The Hua operators on homogeneous line bundle on Bounded Symmetric Domains of Tube Type.

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Abstract

Let $\mathfrak{D}$ be a bounded symmetric domain of tube type . We show that the image of the Poisson transform on the degenerate principal series representation attached to the Shilov boundary of $\mathfrak{D}$ is characterized by a covariant differential operator on a homogeneous line bundle on $\mathfrak{D}$.

1 Introduction

Let G/K be a bounded symmetric domain of tube type with Shilov boundary G/P_Ξ. In [14] Shimeno proved that the Poisson transform maps the space $\mathfrak{B}(G/P_\Xi, L_\lambda)$ of huperfunction-valued sections of a degenerate spherical series representation attached to G/P_Ξ bijectively onto an eigenspace of the Hua operator $\mathcal{H}$ on G/K, under certain conditions on the parameter $\lambda \in (\mathfrak{a}_\Xi)^*_c$.
The aim of this paper is to generalize the above result to a homogeneous line bundle E_ν over G/K.
To state our result in rough form let us fix some notations.
Let E_ν be the homogeneous line bundle on G/K associated to a one dimensional representation τ_ν of K. Let $\mathbb{D}_\nu(G/K)$ be the algebra of G-invariant differential operatos on E_ν. Shimeno [13] proved that the Poisson transform $P_{\mu,\nu}$ is a G-isomorphism from the space $\mathcal{B}(G/P, L_{\mu,\nu})$ of hyperfunction-valued sections of principal series representations attached to the Furstenberg boundary G/P onto the solutions space $\mathcal{A}(G/K, \mathcal{M}_{\mu,\nu})$ of the system of differential equations on E_ν

$$\mathcal{M}_{\mu,\nu}: \quad (D - \chi_{\mu,\nu}(D))F = 0 \quad \forall D \in \mathbb{D}(G/K),$$

under certain conditions on μ and ν. In above $\chi_{\mu,\nu}$ is a certain character of the algebra $\mathbb{D}_\nu(X)$.

Let $P_\Xi = M_\Xi A_\Xi N_\Xi$ be the Langlands decomposition of P_Ξ. Let ξ be the one dimensional representation of P_Ξ defined by

$$\xi(m_1 man) = \tau_\nu(m)a^{\rho_\Xi - \lambda}, \quad m_1 \in M_{\Xi,s}, m \in M, a \in A_\Xi, n \in N_\Xi,$$

where $\lambda \in (\mathfrak{a}_\Xi)^*_c$ ($\mathfrak{a}_\Xi = Lie(A_\Xi)$) and $M_{\Xi,s}$ is the semisimple part of M_Ξ, (see section 4).

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Let L_ξ be the homogeneous line bundle over the Shilov boundary G/P_Ξ associated to ξ. For f in $B(G/P_\Xi, L_{\xi,\lambda})$ the space of hyperfunctions-valued sections of the homogeneous line bundle L_ξ we define its Poisson transform by

$$[P_{\lambda,\nu}f](g) = \int_K f(gk)\tau_\nu(k)dk,$$

where dk is the normalized Haar measure of K.

The degenerate series representation $B(G/P_\Xi, L_{\xi,\lambda})$ attached to G/P_Ξ is a G-submodule of a principal series representation $B(G/P, L_{\mu_\lambda,\nu})$ ($\mu_\lambda = \lambda - \rho_\Xi + \rho$) and the image $P_{\lambda,\nu}(B(G/P_\Xi, L_{\xi,\lambda}))$ is a G-submodule of the solution space $\mathcal{A}(G/K, \mathcal{M}_{\mu_\lambda,\nu})$ of G-invariant differential operators on E_ν.
Therefore it is natural to pose the problem of characterizing this image by differential operators on E_ν.

In the trivial cases the origin of this problem goes back to L. H. Hua [4] who showed in the case of the classical Cartan domain of $n \times n$ matrices that for f in $B(G/P_\Xi)$ $P_{\rho_\Xi}f$ is annihilated by n^2 second order differential operators. Since then many authors considered the problem of constructing differential operators characterizing the image $P_{\rho_\Xi}(B(G/P_\Xi,),$ see [7], [11] in the case of the Siegel upper half plane. In [5] Johnson and Koranyi constructed second order differential operator $\mathcal{H}$-called then after Hua operator- and showed that $\mathcal{H}$ characterizes the image $P_{\rho_\Xi}(B(G/P_\Xi,).$ Lassalle [10] reproved their result introducing the operator $\mathcal{H}_\mathfrak{q}$, cutting down the number of equations.
In this paper we will show that the image of $B(G/P_\Xi, L_{\xi,\lambda})$ under $P_{\lambda,\nu}$ is characterized by a K- covariant differential operator on E_ν .

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of G with Cartan involution θ.
The center $\mathfrak{z}$ of $\mathfrak{k}$ is of dimension one and there exists $Z_0 \in \mathfrak{z}$ such that adZ_0 define a complex structure on $\mathfrak{p}_c$. Let

$$\mathfrak{g}_c = \mathfrak{p}_- \oplus \mathfrak{k}_c \oplus \mathfrak{p}_+,$$

be the corresponding eigenspace decomposition of $\mathfrak{p}_c$.
Let E_i be a basis of $\mathfrak{p}_+$ and E_i^* be the dual basis of $\mathfrak{p}_-$ with respect to the Killing form of $\mathfrak{g}_c$.
We consider the element of $\mathfrak{U}(\mathfrak{g}_c) \otimes \mathfrak{k}_c$ -called here the Hua operator- defined by

$$\mathcal{H} = \sum_{i,j} E_i E_j^* \otimes [E_j, E_i^*].$$

The operator $\mathcal{H}$ is a homogeneous differential operator from the space of C^∞-sections of E_ν to the space of C^∞-sections of the homogeneous vector bundle on G/K associated to the representation $\tau_\nu \otimes Ad_{\mathfrak{k}_c}$, which does not depend on the choice of basis. (see section 4)
The pair $(K, K \cap M_\Xi)$ is a compact symmetric pair. Let $\widetilde{\tau}$ be the corresponding involution of $\mathfrak{k}$ and let $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{k}$ into eigenspaces of the $\widetilde{\tau}$.
Let $\mathcal{H}_q$ be the element of $\mathfrak{U}(\mathfrak{g}_c) \otimes \mathfrak{q}_c$ defined by

$$\mathcal{H}_q = \sum_{i,j} E_i E_j^* \otimes p([E_j, E_i^*]),$$

where p denotes the orthogonal projection of $\mathfrak{k}_c$ onto $\mathfrak{q}_c$.

The main result of this paper can be stated as follows

Theorem 1.1 *Let λ be a complex number and let $\nu \in p\mathbb{Z}$ such that*

$$-\lambda - \frac{m}{2}(-r+2+i) \notin \{1, 2...\} \quad for \quad i = 0, 1,$$

$$e_\nu(\mu_\lambda) \neq 0.$$

Then the Poisson transform $P_{\lambda,\nu}$ is a G-isomorphism from the space $B(G/P_\Xi, L_{\xi,\lambda})$ onto the space of C^∞- section of E_ν satisfying the system

$$\mathcal{H}_q F = \frac{(\lambda^2 - (\eta-\nu)^2)}{4p} F.(-i)Z_0. \tag{1.1}$$

In the above $e_\nu^{-1}(\mu_\lambda)$ denotes the denominator in the Harish-Chandra function $c_\nu(\mu_\lambda)$ and $p = 2\eta$ is the genus of $\mathfrak{D}$, see section 3.

In the case τ is the trivial representation, Theorem 1.1 has been established by Lassalle [10] for $\lambda = \rho_\Xi$ and generalizes to generic λ by Shimeno [14].

To prove our result we first show that every solution of the Hua system is a joint eigenfunction of the algebra $\mathbb{D}_\nu(X)$ (Theorem 6.1). To do so, we consider a subsystem $\mathcal{H}_\mathfrak{h} \in \mathfrak{U}(\mathfrak{g}_c) \otimes \mathfrak{h}_c$, where $\mathfrak{h} \subset \mathfrak{q}$ is a Cartan subalgebra of the symmetric pair $(\mathfrak{k}, \mathfrak{l})$.
Then we prove (Theorem 6.2) that if F is a τ_ν-spherical function satisfying

$$\mathcal{H}_\mathfrak{h} F = \frac{(\lambda^2 - (\eta-\nu)^2)}{4p} F.(-i)Z_0,$$

then the function

$$\phi(t_1, ..., t_r) = \prod_{j=1}^{r} (\cosh t_j)^\nu F(t_1, ..., t_r),$$

satisfies the following system of differential equations

$$\frac{\partial^2 \phi}{\partial t_k^2} + 2\coth 2t_k \frac{\partial \phi}{\partial t_k} - 2\nu \tanh t_k \frac{\partial \phi}{\partial t_k} + \frac{m}{2} \sum_{j=1}^{r} \frac{1}{(\sinh^2 t_j - \sinh^2 t_k)} (\sinh 2t_j \frac{\partial \phi}{\partial t_j} - \sinh 2t_k \frac{\partial \phi}{\partial t_k})$$
$$= \frac{(\lambda^2 - (\eta-\nu)^2)}{4} \phi,$$

for all $k = 1, ..., r$.
Then by using a result of Yan [16] on generalized hypergeometric functions in several variables we deduce that F is given (up to a constant) in terms of the generalized hypergeometric function ${}_2F_1^{(m)}(b, c, d; x_1, ..., x_r)$ associated to the parameter m, see [16]. Namely

$$F(t_1, ..., t_r) = \prod_{j=1}^{r} (1 - \tanh^2 t_j)^{\frac{\lambda+\eta}{2}} \quad {}_2F_1^{(m)}(\frac{\lambda+\eta-\nu}{2}, \frac{\lambda+\eta+\nu}{2}; \eta, \tanh^2 t_1, ..., \tanh^2 t_r),$$

To finish the proof of our main result we show that the induced equations of the Hua system for boundary values on the maximal boundary G/P characterize the space $\mathcal{B}(G/P_\Xi, L_{\lambda,\xi})$.

As a consequence of the method of the proof of Theorem 1.1, we obtain an explicit expression of a Hua type integrals. Namely, we have

$$\int_S [\frac{h(z,z)}{\mid h(z,u)\mid}]^{\frac{\lambda+\eta-\nu}{2}} h(z,u)^{-\nu} du = \Pi_{j=1}^r (1-\tanh^2 t_j)^{\frac{\lambda+\eta-\nu}{2}}$$

$$_2F_1^{(m)}(\frac{\lambda+\eta-\nu}{2}, \frac{\lambda+\eta+\nu}{2}; \eta, \tanh^2 t_1, ..., \tanh^2 t_r),$$

with $z = ka_t.0$.

The above formula has been established for $\nu = 0$ by Faraut and Koranyi in [3].

In the case of the trivial line bundle on $G = SU(n,n)/S(U(n)\times U(n))$, the above Hua type integral can be given explicitly in terms of the classical Gauss hypergeometric functions. More precisely, for $z = \tanh tU$

$$\int_S [\frac{h(z,z)}{\mid h(z,u)\mid}]^{\frac{\lambda+\eta}{2}} du = \frac{n!}{d_m} \det(\phi_{\lambda,|m_i-i+j|}(t))_{i,j},$$

where $d_m = \Pi(1+\frac{m_i-m_j}{j-i})$, and

$$\phi_{\lambda,k}(t) = (1-\tanh^2 t)^{\frac{\lambda+n}{2}} \frac{(\frac{\lambda+n}{2})_k}{(1)_k} \tanh^k t \quad {}_2F_1(\frac{\lambda+n}{2}, \frac{\lambda+n}{2}+k, 1+k; \tanh^2 t),$$

$_2F_1(a,b,c;x)$ being the classical Gauss hypergeometric function and $(a)_k$ the Pochammer symbol. See [2] for more details.

The organization of this paper is as follows. After a preliminaries on Hermitian symmetric spaces we review in section 3 the results of [14] on the Poisson transform on homogeneous line bundles on the Furstenberg boundary. In section 4 we define the Poisson transform on degenerate principal series representation attached to the Shilov boundary and introduce the Hua operator $\mathcal{H}$ on E_ν. Using a trivialization of the space of C^∞-sections of E_ν we give a realization of $\mathcal{H}$ on the Harish-Chandra realization of G/K (Proposition 4.1). Section 5, 6 and 7 are devoted to the proof of our main result .

Recently I was informed by Professor Koufany that he and Professor Zhang have proved the necessity of the conditions in Theorem 1.1 by using a different method, see [9].

Acknowledgement: I thank Professor Koranyi for sending me his preprint [8] which will appear in the volume for J.A. Wolf's 75th birthday, where the necessity of the Hua equations is also proved.

2 Preliminaries and notations

In this section we recall some structural results on Hermitian symmetric space, see [4] for more details.

For a real Lie algebra $\mathfrak{b}$ we shall denote by $\mathfrak{b}_c$ its complexification.

Let G be a connected simple Lie group with finite center and let K be a maximal compact subgroup.

Let $g = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to a Cartan

involution θ. We suppose that G/K is a Hermitian symmetric space of tube type with rank r. Thus $\mathfrak{k}$ has one dimensional center $\mathfrak{z}$, $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}] \neq \mathfrak{k}$ and $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_s$.
Let $Z_0 \in \mathfrak{z}$ such that $(adZ_0)^2 = -1$ on $\mathfrak{p}_c$. Let $\mathfrak{p}_+$ (respectively $\mathfrak{p}_-$) be the i (respectively $-i$) -eigenspace of adZ_0 in $\mathfrak{g}_c$.
Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are Abelian subalgebras. Moreover $[\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_c$. We thus have the Harish-Chandra decomposition of $\mathfrak{g}_c$

$$\mathfrak{g}_c = \mathfrak{p}_+ + \mathfrak{k}_c + \mathfrak{p}_-.$$

Let P_+, P_- and K_c denote the analytic subgroup of G_c corresponding to the Lie subalgebras $\mathfrak{p}_+, \mathfrak{p}_-$ and $\mathfrak{k}_c$ respectively. Then $P_+K_cP_-$ is an open dense subset of G_c containing G. For $z \in \mathfrak{p}_+$ and $g \in G$ we denote by $U(g : z)$ the K_c component of $g \exp z$. That is

$$g \exp z = \exp(g.z)U(g : z)p_-(g). \tag{2.1}$$

Under the above action the G-orbit $\mathcal{D} = G.0$ of $z = 0 \in \mathfrak{p}_+$ is a bounded domain in $\mathfrak{p}_+$ and K is the isotropy subgroup of 0. This is the Harish-Chandra realization of the Hermitian symmetric space G/K.

2.1 The Roots.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ (and hence also of $\mathfrak{g}$). Let Δ denote the root system of $(\mathfrak{g}_c, \mathfrak{t}_c)$.

For $\alpha \in \Delta$ let $\mathfrak{g}_\alpha$ denote the root space for α. A root α is said to be compact (resp. noncompact) if the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}_c$ (resp.$\mathfrak{p}_c$). Let B denote the Killing form of $\mathfrak{g}_c$. For each root α we can choose root vectors $\tilde{H_\alpha} \in \mathfrak{t}_c$, $\tilde{E_\alpha} \in \mathfrak{g}_\alpha$, and $\tilde{E_{-\alpha}} \in \mathfrak{g}_{-\alpha}$ such that

$$\alpha(H) = B(H, \tilde{H_\alpha}), \quad \forall H \in \mathfrak{t}_c,$$

$$[\tilde{E_\alpha}, \tilde{E_{-\alpha}}] = \tilde{H_\alpha}, \quad \tau\tilde{E_\alpha} = -\tilde{E_{-\alpha}},$$

with $B(\tilde{E_\alpha}, \tilde{E_{-\alpha}}) = 1$.
In above τ denotes the conjugation in $\mathfrak{g}_c$ with respect to the real form $\mathfrak{k} + i\mathfrak{p}$.

For α, β in Δ, we set $< \alpha, \beta >= B(\tilde{H_\alpha}, \tilde{H_\beta})$. Then the length $\mid \alpha \mid$ of a root α is defined by $\mid \alpha \mid = \sqrt{< \alpha, \alpha >}$.
Put $c_\alpha = \frac{\sqrt{2}}{|\alpha|}$. Let $H_\alpha = c_\alpha^2 \tilde{H_\alpha}$ and $E_\alpha = c_\alpha \tilde{E_\alpha}$. Then the root vectors H_α, E_α and $E_{-\alpha}$ satisfy

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

$$B(E_\alpha, E_{-\alpha}) = \frac{2}{\mid \alpha \mid^2}.$$

Moreover $\alpha(H_\alpha) = 2$.

We choose an ordering on Δ such that a noncompact root is positive if and only if $\mathfrak{g}_\alpha \subset \mathfrak{p}_+$. We denote by Φ^+ the set of positive noncompact roots.Then we have

$$\mathfrak{p}_+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

$$\mathfrak{p}_- = \sum_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}.$$

For each $\alpha \in \Phi^+$ we set

$$X_\alpha = E_\alpha + E_{-\alpha}, \quad Y_\alpha = i(E_\alpha - E_{-\alpha}).$$

Two roots α and β are said strongly orthogonal if neither $\alpha+\beta$ nor $\alpha-\beta$ are roots.
Let $\Gamma = \{\gamma_1, ..., \gamma_r\}$ be a maximal set of strongly orthogonal noncompact roots, such that γ_j is the highest element of Φ^+ strongly orthogonal to $\gamma_{j+1}, ..., \gamma_r$, for $j = r, ..., 1$. Then $\mathfrak{a} = \sum_{j=1}^r \mathbb{R}X_{\gamma_j}$ is a maximal Abelian subspace of $\mathfrak{p}$.
Let $\mathfrak{h}$ denote the Abelian subalgebra generated by the elements $iH_{\gamma_j}, j = 1, ..., r$ and let $\mathfrak{h}^\perp$ be its orthogonal in $\mathfrak{t}$ with respect to the Killing form B. Then $\mathfrak{h}^\perp = \{H \in \mathfrak{t}; \gamma_j(H) = 0, j = 1, ..., r\}$.

For $\alpha, \beta \in \Delta$ denote $\alpha \sim \beta$ if and only if $\alpha\mid_{\mathfrak{h}} = \beta\mid_{\mathfrak{h}}$. Let

$$\Phi^+_{ij} = \{\alpha \in \Phi^+; \alpha \sim \frac{\gamma_i+\gamma_j}{2}\},$$

for $1 \le i < j \le r$.

Let C denote the set of compact roots in ϕ. Define

$$C_0 = \{\alpha \in C; \alpha \sim 0\},$$

and

$$C_{ij} = \{\alpha \in C; \alpha \sim \frac{\gamma_j - \gamma_i}{2}\}, \qquad \text{for} 1 \le i < j \le r.$$

Then it is known that Δ^+ is the disjoint union of the sets $\Gamma, C_0, C_{i,j}$, Φ^+_{ij} and Φ^+ is the disjoint union of the sets Γ and Φ^+_{ij}.

Let $\alpha \in \Phi^+_{ij}$. Define $\widetilde{\alpha} \in \mathfrak{t}_c^\star$ by $\widetilde{\alpha} = \gamma_i + \gamma_j - \alpha$. Then $\widetilde{\alpha} \in \Phi^+_{ij}$ see [10].
Now we recall a result from [1] which will be helpful later.

Proposition 2.1 *Let k be a fixed integer, $1 \le k \le r$.*
(i) For $\gamma_j \in \Gamma$ we have $\gamma_j(H_{\gamma_k}) = 2\delta_{jk}$.
(ii) If $\alpha \in \Phi^+_{jk}$, then $\alpha(H_{\gamma_k}) = 1$.
(iii) In all other cases $\alpha(H_{\gamma_k}) = 0$.
(iv)Let $\alpha \in \Phi^+_{jk}$. Then $<\alpha, \alpha> = <\gamma_k, \gamma_k>$ if $\alpha \ne \widetilde{\alpha}$ and $<\alpha, \alpha> = \frac{1}{2} <\gamma_k, \gamma_k>$ if $\alpha = \widetilde{\alpha}$.

Let c be the Cayley transform of $\mathfrak{g}_{\mathfrak{c}}$ given by

$$c = \exp\frac{\pi}{4}(\overline{E}_0 - E_0),$$

where $E_0 = \sum_{k=1}^r E_{\gamma_k}$. Then $Adc(H_{\gamma_j}) = X_{\gamma_j}$, and $Adc(\mathfrak{h}) = i\mathfrak{a}$.
It is well known that $Adc^4 = 1$ and Adc^2 is an automorphism of $\mathfrak{k}$. Let $\mathfrak{l}$ (respectively $\mathfrak{q}$) be the $+1$ (respectively -1)eigenspace of Adc^2 in $\mathfrak{k}$. Then we have $\mathfrak{z}$ is a subset of $\mathfrak{q}$. Moreover, let $\mathfrak{q}_s = \mathfrak{q} \cap \mathfrak{k}_s$ then $\mathfrak{k}_s = \mathfrak{q}_s + \mathfrak{l}$ and $\mathfrak{q} = \mathfrak{q}_s + \mathfrak{z}$.

Put $\beta_i = \gamma_j \circ (Adc^{-1}|a)$, for $j = 1, ..., r$. Then the set of restricted roots Σ of the pair $(\mathfrak{g}, \mathfrak{a})$ is given by

$$\Sigma = \{\pm\beta_j \quad (1 \le j \le r), \frac{\pm\beta_j \pm \beta_k}{2} \quad (1 \le j \ne k \le r)\}.$$

Let Σ^+ be the set of positive roots in Σ. Then

$$\Sigma^+ = \{\beta_j \quad (1 \le j \le r), \frac{\beta_j \pm \beta_k}{2} \quad (1 \le k < j \le r)\}.$$

The Weyl group W of Σ acts as the group of all permutations and sign changes of the set $\{\beta_1, ..., \beta_r\}$, so it is isomorphic to the semi-direct product of $(\mathbb{Z}/2\mathbb{Z})^r$ and the symmetric group. We set

$$\alpha_j = \frac{\beta_{r-j+1} - \beta_{r-j}}{2}, \quad (1 \le j \le r-1) \quad \alpha_r = \beta_1.$$

Then $\Gamma = \{\alpha_1, ..., \alpha_r\}$ is the set of simple roots in Σ^+. Let $\{H_1, ..., H_r\}$ denote the basis of $\mathfrak{a}$ which is dual to $\{\alpha_1, ..., \alpha_r\}$.
For $\alpha \in \Sigma$ let $\mathfrak{g}^\alpha$ denote the corresponding root space and let m_α be the multiplicity of α.
The multiplicities of the roots $\frac{\pm\beta_j \pm \beta_k}{2}$ and $\pm\beta_j$ are m and 1 respectively.
As usual set $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$, $\mathfrak{n}^+ = \sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \theta(\mathfrak{n})$.

Let A, N^+ and N^- be the analytic subgroups of G corresponding to $\mathfrak{a}, \mathfrak{n}^+$ and $\mathfrak{n}^-$ respectively.The group G has the Iwasawa decomposition $G = KAN^+$. Let M be the centralizer of $\mathfrak{a}$ in K. Then $P = MAN^+$ is a minimal parabolic subgroup of G.

Let $\Xi = \Gamma \setminus \{\alpha_r\}$ and let P_Ξ be the corresponding standard parabolic subgroup of G with the Langlands decomposition $P_\Xi = M_\Xi A_\Xi N_\Xi$ such that $A_\Xi \subset A$. Then P_Ξ is a maximal standard parabolic subgroup of G and the space G/P_Ξ is the Shilov boundary of X.
If $\mathfrak{a}_\Xi$ denotes the Lie algebra of A_Ξ, then

$$\mathfrak{a}_\Xi = \{H \in \mathfrak{a}; \gamma(H) = 0, \forall\gamma \in \Xi\}.$$

Moreover $\mathfrak{a}_\Xi = \mathbb{R}X_0$ where $X_0 = \sum_{j=1}^r X_{\gamma_j}$.
On $\mathfrak{a}_\Xi$ we define the linear form ρ_0 by $\rho_0(X_0) = r$. Let ρ_Ξ be the restriction of ρ to $\mathfrak{a}_\Xi$. Then

$$\rho_\Xi = (m\frac{r-1}{2} + 1)\rho_0.$$

The algebras $\mathfrak{n}^\pm$ decomposes as $\mathfrak{n}^\pm = \mathfrak{n}^\pm_\Xi + \mathfrak{n}(\Xi)^\pm$, where

$$\mathfrak{n}^\pm_\Xi = \sum_{j<k} \mathfrak{g}^{\pm(\frac{\beta_j+\beta_k}{2})} \quad \mathfrak{n}(\Xi)^\pm = \sum_{k<j} \mathfrak{g}^{\pm(\frac{\beta_j-\beta_k}{2})},$$

and we have $\mathfrak{p}_\Xi = m + a + n^+ + \mathfrak{n}(\Xi)^-$, $\mathfrak{p}_\Xi$ been the Lie algebra of P_Ξ.

3 Eigensections of invariant differential operators

We review the main result of [13] on the image of the Poisson transform on the principal series representation attached to the Furstenberg boundary G/P.
Let us first denote by p the genus of the bounded domain $\mathcal{D}$, given by

$$p = m(r-1) + 2.$$

Then the length $\mid \gamma_j \mid$ of the roots γ_j is such that $\mid \gamma_j \mid = \frac{1}{\sqrt{p}}$.
Let $Z = \frac{p}{n} Z_0$ where $n = dim_{\mathbb{C}} \mathfrak{p}_+$.
Let K_s be the analytic subgroup of K with Lie algebra $\mathfrak{k}_s$. For $\nu \in p\mathbb{Z}$ define $\tau_\nu : K \to \mathbb{C}^\times$ by $\tau_\nu(k) = 1$ if $k \in K_s$ and $\tau_\nu(\exp(tZ)) = e^{-i\nu t}$ for $t \in \mathbb{R}$. Then τ_ν determines a one dimensional representation of K and all one dimensional representation of K have this form, see [12] and the remark below.

Remark 3.1 *Since*

$$B(Z, Z_0) = -2p,$$

it follows that Z is the same as the element

$$\frac{1}{r} \sum_{j=1}^{r} \frac{2i\tilde{H}_{\gamma_j}}{< \gamma_j, \gamma_j >},$$

in [12].

Let E_ν be the homogeneous line bundle on G/K associated to τ_ν.
The space of C^∞-sections of E_ν can be identified with the space $C^\infty(G/K, \tau_\nu)$ of all C^∞-functions on G such that $f(gk) = \tau_\nu(k)^{-1} f(g)$ for all $g \in G$ and $k \in K$.
The group G acts on $C^\infty(G/K, \tau_\nu)$ by the left regular representation $\pi(g)f(x) = f(g^{-1}x)$.
Let $\mathbb{D}_\nu(X)$ be the set of all left-invariant differential operators on G that map $C^\infty(G/K, \tau_\nu)$ into itself. Then, accordingly to Shimeno result [13] $\mathbb{D}_\nu(X)$ is isomorphic,via the Harish-Chandra isomorphism γ_ν, to $\mathcal{U}(\mathfrak{a})^W$ the set of Weyl group invariant elements in $\mathcal{U}(\mathfrak{a})$.
For $\mu \in \mathfrak{a}_c^*$ and $\nu \in p\mathbb{Z}$ we define an algebra homomorphism $\chi_{\mu,\nu}$ of $\mathbb{D}_\nu(X)$ by

$$\chi_{\mu,\nu}(D) = \gamma_\nu(D)(\mu).$$

For $\mu \in \mathfrak{a}_c^*$ we denote by $\mathcal{B}(G/P, L_{\mu,\nu})$ the space of hyperfunction-valued sections of the homogeneous line bundle on G/P associated to the character $\sigma_{\mu,\nu}$ of P given by

$$\sigma_{\mu,\nu}(man) = a^{\rho-\mu} \tau_\nu(m) \quad m \in M, a \in A, n \in N^+.$$

For $f \in \mathcal{B}(G/P, L_{\mu,\nu})$ we define the Poisson transform $\mathcal{P}_{\mu,\nu}$ by

$$\mathcal{P}_{\mu,\nu} f(g) = \int_K f(gk) \tau_\nu(k) dk.$$

A straightforward computation shows that

$$\mathcal{P}_{\mu,\nu} f(g) = \int_K e^{-(\mu+\rho)H(g^{-1}k)} f(k) \tau_\nu(\kappa(g^{-1}k)) dk,$$

where $\kappa : G \to K$ and $H : G \to \mathfrak{a}$ are the projections defined by $g \in \kappa(g) e^{H(g)} N^+$.

Let $\mathcal{B}(G/K, \tau_\nu)$ be the space of hyperfunction-valued sections of the homogeneous line bundle E_ν on G/K. We denote by $\mathcal{A}(G/K, \mathcal{M}_{\mu,\nu})$ the space of all real analytic functions in $\mathcal{B}(G/K, \tau_\nu)$ which satisfy the system of differential equations

$$\mathcal{M}_{\mu,\nu} : DF = \gamma_\nu(D)(\mu) F, \quad D \in \mathbb{D}_\nu(G/K).$$

Let $e_{\mu,\nu}^{-1}$ be the denominator of the c-function associated to $\mathcal{P}_{\mu,\lambda}$. That is

$$e_{\mu,\nu}^{-1} = \prod_{1\leq j<k\leq r} \Gamma(\frac{1}{2}(m+\mu_j+\mu_k))\Gamma(\frac{1}{2}(m+\mu_k-\mu_j))$$

$$\times \prod_{1\leq j\leq r} \Gamma(\frac{1}{2}(1+\mu_j+\nu))\Gamma(\frac{1}{2}(1+\mu_j-\nu)).$$

Theorem 3.1 *[14] Let $\mu \in \mathfrak{a}_c^*$ and $\nu \in p\mathbb{Z}$ satisfying the conditions*

$$-2\frac{<\mu,\alpha>}{<\alpha,\alpha>} \notin \{1,2,...\} \quad \text{for all} \quad \alpha \in \Sigma^+$$

and

$$e_{\mu,\nu} \neq 0,$$

then the Poisson transform $\mathcal{P}_{\mu,\lambda}$ is a G-isomorphism from $\mathcal{B}(G/P, L_{\mu,\nu})$ onto $\mathcal{A}(G/K, \mathcal{M}_{\mu,l})$. The inverse of $\mathcal{P}_{\lambda,\nu}$ is given by the boundary value map up to a non-zero constant multiple.

4 The Poisson transform and the Hua operator

4.1 The Poisson transform on a homogeneous line bundle

Following [15] we define the Poisson transform on degenerate principal series representation attached to the Shilov boundary G/P_Ξ of $\mathcal{D}$.
Let $M_{\Xi,s}$ be the analytic subgroup of M_Ξ with Lie algebra $[\mathfrak{m}_\Xi, \mathfrak{m}_\Xi]$. From here on we suppose that τ_ν is such that $\tau_\nu \mid_{K\cap M_{\Xi,s}}= 1$.
For $\lambda \in \mathbb{C}$ and $\nu \in p\mathbb{Z}$ let $\xi_{\lambda,\nu}$ denote the one dimensional representation of P_Ξ defined by

$$\xi_{\lambda,\nu}(m_1man) = a^{\rho_\Xi-\lambda\rho_0}\tau_\nu(m), \text{for all} \quad m_1 \in M_{\Xi,s}, m \in M, a \in A_\Xi, n \in N_\Xi.$$

Let $B(G/P_\Xi, L_{\lambda,\nu})$ be the space of hyperfunction-valued sections of the line bundle on G/P_Ξ associated to the character $\xi_{\lambda,\nu}$. The Poisson transform $P_{\lambda,\nu}$ of an element $f \in B(G/P_\Xi, L_{\lambda,\nu})$ is defined by

$$P_{\lambda,\nu}f(g) = \int_K f(gk)\tau_\nu(k)dk. \tag{4.1}$$

By the generalized Iwasawa decomposition $G = KP_\Xi$, the restriction map from G to K gives an isomorphism from $B(G/P_\Xi, L_{\lambda,\nu})$ onto the space $B(K/K_\Xi, L_\nu)$ of hyperfunction-valued sections of the homogeneous line bundle on K/K_Ξ associated to the representation τ_ν. Here $K_\Xi = K \cap M_\Xi$.
For $\lambda \in \mathbb{C}$ define the following $\mathbb{C}$-linear form μ_λ on $\mathfrak{a}_c^*$ by

$$\mu_\lambda(H) = (\lambda\rho_0 - \rho_\Xi)(H_\Xi) + \rho(H),$$

where H_Ξ is the $\mathfrak{a}_\Xi$-component of H with respect to the orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_\Xi \oplus \mathfrak{a}(\Xi)$.
We have

$$B(G/P_\Xi, L_{\lambda,\nu}) \subset B(G/P, L_{\mu_\lambda,\nu}). \tag{4.2}$$

From (4.2) we deduce

$$P_{\lambda,\nu}(B(G/P_\Xi, L_{\lambda,\nu})) \subset \mathcal{A}(G/K, \mathcal{M}_{\mu_\lambda,\nu}). \tag{4.3}$$

A straightforward computation shows that the Poisson transform of $f \in B(K/K_\Xi, L_\nu)$ is given by

$$P_{\lambda,\nu}f(g) = \int_K e^{-(\lambda+\eta)\rho_0(H_\Xi(g^{-1}k))} f(k)\tau_\nu(\kappa(g^{-1}k))dk.$$

The space $B(K/K_\Xi, L_\nu)$ can be identified to the space of all hyperfunctions f on K such that

$$f(km) = \tau_\nu^{-1}(m)f(k) \qquad \forall \quad m \in K_\Xi.$$

Let Λ be the map defined from $B(K/K_\Xi), L_\xi)$ into the space $B(K/K_\Xi)$ of all hyperfunctions on the Shilov boundary K/K_Ξ by $\Lambda f(k) = \tau_\nu(k)f(k)$. Then Λ is a K-isomorphism.
Using Λ the Poisson transform (denoted again by $P_{\lambda,\nu}$) of an element $f \in B(K/K_\Xi)$ is given by

$$P_{\lambda,\nu}f(g) = \int_K e^{-(\lambda+\eta)\rho_0 H_\Xi(g^{-1}k)} f(k)\tau_\nu(k^{-1}\kappa(g^{-1}k))dk, \tag{4.4}$$

where $2\eta = p$

4.2 The Hua operator.

If (θ, V) is a finite dimensional representation of the compact group K, we denote by $C^\infty(G/K, \theta)$ the space of C^∞-sections of the homogeneous vector bundle on G/K associated to θ.
Let E_i be a basis of $\mathfrak{p}_+$ and E_i^* be the dual basis of $\mathfrak{p}_-$ with respect to the Killing form B.
Let $U(\mathfrak{g}_c)$ denote the universal enveloping algebra of $\mathfrak{g}_c$. We consider the element of $U(\mathfrak{g}_c) \otimes \mathfrak{k}_c$ defined by

$$\mathcal{H} = \sum_{i,j} E_i E_j^* \otimes [E_j, E_i^*].$$

Then $\mathcal{H}$ defines a homogeneous differential operator from the space $C^\infty(G/K, \tau_\nu)$ to the space $C^\infty(G/K, \tau_\nu \otimes Ad_K \mid_{\mathfrak{k}_c}$, which does not depend on the choice of basis. Let V be a linear subspace of $\mathfrak{k}$ and let V_c be its complexification. We denote by p the orthogonal projection from $\mathfrak{k}_c$ onto V_c. We extend p on $U(\mathfrak{g}_c) \otimes \mathfrak{k}_c$ by setting

$$p(U \otimes X) = U \otimes p(X) \quad (U \in U(\mathfrak{g}_c), X \in \mathfrak{k}_c).$$

We put $\mathcal{H}_V = p(\mathcal{H})$.
If v_j is a basis of V_c and v_j^* is the dual basis with respect to B. Then

$$\mathcal{H}_V = \sum_k U_k \otimes v_k^*,$$

where U_k is the element of $U(\mathfrak{g}_c)$ given by

$$U_k = \sum_j [v_k, E_j] E_j^*,$$

see [10].

Let Λ_ν be the operator defined on $C^\infty(G/K,\tau_\nu)$ by

$$\Lambda_\nu F(z)=\tau_\nu(U(g:0))F(g),\quad z=g.0. \tag{4.5}$$

Then Λ_ν is an isomorphism from $C^\infty(G/K,\tau_\nu)$ onto $C^\infty(\mathcal{D})$. Notice that

$$\tau_{-\nu}(U(g:z))=[J(g,z)]^{\frac{\nu}{p}},$$

where $J(g,z)$ stands for the Jacobian of the transformation g.

Now we define an action T_ν of G on $\mathcal{D}$ as follows:
For each $g\in G$ define $T_\nu(g)$ such that the following diagram

$$\begin{array}{ccc} C^\infty(G/K,\tau_\nu) & \xrightarrow{\pi(g)} & C^\infty(G/K,\tau_\nu) \\ \Lambda_\nu\downarrow & & \downarrow\Lambda_\nu \\ C^\infty(\mathcal{D}) & \xrightarrow[T_\nu(g)]{} & C^\infty(\mathcal{D}) \end{array}$$

is commutative.
The following result can be proved by direct computations

Lemma 4.1 *i) Let $F\in C^\infty(\mathcal{D})$. For any $g\in G$ we have*

$$T_\nu(g)F(z)=\tau_{-\nu}(U(g^{-1}:z))F(g^{-1}z)$$

ii) The operator Λ_ν is a G-intertwining operator from $C^\infty(G/K,\tau_\nu)$ onto $C^\infty(\mathcal{D})$.

Using the above G-intertwining operator the Hua operator may be viewed as acting on $C^\infty(\mathcal{D})$. The new operator which we denote by $\tilde{\mathcal{H}}$ will be given below.
For $f\in C^\infty(G/K,\tau_\nu\otimes Ad)$, we define the function $\Lambda_{\tau_\nu\otimes Ad}f:G/K\to\mathbb{C}\otimes\mathfrak{k}_c$ by

$$\Lambda_{\tau_\nu\otimes Ad}f(z)=(\tau_\nu\otimes Ad)(U(g:0))f(g),\quad z=g.0.$$

We define the Hua operator $\tilde{\mathcal{H}}$ on $C^\infty(\mathcal{D})$ such that the following diagram

$$\begin{array}{ccc} C^\infty(G/K,\tau_\nu) & \xrightarrow{\mathcal{H}} & C^\infty(G/K,\tau_\nu\otimes Ad_K\mid_{\mathfrak{k}_c}) \\ \Lambda_\nu\downarrow & & \downarrow\Lambda_\nu\otimes Ad \\ C^\infty(\mathcal{D}) & \xrightarrow[\tilde{\mathcal{H}}]{} & C^\infty(\mathcal{D},\mathbb{C}\otimes\mathfrak{k}_c) \end{array}$$

is commutative.

Proposition 4.1 *Let $F\in C^\infty(\mathcal{D})$. Then we have*

$$\tilde{\mathcal{H}}F(z)=\tau_\nu(U(g:0))\sum_{i,j}[Ad(U(g:0)^{-1})E_iAd(U(g:0)^{-1})E_j^*]\Lambda_\nu^{-1}F(g)\otimes[E_j,E_i^*]\quad z=g.0 \tag{4.6}$$

Proof. The proof follows by direct computations.

The Hua operator $\tilde{\mathcal{H}}$ has the following invariance property

Proposition 4.2 *For any $h \in G$ and $F \in C^\infty(\mathcal{D})$ we have*

$$T_\nu(h)(\tilde{\mathcal{H}})F(z) = Ad(U(h^{-1} : z))\tilde{\mathcal{H}}(T_\nu(h)F)(z).$$

Proof. Let $g \in G$ such that $g.0 = z$. We have

$$\tilde{\mathcal{H}}F(h^{-1}.z) = \tau_\nu(U(h^{-1}g:0))\sum_{i,j}[Ad(U(h^{-1}g:0)^{-1})E_i Ad(U(h^{-1}g:0)^{-1})E_j^*]\Lambda_\nu^{-1}F(h^{-1}g)\otimes[E_j, E_i^*],$$

by Proposition 4.1.
Use the following identities

$$U(h^{-1}g:0) = U(h^{-1}, g.0)U(g:0),$$

and

$$\sum_{i,j} Ad(k^{-1})E_i Ad(k^{-1})E_j^* f(g) \otimes [E_j, E_i^*] = \sum_{i,j} E_i E_j^* f(g) \otimes [Ad(k)E_j, Ad(k)E_i^*],$$

for every $k \in K_c$ to obtain that

$$\tilde{\mathcal{H}}F(h^{-1}.(z)) = \tau_\nu(U(h^{-1}:z))Ad(U(h^{-1}:z))\tau_\nu(U(g:0))$$

$$\times\sum_{i,j} Ad(U(g:0)^{-1})E_i Ad(U(g:0)^{-1})E_j^*\Lambda_\nu^{-1}F(h^{-1}g)\otimes[E_j, E_i^*].$$

Next by using $\Lambda_\nu^{-1}F(h^{-1}g) = \Lambda_\nu^{-1}(T_\nu(h)F)(g)$ as well as (4.6) we get

$$\tau_{-\nu}(U(h^{-1}:z))\tilde{\mathcal{H}}F(h^{-1}.z) = Ad(U(h^{-1}:z))\tilde{\mathcal{H}}((T_\nu(h)F)(z),$$

and the proposition follows.

5 Necessity of the conditions

In this section we will establish that Poisson integrals are eigenfunctions of the Hua operator.

Lemma 5.1 *Let $F \in C^\infty(G/K, \tau)$. Then*

$$\mathcal{H}F = \frac{(\lambda^2 - (\eta-\nu)^2)}{4p}F.(-i)Z_0.$$

if and only if the function $\tilde{F} = \Lambda_\nu F$ satisfies $\tilde{\mathcal{H}}\tilde{F} = \frac{(\lambda^2-(\eta-\nu)^2)}{4p}\tilde{F}.(-i)Z_0$.

Proof. It follows by direct computations.

Let du be the normalized K-invariant measure on the Shilov boundary S. By using the operator Λ_ν we can rewrite the Poisson transform (4.4) from $B(S)$ to $C^\infty(\mathcal{D})$

$$P_{\lambda,\nu}f(z) = \int_S P_{\lambda,\nu}(z,u)f(u)du,$$

where

$$P_{\lambda,\nu}(z,u) = e^{-(\lambda+\eta)\rho_0(H_\Xi(g^{-1}k))}\tau_\nu(U(g:0))\tau_\nu(k^{-1}\kappa(g^{-1}k)),$$

with $z = g.o$ and $u = k.E_0$.

Next we introduce a K-invariant polynomial $h(z)$ on $\mathfrak{p}_+$ whose restriction on $\sum_{j=1}^r \mathbb{R}E_{\gamma_j}$ is given by

$$h(\sum_{j=1}^r a_j E_{\gamma_j}) = \Pi_{j=1}^r(1-a_j^2).$$

Let $h(z,w)$ denote its polarization, see [3] for more details.

Proposition 5.1 $i)P_{\lambda,\nu)}(z,v) = [\frac{h(z,z)}{|h(z,v)|^2}]^{\frac{\lambda+\eta-\nu}{2}} h(z,v)^{-\nu}$.

$$ii)P_{\lambda,\nu}(g.z,g.u) = P_{\lambda,\nu}(z,u)J_g(z)^{-\frac{\nu}{2p}}J_g(u)^{-\frac{\lambda+\eta-\nu}{2p}}\overline{J_g(u)}^{-\frac{\lambda+\eta+\nu}{2p}}. \tag{5.1}$$

Proof. i) Let

$$\Psi_{\lambda,\nu}(z) = e^{-(\lambda+\eta)\rho_0(H_\Xi(g^{-1}))}\tau_\nu(U(g:0))\tau_\nu(\kappa(g^{-1}) \quad z = g.0 \tag{5.2}$$

Observe that the right hand side of (5.2) is right K-invariant, hence $\Psi_{\lambda,\nu}(z)$ is well defined on $\mathcal{D}$. Define $\mu_\lambda \in \mathfrak{a}_c^*$ by $\mu_\lambda = \lambda\rho_0 + \rho_\Xi - \rho$. Then

$$e^{(\lambda\rho_0+\rho_\Xi)H_\Xi(g^{-1})} = e^{(\mu_\lambda+\rho)H(g^{-1})}.$$

Next recall that if $g = ne^{A(g)}\kappa_1(g)$ with respect to the decomposition $G = NAK$, then

$$A(g) = -H(g^{-1}), \kappa_1(g) = (\kappa(g^{-1}))^{-1}.$$

Henceforth

$$\Psi_{\lambda,\nu}(g) = e^{(\mu_\lambda+\rho)A(g)}\tau_\nu(U(g:0))\tau_{-\nu}(\kappa_1(g)). \tag{5.3}$$

But the right hand-side of (5.3) is nothing but the generalized Harish-Chandra c-function $e_{\lambda,\nu}$ on G/K, introduced in [17]. Accordingly to [17], in the Siegel domain realization T_Ω of G/K the function $e_{\lambda,\nu}$ is given by

$$\tilde{e}_{\lambda,\nu}(w) = \Delta_{\mu_\lambda+\rho}(\omega(w))\Delta(\omega(w))^{-\frac{\nu}{2}}, \quad w \in T_\Omega.$$

In above Δ denotes the Koecher norm function on $\mathfrak{p}_+$, and $\omega(w) = \frac{w+\bar{w}}{2}$, see [3] for more details. Let γ be the Cayley transform from $\mathcal{D}$ onto the Siegel realization T_Ω of G/K. Then we have

$$\Psi_{\lambda,\nu}(z) = \Delta(e-z)^{-\nu}\Delta_{\mu_\lambda+\rho}(\omega(\gamma(z)))\Delta(\omega(\gamma(z)))^{-\frac{\nu}{2}},$$

and since

$$h(z,z) = \Delta(e-z)\Delta(\omega(\gamma(z))\overline{\Delta(e-z)},$$

we get

$$\Psi_{\lambda,\nu}(z) = [\frac{h(z,z)}{\mid h(z,e)\mid}]^{-\frac{\nu}{2}} h(z,e)^{-\nu}.$$

Observing that $P_{\lambda,\nu}(z,v) = \Psi_{\lambda,\nu}(k^{-1}g)$ and that $h(z,w)$ is K bi-invariant we obtain the desired result.
ii) The identity (5.1) is easily derived from the following identity on the Jordan polynomial $h(z,w)$

$$h(g.z, g.w) = J_g(z)^{\frac{1}{p}} h(z,w)\overline{J_g(u)^{\frac{1}{p}}},$$

and the proof of Proposition 5.1 is finished.

Now we are ready to prove the main result of this section.

Proposition 5.2 *Let $F = P_{\lambda,\nu}f$ with $f \in B(G/P_\Xi, L_{\lambda,\xi})$. Then*

$$\mathcal{H}F = \frac{(\lambda^2 - (\eta-\nu)^2)}{4p} F.(-i)Z_0 \tag{5.4}$$

Proof. In view of Lemma 5.1 and the invariance property (5.1) of the Poisson kernel, it suffices to show that $\tilde{\mathcal{H}}P_{\lambda,\nu}(z,v)_{|z=0} = \frac{(\lambda^2-(\eta-\nu)^2)}{4p}.(-i)Z_0$.

Let E_j be an orthonormal basis of $\mathfrak{p}_+$ and E_j^* be a dual basis of $\mathfrak{p}_-$ with respect to B (for example $\{\tilde{E}_\alpha\}_{\alpha\in\Phi^+}$ and $\{\tilde{E}_{-\alpha}\}_{\alpha\in\Phi^+}$ are such basis). Let $z_1, ..., z_n$ be coordinates for $\mathfrak{p}_+$ with respect to E_j.

Regarding K-invariant functions on G as functions on $\mathcal{D}$ and vice versa, we have

$$E_i E_j^* F(e) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} F(0).$$

We know that

$$h(z,v)^{-\nu} = 1 + \nu < z, v > + \text{higher order homogeneous terms},$$

where $< z, v >= -B(z, \tau v)$. A simple computation gives

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} P_{\lambda,\nu}(z,v)_{|z=0} = \frac{(\lambda+\eta-\nu)(\lambda+\eta+\nu)}{4} v_j \bar{v}_i - \frac{\lambda+\eta-\nu}{2}\delta_{ij}.$$

Therefore

$$\tilde{\mathcal{H}}P_{\lambda,\nu}(z,v)_{|z=0} = \sum_{i,j}[E_j, E_i^*](\frac{(\lambda+\eta-\nu)(\lambda+\eta+\nu)}{4} v_j \bar{v}_i - \frac{\lambda+\eta-\nu}{2}\delta_{ij})$$

$$= \frac{(\lambda+\eta-\nu)(\lambda+\eta+\nu)}{4}[v, \bar{v}] - \frac{\lambda+\eta-\nu}{2} \sum_{\alpha\in\Phi^+} \tilde{H}_\alpha.$$

Notice that $[v,\bar{v}] = [Ad(k)\tilde{E_0}, Ad(\bar{k})\tilde{E_0}] = \sum_{j=1}^r \tilde{H_{\gamma_j}}$.
Since

$$\frac{1}{\eta}\sum_{\alpha\in\Phi^+}\tilde{H_\alpha} = \sum_{j=1}^r \tilde{H_{\gamma_j}},$$

we get

$$\tilde{\mathcal{H}}P_{\lambda,\nu}(z,v)_{|z=0} = \frac{(\lambda+\eta-\nu)(\lambda-\eta+\nu)}{4}\tilde{H_0},$$

As $\sum_{j=1}^r \tilde{H_{\gamma_j}} = \frac{-i}{p}Z_0$, the result follows.

6 The Hua eigensections

In this section we shall consider the following subsystem of the system (5.4)

$$\mathcal{H}_{\mathfrak{q}}F = \frac{(\lambda^2-(\eta-\nu)^2)}{4p}F.(-i)Z_0.$$

We prove the following result

Theorem 6.1 *Let $F\in B(G/K,\tau)$ such that*

$$\mathcal{H}_{\mathfrak{q}}F = \frac{(\lambda^2-(\eta-\nu)^2)}{4p}F.(-i)Z_0.$$

Then $F\in\mathcal{A}(G/K,\mathcal{M}_{\mu_\lambda,\nu})$.

Most of the Proof of Theorem 6.1 consists in proving the following

Theorem 6.2 *Let F be a τ-spherical function satisfying*

$$\mathcal{H}_{\mathfrak{h}}F = \frac{\lambda^2-(\eta-\nu)^2}{4p}F.(-i)Z_0. \tag{6.1}$$

Then up to a constant multiple F is given by ($a_t = \exp(\sum_{j=1}^r t_jX_{\gamma_j})$)

$$F(a_t) = \prod_{j=1}^r(1-\tanh^2 t_j)^{\frac{\lambda+\eta}{2}}\quad {}_2F_1^{(m)}(\frac{\lambda+\eta-\nu}{2},\frac{\lambda+\eta+\nu}{2},\eta;\tanh^2 t_1,...,\tanh^2 t_r).$$

In the above ${}_2F_1^{(m)}(\alpha,\beta,\gamma;x_1,....,x_r)$ is the generalized Gauss hypergeometric function, see [16] for more details.

Proof of Theorem 6.1. Let $F\in B(G/K,\tau_\nu)$ such that

$$\mathcal{H}_{\mathfrak{q}}F = \frac{\lambda^2-(\eta-\nu)^2}{4p}F.(-i)Z_0. \tag{6.2}$$

Let L be the Casimir operator. Then $\mathcal{H}_{\mathfrak{z}} = iL\otimes Z_0^*$.
Since $\mathfrak{z}\subset\mathfrak{h}$, then F satisfies

$$LF = r\frac{\lambda^2-(\eta-\nu)^2}{4}F.$$

Hence F is a real analytic function on G.
Next fix $g \in G$ and put $F_g(x) = \int_K F(gkx)\tau_\nu(k)dk$. Then F_g satisfies

$$F_g(k_1xk_2) = \tau_\nu^{-1}(k_1)F(_g(x)\tau_\nu^{-1}(k_2).$$

Since $\mathcal{U}(g_c)$ acts as left G-invariant differential operators, then F_g satisfies (6.2).
Let $\mu_\lambda = \lambda\rho_0 - \rho_\Xi + \rho$. Let $w \in W$ such that $w.H = H, \forall H \in a_\Xi$. Then we have

$$\Phi_{w\mu_\lambda,\nu} = P_{\lambda,\nu}1.$$

As $\Phi_{w\mu_\lambda,\nu} = \Phi_{\mu_\lambda,\nu}$ we deduce that $\Phi_{\mu_\lambda,\nu}$ is a τ_ν-spherical function satisfying (6.2). Therefore $F_g = c_g\Phi_{\mu_\lambda,\nu}$, by Theorem 6.2.
Hence

$$\int_K F(gkx)\tau_\nu(k)dk = \Phi_{\mu_\lambda,\nu}(x)\int_K F(gk)\tau_\nu(k)dk.$$

As in the case $\nu = 0$ the above functional equation characterizes the joint eigensections of $D_\nu(G/K)$, so $F \in \mathcal{A}(G/K, \mathcal{M}_{\mu_\lambda,\nu})$ and the proof of Theorem 6.1 is finished.

To prove Theorem 6.2 we shall need an explicit form of the radial components of the operator $\mathcal{H}_\mathfrak{h}$.

6.1 Radial components.

A function F on G will be called τ_ν-spherical if it satisfies

$$F(k_1gk_2) = \tau_\nu(k_2)^{-1}F(g)\tau_\nu(k_1)^{-1}.$$

Let $\overline{F}$ denote the restriction to A^+ of a τ-spherical function F. By the Cartan decomposition $G = KCl(A^+)K$ a τ_ν-spherical function F is essentially determined by $\overline{F}$.
For $U \in U(\mathfrak{g}_c)$, we denote by $\Delta_{\tau_\nu}(U)$ its τ-radial component. That is $\Delta_{\tau_\nu}(U) \in U(\mathfrak{a}_c)$ and satisfies

$$\overline{(UF)} = \Delta_{\tau_\nu}(U)\overline{F},$$

for every τ-spherical function F on G.
Let $H^*_{\gamma_k}$ $(k = 1, ..., r)$ be the basis of $\mathfrak{h}$ which is dual to H_{γ_k}. Then we have

$$\mathcal{H}_\mathfrak{h} = \sum_{k=1}^{r} U_k \otimes H^*_{\gamma_k},$$

where the components U_k are given by

$$U_k = \sum_{\alpha\in\Phi^+} \alpha(H_{\gamma_k})E_\alpha E^*_\alpha.$$

We have

$$\Delta_{\tau_\nu}(U_k) = \sum_{\alpha\in\Phi^+} \alpha(H_{\gamma_k})\Delta_\tau(E_\alpha E^*_\alpha).$$

Let F be a τ_ν-spherical function on G. Then the system $\mathcal{H}_\mathfrak{h}F = \frac{(\lambda^2-(\eta-\nu)^2)}{4p}F.(-i)Z_0$ reads as

$$(\Delta_\tau(U_k))\overline{F}(a) = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p}F, \tag{6.3}$$

for $k = 1, 2, ..., r$.
Recall that the dual basis E^*_α of $\mathfrak{p}_-$ is given by

$$E^*_\alpha = \frac{<\alpha, \alpha>}{2} E_{-\alpha}.$$

In view of Proposition 2.1, U_k can be rewritten as

$$U_k = \mid \gamma_k \mid^2 E_{\gamma_k} E_{-\gamma_k} + \frac{1}{2} \sum_j \sum_{\alpha \in \Phi^+_{jk}} <\alpha, \alpha> E_\alpha E_{-\alpha}.$$

To compute the τ-radial parts $\Delta_{\tau_\nu}(U_k)$ explicitly, we should determine $\Delta_\tau(E_\alpha E_{-\alpha})$ for all positive noncompact roots α.

The representation τ_ν of K induces differentiated representation of the Lie algebra $\mathfrak{k}$. We shall denote this representation by the same latter τ_ν.

Lemma 6.1 *Let F be a τ_ν-spherical function on G and let $\alpha \in \Phi_+$.*
i) If $\alpha \in \Gamma$ or $\alpha \in \Phi^+_{ij}$ with $\alpha \neq \tilde{\alpha}$, then

$$\tau_\nu(-iH_\alpha) F(a) = -i\nu$$

ii) If $\alpha \in \Phi^+_{ij}$ with $\alpha = \tilde{\alpha}$, then

$$\tau_\nu(-iH_\alpha) F(a) = -2i\nu$$

Proof. We have

$$B(-iH_\alpha, Z_0) = \frac{2}{\mid \alpha \mid^2}, \tag{6.4}$$

i) If $\alpha \in \gamma_k$ for some $k = 1, ..., r$ or $\alpha \in \Phi^+_{ij}$ with $\alpha \neq \tilde{\alpha}$ then

$$< Z - iH_\alpha, Z_0 >= 0,$$

by iv) of Proposition 2.1. Thus $Z - iH_\alpha \in \mathfrak{t}_s$. Therefore

$$\tau_\nu^{-1}(\exp t(-iH_\alpha) = e^{-i\nu t},$$

and i) follows.
ii) In the case $\alpha \in \Phi^+_{ij}$ with $\alpha = \tilde{\alpha}$ then

$$< Z - \frac{i}{2} H_\alpha, Z_0 >= 0,$$

from which we deduce that $\tau_\nu^{-1}(\exp t(\frac{i}{2} H_\alpha) = e^{-i\nu t}$. Hence ii).

Proposition 6.1

$$4\Delta_\tau(E_{\gamma_k} E_{-\gamma_k}) = X^2_{\gamma_k} + 2 \coth 2t_k X_{\gamma_k} - \nu^2 \tanh^2 t_k + 2\nu.$$

Proof.Write

$$E_{\gamma_k}E_{-\gamma_k} = \frac{1}{4}(X_{\gamma_k}^2 + Y_{\gamma_k}^2 + i[X_{\gamma_k}, Y_{\gamma_k}]),$$

and since

$$[X_{\gamma_k}, Y_{\gamma_k}] = -2iH_{\gamma_k},$$

we get

$$E_{\gamma_k}E_{-\gamma_k} = \frac{1}{4}(X_{\gamma_k}^2 + Y_{\gamma_k}^2 + i(-2iH_{\gamma_k})).$$

Hence we need to compute only $\Delta_\tau(Y_{\gamma_k}^2)$.
Let $a = \exp\sum_{j=1}^r t_j X_{\gamma_j}$. Then we have

$$Ad(a^{-1})iH_{\gamma_k} = (\cosh 2t_k)iH_{\gamma_k} + (\sinh 2t_k)Y_{\gamma_k}.$$

Thus

$$Y_{\gamma_k}^2 = (\coth 2t_{\gamma_k})(iH_{\gamma_k})^2 + \sinh^{-2} 2t_k(Ad(a^{-1})iH_{\gamma_k})^2$$
$$-\coth 2t_{\gamma_k}\sinh^{-1} 2t_k(iH_{\gamma_k}Ad(a^{-1})iH_{\gamma_k}) + (Ad(a^{-1})iH_{\gamma_k})iH_{\gamma_k}.$$

Observe that

$$[Ad(a^{-1})iH_{\gamma_k}, iH_{\gamma_k}] = (2\sinh 2t_k)X_{\gamma_k},$$

since $[Y_{\gamma_k}, iH_{\gamma_k}] = 2X_{\gamma_k}$.
Next, since F is a τ_ν-spherical function we have

$$(Ad(a^{-1})iH_{\gamma_k})iH_{\gamma_k}F(a) = -\nu^2 F(a),$$

hence

$$(iH_{\gamma_k}Ad(a^{-1})iH_{\gamma_k})F(a) = -2\sinh 2t_k X_{\gamma_k} - \nu^2,$$

from which we deduce that

$$Y_{\gamma_k}^2 = 2\coth 2t_k X_{\gamma_k} - \nu^2\tanh^2 t_k,$$

and the proof of Proposition 6.1 is finished.

Proposition 6.2 *let $\alpha \in \Phi^+$ such that $\alpha \sim \frac{\gamma_i+\gamma_j}{2}$ ($i \neq j$), with $\alpha \neq \widetilde{\alpha}$. Then we have*

$$\Delta_\tau(E_\alpha E_{-\alpha} + E_{\widetilde{\alpha}}E_{\widetilde{-\alpha}}) = \frac{1}{2}[\coth(t_i+t_j)(X_{\gamma_i} + X_{\gamma_j}) + \coth(t_i - t_j)(X_{\gamma_i} - X_{\gamma_j}) + 2\nu].$$

We first prepare the following Lemma

Lemma 6.2

$$i) Ad(a^{-1})U_1 = \cosh(t_i+t_j)U_1 - \sinh(t_i+t_j)(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}),$$
$$ii) Ad(a^{-1})U_2 = \cosh(t_i+t_j)U_2 - \sinh(t_i+t_j)(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}}),$$
$$iii) Ad(a^{-1})V_1 = \cosh(t_i-t_j)V_1 - \sinh(t_i-t_j)(X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}}),$$
$$iv) Ad(a^{-1})V_2 = \cosh(t_i-t_j)V_2 - \sinh(t_i-t_j)(Y_\alpha + \epsilon_\alpha Y_{\tilde{\alpha}}).$$

To prove the above lemma we need the following result from [10]

Lemma 6.3 *[10]. Let $\alpha \in \Phi_{ij}^+$. Then there exists U_1, U_2 in $\mathfrak{q}$ and V_1, V_2 in $\mathfrak{l}$ such that*

$$i) ad(X_{\gamma_k})U_1 = (\delta_{ik} + \delta_{jk})(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}),$$

$$ii) ad(X_{\gamma_k})(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}) = (\delta_{ik} + \delta_{jk})U_1,$$
$$iii) ad(X_{\gamma_k})U_2 = (\delta_{ik} + \delta_{jk})(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}}),$$
$$iv) ad(X_{\gamma_k})(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}}) = (\delta_{ik} + \delta_{jk})U_2,$$
$$v) ad(X_{\gamma_k})V_1 = (\delta_{ik} - \delta_{jk})(X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}}),$$
$$vi) ad(X_{\gamma_k})(X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}}) = (\delta_{ik} - \delta_{jk})V_1,$$
$$vii) ad(X_{\gamma_k})V_2 = (\delta_{ik} - \delta_{jk})(Y_\alpha + \epsilon_\alpha Y_{\tilde{\alpha}}),$$
$$viii) ad(X_{\gamma_k})(Y_\alpha + \epsilon_\alpha Y_{\tilde{\alpha}}) = (\delta_{ik} - \delta_{jk})V_2.$$

Proof of Lemma 6.2. We shall prove only i), the others assertions can be proved in a similar way.
we have

$$Ad(\exp(-\sum_{k=1}^{r} t_k X_{\gamma_k})U_1 = \exp(ad(-\sum_{k=1}^{r} X_{\gamma_k}))U_1.$$

By i) of the above lemma, we have

$$ad(-\sum_{k=1}^{r} X_{\gamma_k})^{2n}U_1 = (t_i + t_j)^{2n}U_1,$$

and

$$ad(-\sum_{k=1}^{r} X_{\gamma_k})^{2n+1}U_1 = (t_i + t_j)^{2n+1}(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}),$$

from which we get

$$Ad(a^{-1})U_1 = \cosh(t_i + t_j)U_1 - \sinh(t_i + t_j)(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}).$$

This finishes the proof.
Now we prove Proposition 6.2 giving the τ_ν-radial part of $E_\alpha E_{-\alpha} + E_{\tilde{\alpha}} E_{\widetilde{-\alpha}}$.

Proof of Proposition 6.2. First observe that

$$4(E_\alpha E_{-\alpha} + E_{\tilde{\alpha}} E_{-\tilde{\alpha}}) = X_\alpha^2 + X_{\tilde{\alpha}}^2 + Y_\alpha^2 + Y_{\tilde{\alpha}}^2 + i(2iH_\alpha + 2iH_{\tilde{\alpha}})).$$

Next, since

$$X_\alpha^2 + X_{\tilde{\alpha}}^2 = \frac{1}{2}[(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}})^2 + (X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}})^2],$$

we will compute $\Delta_\tau((X_\alpha \pm \epsilon_\alpha X_{\tilde{\alpha}})^2$.
To this end, consider the element U_2 given by Lemma 6.3 and let $a = \exp(\sum_{k=1}^{r} t_j X_{\gamma_j})$. Then we have

$$Ad(a^{-1})U_2 = \cosh(t_i + t_j)U_2 - \sinh(t_i + t_j)(X_\alpha + \epsilon X_{\tilde{\alpha}}),$$

from which we get

$$\sinh^2(t_i+t_j)(X_\alpha+\epsilon X_{\tilde{\alpha}})^2 = \cosh^2(t_i+t_j)U_2^2 - \cosh(t_i+t_j)U_2 Ad(a^{-1})U_2 - \\ \cosh(t_i+t_j)(Ad(a^{-1})U_2)U_2 + (Ad(a^{-1})U_2)^2.$$

Recall from [10], that $U_2 = i(Q_\alpha - \bar{Q_\alpha})$ where

$$Q_\alpha = N_{\alpha,-\gamma_i}E_{\alpha-\gamma_i} + N_{\alpha,-\gamma_j}E_{\alpha-\gamma_j},$$

from which we obtain $< U_2, Z_0 >= 0$, therefore $U_2 \in \mathfrak{q}\cap\mathfrak{t}_s$ and $\tau_\nu(U_2) = 0$.
By ii) in Lemma 6.2, we have

$$[Ad(a^{-1})U_2), U_2] = -2\sinh(t_i+t_j)(X_{\gamma_i}+X_{\gamma_j}),$$

which gives us

$$\Delta_\tau((X_\alpha+\epsilon_\alpha X_{\tilde{\alpha}})^2 = 2\coth(t_i+t_j)(X_{\gamma_i}+X_{\gamma_j}).$$

Similarly by considering V_1 and noticing that $V_1 \in \mathfrak{l}\cap\mathfrak{t}_s$ we get

$$\Delta_\tau((X_\alpha-\epsilon_\alpha X_{\tilde{\alpha}})^2 = 2\coth(t_i-t_j)(X_{\gamma_i}-X_{\gamma_j}).$$

Finally since $-2iH_{\tilde{\alpha}}F(a) = -2i\nu F(a)$,by Lemma 6.1, the result follows.

Proposition 6.3 *Let $\alpha\in\Phi^+$ such that $\alpha = \frac{\gamma_i+\gamma_j}{2}$ $(i\neq j)$, that is $\alpha = \widetilde{\alpha}$. Then we have*

$$\Delta_\tau(E_\alpha E_{-\alpha}) = \frac{1}{2}[\coth(t_i+t_j)(X_{\gamma_i}+X_{\gamma_j}) + \coth(t_i-t_j)(X_{\gamma_i}-X_{\gamma_j}) + 2\nu].$$

Proof. We first suppose that $\epsilon_\alpha = 1$. Accordingly to Proposition 13 and Proposition 14 in [10], $U_1 = V_1 = 0$.
We compute first $\Delta_\nu(X_\alpha^2)$ and $\Delta_\nu(Y_\alpha^2)$ in the case $\epsilon_\alpha = 1$.
We have

$$Ad(a^{-1})U_2 = \cosh(t_i+t_j)U_2 - 2\sinh(t_i+t_j)X_\alpha,$$

hence

$$4\sinh^2(t_i+t_j)X_\alpha^2 = \cosh^2(t_i+t_j)U_2^2 + (Ad(a^{-1})U_2)^2 - \cosh(t_i+t_j)(U_2Ad(a^{-1})U_2 + (Ad(a^{-1})U_2)U_2).$$

Noticing that

$$U_2Ad(a^{-1})U_2 = -[Ad(a^{-1})U_2, U_2] \\ = -8\sinh(t_i+t_j)(X_{\gamma_i}+X_{\gamma_j}),$$

we get

$$\Delta_\nu(X_\alpha^2) = 2\coth(t_i+t_j)(X_{\gamma_i}+X_{\gamma_j}),$$

and similarly

$$\Delta_\nu(Y_\alpha^2) = 2\coth(t_i-t_j)(X_{\gamma_i}-X_{\gamma_j}).$$

in the case $\epsilon_\alpha = -1$, analogous computations give

$$\Delta_\nu(Y_\alpha^2) = 2\coth(t_i+t_j)(X_{\gamma_i}+X_{\gamma_j}),$$

and

$$\Delta_\nu(X_\alpha^2) = 2\coth(t_i-t_j)(X_{\gamma_i}-X_{\gamma_j}).$$

To finish the proof of Proposition 6.3, notice that in the case $\alpha = \tilde{\alpha}$

$$-i2H_\alpha F(a) = -4i\nu.$$

Proposition 6.4 *The τ-radial part of the operator U_k is given by*

$$\frac{4}{|\gamma_k|^2}\Delta_\tau(U_k) = \frac{\partial^2}{\partial t_k^2} + 2\coth 2t_k \frac{\partial}{\partial t_k} - \nu^2 \tanh^2 t_k + 2\nu$$

$$+\frac{m}{2}\sum_{j\neq k}^{r}[\coth(t_j+t_k)(\frac{\partial}{\partial t_j}+\frac{\partial}{\partial t_k}) + \coth(t_j-t_k)(\frac{\partial}{\partial t_j}-\frac{\partial}{\partial t_k}) + 2\nu].$$

Proof. Recall that

$$\Delta_\tau(U_k) = \sum_{\alpha\in\Phi^+}\alpha(H_{\gamma_k})\Delta_\tau(E_\alpha E_\alpha^*).$$

By using (i), (ii) and (iii) of proposition 2.1, we obtain

$$\Delta_\tau(U_k) =< \gamma_k, \gamma_k > \Delta_\tau(E_{\gamma_k}E_{-\gamma_k}) + \frac{1}{2}\sum_{j\neq k}\sum_{\alpha\in\Phi_{jk}^+} < \alpha, \alpha > \Delta_\tau(E_\alpha E_{-\alpha}),$$

from which we deduce

$$\frac{4}{|\gamma_k|^2}\Delta_\tau(U_k) = \Delta_\tau(E_{\gamma_k}E_{-\gamma_k}) + \frac{m}{2}\sum_{j\neq k}[\coth(t_j+t_k)(X_{\gamma_j}+X_{\gamma_k}) + \coth(t_i-t_j)(X_{\gamma_j}-X_{\gamma_k}) + 2\nu],$$

by the results of Proposition 6.1, Proposition 6.2 and Proposition 6.3.

Next consider a coordinate system $t = (t_1, ..., t_r) \in \mathbb{R}^r \longmapsto \exp(\sum_{j=1}^r t_j X_{\gamma_j}) \in A$, such that the Weyl group W acts as the group of all permutations and sign changes of the coordinates $(t_1, ..., t_r)$ to get the result. This finishes the proof of Proposition 6.4.

6.2 Proof of Theorem 6.2

In this subsection we give the proof Theorem 6.2 which is the main step in the proof of our main result.

Proof of Theorem 6.2 . Let F be a τ-spherical function satisfying the system (6.1). Recall from subsection 6.1 (equation (6.3)) that F satisfies

$$(\Delta_\tau(U_k))\overline{F}(a) = \frac{(\lambda^2 - (\eta-\nu)^2)}{4p}F, \quad k = 1, ..., r$$

Let

$$\phi(t_1, ..., t_r) = (\prod_{j=1}^{r}\cosh t_j)^\nu F(t_1, ..., t_r).$$

It follows from Proposition 6.4 that the function $\phi(t_1, ..., t_r)$ satisfies the system of differential equations

$$\frac{\partial^2\phi}{\partial t_k^2} + 2\coth 2t_k\frac{\partial\phi}{\partial t_k} - 2\nu\tanh t_k\frac{\partial\phi}{\partial t_k} + \frac{m}{2}\sum_{j=1}^{r}\frac{1}{(\sinh^2 t_j - \sinh^2 t_k)}(\sinh 2t_j\frac{\partial\phi}{\partial t_j} - \sinh 2t_k\frac{\partial\phi}{\partial t_k})$$

$$= \frac{(\lambda^2-(\eta-\nu)^2)}{4}\phi,$$

for all $k=1,...,r$.

Put $x_i = -\sinh^2 t_i$ and $\psi(x_1,....,x_r) = \phi((t_1,...,t_r)$. Then the function ψ satisfies the system

$$x_k(1-x_k)\frac{\partial^2\psi}{\partial x_k^2} + (1-(2-\nu)x_k)\frac{\partial\psi}{\partial x_k} - \frac{m}{2}\sum_{j\neq k}\frac{x_j(1-x_j)}{x_k-x_j}\frac{\partial\psi}{\partial x_j} - \frac{x_k(1-x_k)}{x_k-x_j}\frac{\partial\psi}{\partial x_k}$$

$$= \frac{(\eta-\nu)^2-\lambda^2}{4}\psi,$$

for all $k=1,...,r$.
Since $\mathfrak{z}\subset\mathfrak{h}$, F is an eigenfunction of the Laplace-Beltrami operator (see the proof of Theorem 6.1). Hence F is analytic. Being τ-spherical F is W-invariant. Since the Weyl group W acts as the group of all permutations and sign changes of the coordinates $(t_1,...,t_r)$, it follows that ψ is a symmetric function of $x_1,...,x_r$ and analytic at $x_1=...=x_r=0$.
From Theorem 2.1 [16] we deduce that

$$\psi(x_1,...,x_r) = c \quad {}_2F_1^{(m)}(\frac{\lambda+\eta-\nu}{2}, \frac{-\lambda+\eta-\nu}{2}, \eta; x_1,...,x_r),$$

where c is some numerical constant.
Thus

$$F(a_t) = (\prod_{j=1}^{r}\cosh t_j)^{-\nu} \quad {}_2F_1^{m}(\frac{\lambda+\eta-\nu}{2}, \frac{-\lambda+\eta-\nu}{2}, \eta; -\sinh^2 t_1,..., -\sinh^2 t_r).$$

Next using the following Formula on the generalized Gauss hypergeometric function [16]

$${}_2F_1^{(m)}(\alpha,\beta,\gamma;y_1,....,y_r) = \prod_{j=1}^{r}(1-y_j)^{-a} \quad {}_2F_1^{(m)}(\alpha,\gamma-\beta,\gamma;\frac{y_1}{y_1-1},....,\frac{y_r}{y_r-1}).$$

we get

$$F(a_t) = \prod_{j=1}^{r}(1-\tanh^2 t_j)^{\frac{\lambda+\eta}{2}} \quad {}_2F_1^{(m)}(\frac{\lambda+\eta-\nu}{2}, \frac{\lambda+\eta+\nu}{2}, \eta; \tanh^2 t_1,...,\tanh^2 t_r),$$

and the proof of Theorem 6.2 is finished.

7 The sufficiency of the conditions

In this section we shall complete the proof of our main result.

Let $F\in\mathcal{B}(G/K,\tau_\nu)$ such that F satisfies the Hua system

$$\mathcal{H}F = \frac{(\lambda^2-(\eta-\nu)^2)}{4p}F.(-i)Z_0. \tag{7.1}$$

Then $F\in\mathcal{A}(G/K,\mathcal{M}_{\mu_\lambda,\nu})$, by Theorem 6.1.
By Theorem 3.1, it suffices to prove that the boundary value $\tilde{\beta}_{\mu_\lambda,\nu}F$ is in $\mathcal{B}(G/P_\Xi, L_{\lambda,\nu})$. To

do so, we will show that the induced equations of the subsystem $\mathcal{H}_s F = 0$ of (7.1)for boundary values on G/P characterize $\mathcal{B}(G/P_\Xi, L_{\lambda,\nu})$, see [6] for more details on induced equations that boundary values satisfy.
The method of the proof for $\nu = 0$ in [14] can be generalized to our situation, see also [10].
Below, we give an outline of the proof. Let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{q}$ with respect to B. Denote by C^+ the set of positive compact roots β such that $\beta \sim \frac{\gamma_j - \gamma_i}{2}$ for some $j > i$.
Let $\{S^*_\beta, \beta \in C^+\}$ be the basis of $\mathfrak{s}_c$ that is dual to $\{S_\beta, \beta \in C^+\}$, with respect to B. We have

$$\mathcal{H}_\mathfrak{s} = \sum_{\beta \in C^+} U_\beta \otimes S^*_\beta,$$

where $U_\beta \in \mathcal{U}(\mathfrak{g}_c)$ is given by

$$U_\beta = \sum_{\alpha \in \Phi^+} [S_\beta, E_\alpha] E^*_\alpha.$$

The condition $\mathcal{H}_\mathfrak{s} F = 0$ implies

$$U_\beta F = 0, \quad \forall \beta \in C^+,$$

Consider the Poincaré-Birkhoff-Witt Theorem decomposition

$$U(\mathfrak{g}_c) = U(\mathfrak{n}_c^- + \mathfrak{a}_c) + \sum_{X \in \mathfrak{k}_c} U(\mathfrak{g}_c)(X - \tau_\nu(X)), \tag{7.2}$$

and let Π_1 denote the projection of $U(\mathfrak{g}_c)$ to $U(\mathfrak{n}_c^- + \mathfrak{a}_c)$ with respect to the decomposition (7.2).
For $U \in U(\mathfrak{g}_c)$, let $\tilde{U}$ be the differential operator on $N^- \times \mathbb{R}^r$ defined by $\tilde{U} = P(\Pi_1(U))$ where P denotes the operator defined by $P(X_\alpha) = t^\alpha X_{-\alpha}$ and $P(H_j) = -t_j \frac{\partial}{\partial t_j}$, under the isomorphism from $\mathbb{R}^r$ onto A given by $t \to a(t) = \exp(-\sum_{t_j \neq 0} \log \mid t_j \mid H_j)$.
By the Iwasawa decomposition $G = N^- AK$, the restriction map from G to $N^- A$ gives an isomorphism from $\mathcal{B}(G/K, \tau_\nu)$ to $\mathcal{B}(N^- A)$.
Now F regarded as an element of $\mathcal{B}(N^- A)$ satisfies the differential equation

$$\tilde{U}_\beta F = 0, \quad \beta \in C^+.$$

Fix $\beta \in C^+$ such that $\beta \sim \frac{\gamma_i - \gamma_{i-1}}{2} \quad (2 \leq i \leq r)$.

Similar computations as in [[14], Proposition 4.4 and Proposition 4.5] show that the operator

$$t^{\frac{-1}{2}(\beta_i - \beta_{i-1})} \tilde{U}_\beta, \tag{7.3}$$

is well defined on $N^- \times \mathbb{R}$ and has analytic coefficients near $t = 0$,and that the induced equations for the system $\mathcal{H}_\mathfrak{s} F = 0$ are

$$Adc(E_{-\tilde{\beta}}) \beta_{\mu_\lambda, \nu}(F) = 0, \quad \forall \beta \in C^+; \beta \sim \frac{\gamma_i - \gamma_{i-1}}{2} \quad (2 \leq i \leq r).$$

To conclude recall that the vectors $\{Adc(E_{-\tilde{\beta}}), \beta \sim \frac{\gamma_i - \gamma_{i-1}}{2}\}$ span the root space $g_{\frac{\beta_i - \beta_{i-1}}{2}}$ and that $\{\frac{\beta_i - \beta_{i-1}}{2}, 2 \leq i \leq r\}$ are the simple roots of $\{\frac{\beta_i - \beta_j}{2}, 1 \leq j < i \leq r$. Thus

$$X_\alpha \beta_{\mu_\lambda, \nu}(F) = 0 \quad \forall \alpha; \quad \alpha = \frac{\beta_i - \beta_j}{2},$$

with $1 \leq j < i \leq r$.
This shows that $\tilde{\beta}_{\mu_\lambda, \nu} F \in \mathcal{B}(G/P_\Xi, L_{\lambda,\nu})$ and the proof the main result of this paper is finished.

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