Coherent state transforms attached to generalized Bargmann spaces on the complex plane

Zouhaïr MOUAYN

Department of Mathematics, Faculty of Sciences and Technics (M'Ghila), Sultan Moulay Slimane University, BP 523, Béni Mellal, Morocco
(e-mail: mouayn@gmail.com)

Abstract

We construct a family of coherent states transforms attached to generalized Bargmann spaces [*C.R. Acad.Sci.Paris, t.325,1997*] in the complex plane. This constitutes another way of obtaining the kernel of an isometric operator linking the space of square integrable functions on the real line with the *true-poly*-Fock spaces [*Oper.Theory. Adv.Appl.,v.117,2000*].

1 Introduction

The Bargmann transform, was originally introduced in 1961 by V. Bargmann [1] and was closely connected to the Heisenberg group. It has found many applications in quantum optics. Another interest on this transform lies in that it is a windowed Fourier transform [2] and as such it plays an important role in signal processing and harmonic analysis on phase space [3].

This transform can be defined as

$$\mathcal{B}[f](z) := \pi^{-\frac{1}{4}} \int_{\mathbb{R}} f(\xi)\, e^{-\frac{1}{2}\xi^2+\sqrt{2}\xi z-\frac{1}{2}z^2} d\xi, z \in \mathbb{C}. \tag{1.1}$$

It maps isometrically the space $L^2(\mathbb{R}, d\xi)$ of square integrable functions f on the real line onto the Fock space $\mathfrak{F}(\mathbb{C})$ of entire complex-valued functions which are $e^{-|z|^2}d\lambda-$square integrable, $d\lambda$ denotes the ordinary planar Lebesgue measure.

Note also that the Fock space $\mathfrak{F}(\mathbb{C})$ coincides with the null space

$$\mathcal{A}_0(\mathbb{C}) := \left\{ \varphi \in L^2(\mathbb{C}, e^{-|z|^2}d\lambda), \widetilde{\Delta}\varphi = 0 \right\} \tag{1.2}$$

of the second order differential operator [4]:

$$\widetilde{\Delta} := -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}}. \tag{1.3}$$

The latter constitutes (in suitable units and up to additive constant) a realization in $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$ of the Schrödinger operator describing the motion of a charged particle evolving in the complex plane $\mathbb{C}$ under influence of a normal uniform magnetic field. Its spectrum consists of eigenvalues of infinite multiplicity (*Landau levels*) of the form :

$$\epsilon_m = m,\ m = 0, 1, 2, ...$$

The corresponding eigenspaces

$$\mathcal{A}_m(\mathbb{C}) := \left\{ \varphi \in L^2(\mathbb{C}, e^{-|z|^2} d\lambda), \widetilde{\Delta}\varphi = \epsilon_m \varphi \right\} \tag{1.4}$$

are pairwise orthogonal in the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} d\lambda)$ which decomposes as

$$L^2(\mathbb{C}, e^{-|z|^2} d\lambda) = \bigoplus_{m \geq 0} \mathcal{A}_m(\mathbb{C}).$$

In this Note, the main objective is to construct for each Hilbert space $\mathcal{A}_m(\mathbb{C})$, $m = 0, 1, 2, ...$ a unitary transformation, $\mathcal{B}_m : L^2(\mathbb{R}) \to \mathcal{A}_m(\mathbb{C})$ in a such a way that for the first Hilbert space $\mathcal{A}_0(\mathbb{C})$, which is the Fock space, the constructed transform $\mathcal{B}_0$ coincides with the classical Bargmann transform $\mathcal{B}$. This will be achieved by adopting a coherent states analysis. Precisely, the constructed transforms are of the form

$$\mathcal{B}_m[f](z) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2}\xi^2 + \sqrt{2}\xi z - \frac{1}{2}z^2} H_m\left(\xi - \frac{z + \bar{z}}{2}\right) d\xi,$$

where $H_m(\xi) = (-1)^m e^{\xi^2} \left(\frac{d}{d\xi}\right)^m e^{-\xi^2}$is the mth Hermite polynomial.

We should note that the expression of the transforms $\mathcal{B}_m$ coincides with the expression of a family of isometric operators linking the space $L^2(\mathbb{R})$ with the *true-poly-Fock* spaces introduced by N. L. Vasilevski [5] . Thereby, the present work constitutes another way to arrive at the result of theorem 2.5 in [5], by using a coherent states method exploiting tools of the L^2-spectral theory of the Schrödinger operator given in (1.3) .

In the next section, We review briefly the coherent states formalism we will be using. Section 3 deals with some needed facts on the L^2-spectral theory of the Schrödinger operator $\widetilde{\Delta}$. In section 4 we define a family of coherent state transforms attached to the generalized Bargmann spaces $\mathcal{A}_m(\mathbb{C})$.

2 Coherent states formalism

Here, we follow the generalization of the canonical coherent states according to the procedure in [6].

Let (X, μ) be a measure space and let $\mathfrak{H}^2 \subset L^2(X, \mu)$ be a closed subspace of infinite dimension. Let $\{\Phi_n\}_{n=0}^{\infty}$ be an orthogonal basis of $\mathfrak{H}^2$ satisfying, for arbitrary $x \in X$,

$$\omega(x) := \sum_{n=0}^{\infty} \rho_n^{-1} |\Phi_n(x)|^2 < +\infty,$$

where $\rho_n := \|\Phi_n\|^2_{L^2(X)}$. Define

$$\mathfrak{K}(x, y) := \sum_{n=0}^{\infty} \rho_n^{-1} \Phi_n(x) \overline{\Phi_n(y)}, \; x, y \in X.$$

Then, $\mathfrak{K}(x, y)$ is a reproducing kernel, $\mathfrak{H}^2$ is the corresponding reproducing kernel Hilbert space and $\omega(x) := \mathfrak{K}(x, x)$, $x \in X$.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space with* $\dim \mathcal{H} = \infty$ *and* $\{\phi_n\}_{n=0}^{\infty}$ *be an orthonormal basis of $\mathcal{H}$. The coherent states labeled by points $x \in X$ are defined as the ket-vectors $\vartheta_x \equiv \mid x > \in \mathcal{H}$:*

$$\vartheta_x \equiv \mid x >:= (\omega(x))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(x)}{\sqrt{\rho_n}} \phi_n. \tag{2.1}$$

By definition, it is straightforward to show that $< \vartheta_x, \vartheta_x >_{\mathcal{H}} = 1$.

Definition 2.2. *The coherent state transform associated to the set of coherent states $(\vartheta_x)_{x \in X}$ is the isometric map*

$$W : \mathcal{H} \to \mathfrak{H}^2 \subset L^2(X, \mu) \tag{2.2}$$

defined for every $x \in X$ by

$$W[\phi](x) := (\omega(x))^{\frac{1}{2}} < \phi, \vartheta_x >_{\mathcal{H}} .$$

Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$< \phi, \psi >_{\mathcal{H}} = < W[\phi], W[\psi] >_{L^2(X)} = \int_X d\mu(x)\, \omega(x) < \phi, \vartheta_x >< \vartheta_x, \psi > .$$

Thereby, we have a resolution of the identity of $\mathcal{H}$ which can be expressed in Dirac's bra-ket notation as:

$$\mathbf{1}_{\mathcal{H}} = \int_X d\mu(x)\, \omega(x) \mid x >< x \mid,$$

and where $\omega(x)$ appears as a weight function.

Remark 2.1. Note that formula (2.1) can be considered as a generalization of the series expansion of the canonical coherent states

$$\vartheta_\zeta \equiv \mid \zeta >:= e^{-\frac{1}{2}|\zeta|^2} \sum_{k=0}^{+\infty} \frac{\zeta^k}{\sqrt{k!}} \phi_k, \zeta \in \mathbb{C}$$

with $\{\phi_k\}_{k=0}^{+\infty}$ being an orthonormal basis of eigenstates of the quantum harmonic oscillator. Here, the space $\mathfrak{H}^2$ is the Fock space $\mathfrak{F}(\mathbb{C})$ and $\omega(\zeta) = \pi^{-1} e^{|\zeta|^2}, \zeta \in \mathbb{C}$.

3 The generalized Fock spaces $\mathcal{A}_m(\mathbb{C})$

As the Fock space $\mathfrak{F}(\mathbb{C})$ has $K_0(z, w) := \pi^{-1} e^{z\overline{w}}$ as reproducing kernel, we have shown [4] that the Hilbert spaces $\mathcal{A}_m(\mathbb{C})$ also have explicit reproducing kernel of the form

$$K_m(z, w) := \pi^{-1} e^{\langle z, w \rangle} L_m^{(0)}\left(|z - w|^2\right), z, w \in \mathbb{C}, \tag{3.1}$$

where $L_m^{(\alpha)}(t)$ is the Laguerre polynomial defined by the Rodriguez formula as

$$L_m^{(\alpha)}(t) = \frac{1}{m!} t^{-\alpha} e^t \left(\frac{d}{dt}\right)^m \left(t^{\alpha+m} e^{-t}\right), t \in \mathbb{R}$$

In particular, if we set $\omega_m(z) := K_m(z, z)$, then $\omega_m(z) = \pi^{-1} e^{|z|^2}$, $z \in \mathbb{C}$.

The spaces $\mathcal{A}_m(\mathbb{C})$ have been also used to study the spectral properties of the Cauchy transform on $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$; see [7] where the authors exhibited for each fixed $m = 0, 1, 2, ...$ an orthogonal basis denoted $\{h_{m,p}\}_{p=0}^{+\infty}$ and defined by

$$h_{m,p}(z) := \gamma_{m,p}\ {}_1F_1\left(-\min(m,p), |m-p|+1, |z|^2\right)|z|^{|m-p|}e^{-i(m-p)\arg z} \tag{3.2}$$

where

$$\gamma_{m,p} := \frac{(-1)^{\min(m,p)}(\max(m,p))!}{(|m-p|)!},$$

and ${}_1F_1$ is the confluent hypergeometric function given by [8]:

$${}_1F_1(a,b;u) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{j=0}^{+\infty}\frac{\Gamma(a+j)}{\Gamma(b+j)}\frac{u^j}{j!}, |u| < +\infty, b \neq 0, -1, -2, ...$$

Here $\Gamma(a)$ is the Euler's Gamma function such that $\Gamma(j+1) = j!$ if $j = 0, 1, 2, ...$

Note that for $a = -n$ with n being a positive integer, the hypergeometric function ${}_1F_1$ becomes a polynomial and can be expressible in term of Laguerre polynomial according to [8]:

$${}_1F_1(-n, \alpha+1; u) = \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(u).$$

For our purpose we shall consider the orthogonal basis of $\mathcal{A}_m(\mathbb{C})$ in the following form

$$h_{m,p}(z) = (-1)^{\min(m,p)}(\min(m,p))!\,|z|^{|m-p|}e^{-i(m-p)\arg z}L_{\min(m,p)}^{(|m-p|)}\left(|z|^2\right), z \in \mathbb{C}, \tag{3.3}$$

with the square norm in $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$ given by

$$\rho_{m,p} := \left\|h_{m,p}\right\|^2 = \pi m! p!.$$

Remark 3.1. In [7, p. 404] the elements of the orthogonal basis given in (3.2) have been also expressed as

$$h_{m,p}(z) = \sum_{j=0}^{\min(m,p)}(-1)^j\frac{m!p!}{j!(m-j)!(p-j)!}z^{m-j}\overline{z}^{p-j}. \tag{3.4}$$

We should note these complex polynomials in (3.4) were considered also by Itô [9] in the context of complex Markov process.

4 Coherent states attached to $\mathcal{A}_m(\mathbb{C})$

In this section, we shall attach to each space $\mathcal{A}_m(\mathbb{C})$ a set coherent states via series expansion according to the procedure presented in section 2. We will also give expressions of these coherent states in a closed form by using direct calculations.

Definition 4.1. *For $m = 0, 1, 2, ...$, the coherent states associated with the space $\mathcal{A}_m(\mathbb{C})$ and labelled by points $z \in \mathbb{C}$ are defined formally according to formula* (2.1) *as*

$$\vartheta_{z,m} \equiv | z, m >:= (\omega_m(z))^{-\frac{1}{2}}\sum_{p=0}^{+\infty}\frac{h_{m,p}(z)}{\sqrt{\rho_{m,p}}}\psi_p$$

where ψ_p are elements of a total orthonormal system of $L^2(\mathbb{R}, d\xi)$ given

$$\psi_p(\xi) := \left(\sqrt{\pi}2^p p!\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\xi^2} H_p(\xi), \; p = 0, 1, 2, ..., \quad \xi \in \mathbb{R},$$

and $H_p(\xi)$ is the pth Hermite polynomial.

Proposition 4.1. *The wave functions of these coherent states are expressed as*

$$\vartheta_{z,m}(\xi) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\bar{z}^2 + \sqrt{2}\xi\bar{z} - \frac{1}{2}|z|^2 - \frac{1}{2}\xi^2} H_m\left(\xi - \frac{z + \bar{z}}{2}\right), \xi \in \mathbb{R}.$$

Proof. According to Definition 4.1, we start by writing

$$\vartheta_{z,m}(\xi) = \left(\frac{1}{\pi} e^{|z|^2}\right)^{-\frac{1}{2}} \sum_{p=0}^{+\infty} \frac{h_{m,p}(z)}{\sqrt{\pi m! p!}} \psi_p(\xi).$$

Recalling the expression of $h_{m,p}(z)$ in (3.3), then these wave functions can be rewritten as

$$\vartheta_{z,m}(\xi) = \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}} \sum_{p=0}^{+\infty} \frac{(-1)^{\min(m,p)}}{\sqrt{p!}} (\min(m,p))! \, |z|^{|m-p|} e^{-i(m-p)\arg z} L_{\min(m,p)}^{(|m-p|)}\left(|z|^2\right) \psi_p(\xi).$$

The integer m being fixed, we denote by $\mathfrak{S}_m(z, \xi)$ the following series:

$$\mathfrak{S}_m(z, \xi) := \sum_{p=0}^{+\infty} \frac{(-1)^{\min(m,p)}}{\sqrt{p!}} (\min(m,p))! \, |z|^{|m-p|} e^{-i(m-p)\arg z} L_{\min(m,p)}^{(|m-p|)}\left(|z|^2\right) \psi_p(\xi)$$

and we split it into two part as

$$\begin{aligned} \mathfrak{S}_m(z, \xi) &= \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^p p! \, |z|^{m-p} e^{-i(m-p)\arg z} L_p^{(m-p)}\left(|z|^2\right) \psi_p(\xi) \\ &+ \sum_{p=m}^{+\infty} \frac{1}{\sqrt{p!}} (-1)^m m! \, |z|^{p-m} e^{-i(m-p)\arg z} L_m^{(p-m)}\left(|z|^2\right) \psi_p(\xi) \end{aligned}$$

This can also be written as

$$\mathfrak{S}(m, z, \xi) = \mathcal{S}_{(<\infty)}(m, z, \xi) + \mathcal{S}_{(\infty)}(m, z, \xi)$$

with

$$\begin{aligned} \mathcal{S}_{(<\infty)}(m, z, \xi) &= \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^p p! \, |z|^{m-p} e^{-i(m-p)\arg z} L_p^{(m-p)}\left(|z|^2\right) \psi_p(\xi) \\ &- \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^m m! \, |z|^{p-m} e^{-i(m-p)\arg z} L_m^{(p-m)}\left(|z|^2\right) \psi_p(\xi) \end{aligned}$$

and

$$\mathcal{S}_{(\infty)}(m, z, \xi) = \sum_{p=0}^{+\infty} \frac{1}{\sqrt{p!}} (-1)^m m! \, |z|^{p-m} e^{-i(m-p)\arg z} L_m^{(p-m)}\left(|z|^2\right) \psi_p(\xi).$$

The finite sum $\mathcal{S}_{(<\infty)}(m,z,\xi)$ reads

$$\mathcal{S}_{(<\infty)}(m,z,\xi)=\sum_{p=0}^{m-1}\left((-1)^p\sqrt{p!}\bar{z}^{m-p}L_p^{(m-p)}\left(|z|^2\right)-(-1)^m\frac{m!}{\sqrt{p!}}z^{p-m}L_m^{(p-m)}\left(|z|^2\right)\right)\psi_p(\xi)$$

Making use of the identity [10, p. 98]:

$$L_m^{(-k)}(t)=(-t)^k\frac{(m-k)!}{m!}L_{m-k}^{(k)}(t),\ 1\leq k\leq m$$

for $k=p-m$, we write the Laguerre polynomial with upper indice $p-m<0$ as

$$L_m^{(p-m)}\left(|z|^2\right)=\left(-|z|^2\right)^{m-p}\frac{p!}{m!}L_p^{(m-p)}\left(|z|^2\right),$$

and we obtain after calculation that $\mathcal{S}_{(<\infty)}(m,z,\xi)=0$.

Now, for the infinite sum $\mathcal{S}_{(\infty)}(m,z,\xi)$, we make use of the explicit expression of the Gaussian-Hermite functions

$$\psi_p(\xi)=\left(\sqrt{\pi}2^p p!\right)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2}H_p(\xi),\ p=0,1,2,...$$

and we obtain that

$$\begin{aligned}\mathcal{S}_{(\infty)}(m,z,\xi) &= \sum_{p=0}^{+\infty}\frac{1}{\sqrt{p!}}(-1)^m m!\,|z|^{p-m}e^{-i(m-p)\arg z}L_m^{(p-m)}\left(|z|^2\right)\frac{e^{-\frac{1}{2}\xi^2}H_p(\xi)}{\left(\sqrt{\pi}2^p p!\right)^{\frac{1}{2}}}\\ &= \left(\sqrt{\pi}\right)^{-\frac{1}{2}}(-1)^m m!e^{-\frac{1}{2}\xi^2}\mathfrak{T}_{(\infty)}(m,z,\xi)\end{aligned}$$

where

$$\mathfrak{T}_{(\infty)}(m,z,\xi):=\sum_{p=0}^{+\infty}\frac{\left(2^p\right)^{-\frac{1}{2}}}{p!}z^{p-m}L_m^{(p-m)}\left(|z|^2\right)H_p(\xi)$$

Next, we make use of following addition formula involving Laguerre and Hermite polynomials [11]:

$$\begin{aligned}&\sum_{j=-n}^{+\infty}\frac{2^{-j}\beta^{\frac{j}{2}}}{(j+n)!}(a+ib)^j L_n^{(j)}\left(\frac{\beta}{2}\left(a^2+b^2\right)\right)H_{j+n}(\xi)\\ &= \frac{1}{m!}\exp\left(-\frac{\beta}{4}(a-ib)^2+\sqrt{\beta}\xi(a-ib)\right)H_n\left(\xi-\sqrt{\beta}a\right)\end{aligned}$$

for $n=m$, $j=p-n$, $\beta=2$ and $z=a+ib\in\mathbb{C}$. This gives that

$$\mathfrak{T}_{(\infty)}(m,z,\xi)=\frac{2^{-\frac{m}{2}}}{m!}e^{-\frac{1}{2}\bar{z}^2+\sqrt{2}\xi\bar{z}}H_n\left(\xi-\frac{z+\bar{z}}{2}\right)$$

Summarizing up the above calculations, we can write successively

$$\begin{aligned}\vartheta_{z,m}(\xi) &= \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}}\left(\sqrt{\pi}\right)^{-\frac{1}{2}}(-1)^m m!e^{-\frac{1}{2}\xi^2}\mathfrak{T}_{(\infty)}(m,z,\xi)\\ &= \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}}\left(\sqrt{\pi}\right)^{-\frac{1}{2}}(-1)^m m!e^{-\frac{1}{2}\xi^2}\left(\frac{2^{-\frac{m}{2}}}{m!}e^{-\frac{1}{2}\bar{z}^2+\sqrt{2}\xi\bar{z}}H_m\left(\xi-\frac{z+\bar{z}}{2}\right)\right)\\ &= (-1)^m\left(2^m m!\sqrt{\pi}\right)^{-\frac{1}{2}}e^{-\frac{1}{2}\bar{z}^2+\sqrt{2}\xi\bar{z}-\frac{1}{2}|z|^2-\frac{1}{2}\xi^2}H_m\left(\xi-\frac{z+\bar{z}}{2}\right).\end{aligned}$$

The proof of Proposition 4.1 is finished. ■

Finally, according to Definition 2.2, the coherent state transform associated with the coherent states $\vartheta_{z,m}$ is the unitary map:

$$\mathcal{B}_m : L^2(\mathbb{R}, d\xi) \rightarrow \mathcal{A}_m(\mathbb{C})$$

defined by

$$\mathcal{B}_m[f](z) := (\omega_m(z))^{\frac{1}{2}} \langle f, \vartheta_{z,m} \rangle_{L^2(\mathbb{R})}, f \in L^2(\mathbb{R}, d\xi), z \in \mathbb{C}$$

Explicitly,

$$\mathcal{B}_m[f](z) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2}\xi^2 + \sqrt{2}\xi z - \frac{1}{2}z^2} H_m\left(\xi - \frac{z + \bar{z}}{2}\right) d\xi$$

which can be called the extended Bargmann transform of index $m = 0, 1, 2, ...$

References

[1] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, Part I, *Comm. Pure Appl. Math.*, 14, 187-214 (1961)

[2] B.C. Hall, Bounds on the Segal-Bargmann transform of L^p functions, *J. Fourier Anal & App*, 7, issue 6, 553-569 (2001)

[3] G.B. Folland, Harmonic analyse on phase space, *Annals of Math Studies* No.122, Princeton U.P., Princeton NJ (1989)

[4] N. Askour, A. Intissar and Z. Mouayn, Espaces de Bargmann généralisés et formules explicites pour leurs noyaux reproduisants, *Compt. Rend. Acad. Sci. Paris*, t.325, Série I, pp.707-712 (1997)

[5] N.L. Vasilevski, Poly-Fock spaces, *Oper. Theory, Adv. App*. vol.117, pp.371-386 (2000)

[6] K Thirulogasantar and N Saad: Coherent states associated with the wavefunctions and the spectrum of the isotonic oscillator, J. *Phys. A: Math. Gen*. 37, 4567-4577 (2004)

[7] Abdelkader Intissar and Ahmed Intissar, Spectral properties of the Cauchy transform on $L_2(\mathbb{C}, e^{-|z|^2} d\lambda(z))$, *J. Math. Anal. Appl*, 313, 400-418 (2006)

[8] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, INC (1980)

[9] K Itô, Complex multiple Wiener integral, *Jap. J. Math.*, 22, 63-86 (1953)

[10] Szegö G, Orthogonal polynomials. American Mathematical Society; Providence, R.I. (1975)

[11] Z Mouayn, A generating function for Hermite polynomials in connection with Euclidean Landau levels, (submitted).