_{m} \left( |z|^2 \right) \psi_p(\xi)
\]

and

\[
S_{(\infty)}(m, z, \xi) = \sum_{p=0}^{+\infty} \frac{1}{\sqrt{p!}} (-1)^m m! |z|^{p-m} e^{-i(m-p) \arg z} L^{(p-m)}_{m} \left( |z|^2 \right) \psi_p(\xi).
\]
The finite sum \( S_{(<\infty)} (m, z, \xi) \) reads

\[
S_{(<\infty)} (m, z, \xi) = \sum_{p=0}^{m-1} \left( (-1)^p \sqrt{p!} z^n L_p^{(m-p)} \left( |z|^2 \right) - (-1)^m \frac{m!}{\sqrt{p!}} z^{p-m} L_m^{(p-m)} \left( |z|^2 \right) \right) \psi_p (\xi)
\]

Making use of the identity \([10] \text{ p. 98}:\)

\[
L_m^{(-k)} (t) = (-t)^k \frac{(m-k)!}{m!} L_m^{(k)} (t), \quad 1 \leq k \leq m
\]

for \( k = p - m \), we write the Laguerre polynomial with upper indice \( p - m < 0 \) as

\[
L_m^{(p-m)} \left( |z|^2 \right) = \left( - |z|^2 \right)^{m-p} \frac{p!}{m!} L_m^{(p-m)} \left( |z|^2 \right),
\]

and we obtain after calculation that \( S_{(<\infty)} (m, z, \xi) = 0. \)

Now, for the infinite sum \( S_{(\infty)} (m, z, \xi) \), we make use of the explicit expression of the Gaussian-Hermite functions

\[
\psi_p (\xi) = \left( \sqrt{\pi} 2^p p! \right)^{-\frac{1}{2}} e^{-\frac{1}{4} z^2} H_p (\xi), \quad p = 0, 1, 2, ...
\]

and we obtain that

\[
S_{(\infty)} (m, z, \xi) = \sum_{p=0}^{\infty} \frac{1}{\sqrt{p!}} (-1)^m m! |z|^{p-m} e^{-i(m-p) \arg z} L_m^{(p-m)} \left( |z|^2 \right) \frac{e^{-\frac{1}{4} z^2} H_p (\xi)}{\left( \sqrt{\pi} 2^p p! \right)^{\frac{1}{2}}}
\]

where

\[
\xi_{(\infty)} (m, z, \xi) := \sum_{p=0}^{\infty} \frac{(2^p)^{-\frac{1}{2}}}{p!} z^{p-m} L_m^{(p-m)} \left( |z|^2 \right) H_p (\xi)
\]

Next, we make use of following addition formula involving Laguerre and Hermite polynomials \([11]:\)

\[
\sum_{j=-n}^{\infty} \frac{2^{-j} \beta^j}{(j+n)!} (a + ib)^j L_n^{(j)} \left( \frac{\beta}{2} (a^2 + b^2) \right) H_{j+n} (\xi) = \frac{1}{m!} \exp \left( -\frac{\beta}{4} (a - ib)^2 + \sqrt{\beta} \xi (a - ib) \right) H_n \left( \xi - \sqrt{\beta} a \right)
\]

for \( n = m, j = p - n, \beta = 2 \) and \( z = a + ib \in \mathbb{C} \). This gives that

\[
\xi_{(\infty)} (m, z, \xi) = \frac{2^{-m \frac{z}{m}}}{m!} e^{-\frac{1}{2} z^2 + \sqrt{2} \xi z} H_m \left( \xi - \frac{z + \bar{z}}{2} \right)
\]

Summarizing up the above calculations, we can write successively

\[
\theta_{z,m} (\xi) = \frac{e^{-\frac{1}{2} z^2}}{\sqrt{m!}} \left( \sqrt{\pi} \right)^{-\frac{1}{2}} (-1)^m m! e^{-\frac{1}{2} z^2} \xi_{(\infty)} (m, z, \xi)
\]

\[
= \frac{e^{-\frac{1}{2} z^2}}{\sqrt{m!}} \left( \sqrt{\pi} \right)^{-\frac{1}{2}} (-1)^m m! e^{-\frac{1}{2} z^2} \left( \frac{2^{-\frac{m}{2}}}{m!} e^{-\frac{1}{2} z^2 + \sqrt{2} \xi z} H_m \left( \xi - \frac{z + \bar{z}}{2} \right) \right)
\]

\[
= (-1)^m \left( 2^m m! \sqrt{\pi} \right)^{-\frac{1}{2}} e^{-\frac{1}{2} z^2 + \sqrt{2} \xi z - \frac{1}{2} |z|^2} H_m \left( \xi - \frac{z + \bar{z}}{2} \right).
\]
The proof of Proposition 4.1 is finished. ■

Finally, according to Definition 2.2, the coherent state transform associated with the coherent states $\vartheta_{z,m}$ is the unitary map:

$$B_m : L^2(\mathbb{R}, d\xi) \to A_m(\mathbb{C})$$

defined by

$$B_m[f](z) := (\omega_m(z))^{1/2} \langle f, \vartheta_{z,m} \rangle_{L^2(\mathbb{R})}, f \in L^2(\mathbb{R}, d\xi), z \in \mathbb{C}$$

Explicitly,

$$B_m[f](z) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2} \xi^2 + \sqrt{2} \xi z - \frac{1}{4} z^2} H_m \left( \xi - \frac{z + \overline{z}}{2} \right) d\xi$$

which can be called the extended Bargmann transform of index $m = 0, 1, 2, \ldots$.

References


