Invariant Subspace Problem: $X$ is a Hilbert or Banach space ①

$T: X \to X$ linear and cts Does there exist a closed $T$ invariant subspace $A$ which is non-trivial? ($\bar{A} = A$, $T(A) \subset A$, $A \neq \{0\}$ or $X$)

Let $x \in X$, What is smallest closed $T$-invariant subspace?

$\overline{span}\{x, Tx, T^2x, T^3x, --\} := \overline{span}(orbit(x,T))$

Can $\overline{span}(orbit(x,T)) = X$? , If equality holds then $x$ is called a cyclic vector for $T$. also $T$ is said to be cyclic.

$T$ is a counterexample to ISP iff each $x \in X$ with $x \neq 0$ is cyclic for $T$.

P. Enflo (1987): Gave a counter-example on a Banach space.

What about a $T$-invariant closed non-trivial subset?

$x \in X$, smallest $T$-invariant closed subset $\overline{\{x, Tx, T^2x, \dots\}} = X$ $(x \neq 0)$.

If equality holds then $x$ is said to be a hypercyclic vector for $T$ and $T$ is said to be hypercyclic.

$T$ is a counterexample to ISP iff the set of hypercyclic vectors for $T = X \setminus \{0\}$

C Read (1988) $\exists T: \ell^1 \to \ell^1$ s.t $HC(T) = \ell^1 \setminus \{0\}$.

Both problems are open in Hilbert spaces.

Having dense orbit appears in Topological Dynamics:

$(X,T)$ is a dynamical system if $T: X \to X$ is cts and $X$ is a complete metric space.

Definition: $(X,T)$ a dynamical system. $T$ is said to be topologically transitive if for any two non-empty open subset $U, V$ of $X$, $\exists n \in \mathbb{N}$ s.t

$$T^n(U) \cap V \neq \emptyset$$

Birkhoff's theorem: If $X$ is separable & has no isolated points then $T$ is topologically transitive iff $T$ has a dense orbit. In this case the set vectors with a dense orbit is a dense $G_\delta$-set.

Proof: Let $\{U_k\}_k$ be a basis for the topology for $X$, the set of vectors with a dense orbit $= \bigcap_{k=1}^{\infty} \underbrace{\bigcup_{n=0}^{\infty} T^{-n}(U_k)}_{dense}$ by Baire Category this set is dense and $G_\delta$.

Dense orbits appears in most definitions of Chaos:

Let $(X,T)$ be a dynamical system $T$ is chaotic if

1) $T$ has a dense orbit ($T$ is transitive)
2) $T$'s set of periodic points of $T$ is dense ($x \in per(T)$ if $T^n x = x$ for some $n \in \mathbb{N}$)
3) $T$ has sensitive dependence to initial conditions:
   $\exists \delta > 0$ s.t for any $x \in X$, $\varepsilon > 0$, $\exists n \in \mathbb{N}$ and $y \in X$ s.t

$$d(x,y) < \varepsilon, \text{ and } d(T^n x, T^n y) > \delta.$$

Bank et al (1992) (i) and (ii) $\Rightarrow$ (iii)

**Definition**: Let $S: Y \to Y$ and $T: X \to X$ be Dynamical systems.

1) $T$ is called quasiconjugate to $S$ if $\exists\, \phi: X \to X$ cts with a dense range s.t $T \circ \phi = \phi \circ S$ i.e the diagram commute.

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \phi \downarrow & & \downarrow \phi \\ X & \xrightarrow{T} & X \end{array}$$

2) If $\phi$ is a homeomorphism then $T$ is conjugate to $S$.

**Thm**: If $T$ is a quasiconjugate to $S$ then $T$ is topologically transitive, per$(T)$ is dense or is chaotic, if $S$ is top. trans. per$(S)$ is dense or is Chaotic.

(i) Assume $S$ is top. trans. choose $U, V \subset X$ s.t $U, V$ open, non-empty. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are non-empty open. Thus $\exists n \in \mathbb{N}$, $\phi$ and $y \in \phi^{-1}(U)$ s.t $S^n y \in \phi^{-1}(V)$ Then $\phi(y) \in U$ and $T^n(\phi(y)) = \phi(S^n y) \in V$.

(ii) Let per$(S)$ be dense in $Y$. Let $U \subset X$ be non-empty open so is $\phi^{-1}(U)$. $\exists y \in \phi^{-1}(U)$ s.t $S^n y = y$ but then $\phi(y) \in U$ and $T^n(\phi(y)) = \phi(S^n y) = \phi(y)$, $\phi(y) \in$ per$(T)$

**Corollary**: If $S$ is chaotic then $T$ is chaotic.

**Definition**: $(X, T)$ is dynamical system. $T$ is mixing if for any non-empty open $U, V \subset X$ $\exists N \in \mathbb{N}$ s.t $T^n(U) \cap V \neq \emptyset$ $\forall n \geq N$.

$S: Y \to Y$, $T: X \to X$

$S \times T$ is mixing on $Y \times X$ iff $S$ and $T$ are mixing. $S \times T$ is topologically transitive or chaotic or has a dense orbits $\underset{\not\Leftarrow \text{ not always true}}{\Rightarrow}$ $S$ and $T$ are top-trans. chaotic have a dense orbit.

$$T \text{ is mixing} \underset{\Leftarrow ?}{\Rightarrow} \underbrace{T \times T \text{ is top. transitive}}_{\text{weakly mixing property}} \underset{\Leftarrow}{\overset{?}{\Rightarrow}} T \text{ is top. transitive}$$

counterexample

**Ex**:

$T: \mathbb{T} \to \mathbb{T}$ is the $\partial \mathbb{D} \subset \mathbb{C}$
unit circle

$z \to e^{2\pi i \alpha} z$ $\alpha$ is irrational.

$\omega = e^{2\pi i \alpha}$ orb$(1, T) = \{1, \omega, \omega^2, \omega^3, \dots\}$ since $\alpha$ is irrational ↳ these are all distinct.

$\omega$ is not a root of unity.

orb$(1,T)$ has a limit point in $\mathbb{T}$

$$0 = \lim_{k \to \infty} |\omega^{n_{k+1}} - \omega^{n_k}| = \lim_k \underbrace{|\omega^{n_k}|}_{1} |\omega^{n_{k+1} - n_k} - 1| \quad \forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ s.t } |\omega^n - 1| < \varepsilon.$$

$y' = \omega^n$.

$\{y, y^2, y^3, \dots\} \subset$ orb$(1, T)$ is a partition of $\mathbb{T}$ with arcs with length $< \varepsilon$.

thus $T$ has a dense orbit. orb$(x, T) = x\,$orb$(1, T)$ is also dense.

$T \times T: \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ $(T^n \times T^n)(z_1, z_2) = (T^n z_1, T^n z_2) = (e^{2\pi i n \alpha} z_1, e^{2\pi i n \alpha} z_2)$

$\dfrac{e^{2\pi i n \alpha} z_1}{e^{2\pi i n \alpha} z_2} = \dfrac{z_1}{z_2}$ doesn't depend on $n$. thus $T \times T$ is not a weakly-mixing.

Thm: (Blow up/Collapse): $T$ is weakly mixing iff For any non-empty open $U, V, W \subset X$ with $0 \in W$, $\exists n \in \mathbb{N}$ s.t

$$T^n(U) \cap W \neq \emptyset$$
$$T^n(W) \cap V \neq \emptyset$$

When $T$ is linear.

LINEAR DYNAMICAL SYSTEMS:

Definition: A functional $p: X \to \mathbb{R}_+$ (vector space) is a seminorm if

(i) $p(x+y) \leq p(x) + p(y)$

(ii) $p(\lambda x) = |\lambda| p(x)$

$p$ is a norm if $p(x)=0 \Rightarrow x=0$.

Definition: $(p_n)_n$, a sequence of seminorms, is separating if $p_n(x)=0 \;\forall n \geq 1 \Rightarrow x=0$.

Note: We can always assume $(p_n)$'s are increasing defining $q_n = \max_{k \leq n} p_k$

Definition: $X$ is a Fréchet space if it's endowed with an increasing seperating sequence of seminorms and which is complete in the metric given by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x-y)).$$

Prop: $x_n \to x$ in Fréchet space $X$ $\Leftrightarrow$ $p_k(x_n - x) \to 0$ as $n \to \infty$ for all $k \geq 1$.

$$d(x,y) = d(x+z, y+z)$$

Ex: $H(\mathbb{C}) = \{ f : f \text{ is holomorphic on } \mathbb{C} \}$.

$$p_n = \sup_{|z| \leq n} |f(z)|$$

Definition: $(X,T)$ is a linear dynamical system if $T: X \to X$ linear, cts and $X$ Fréchet

Definition: $T$ is hypercyclic if $\exists x \in X$ s.t $orb(x,T)$ is dense in $X$. Such an $x$ is called hypercyclic vector of $T$. $HC(T)$ is the set of hypercyclic vectors of $T$.

Hypercyclic $\Leftrightarrow$ Top. Transitive.

Ex: $T_a : H(\mathbb{C}) \to H(\mathbb{C})$
$f(z) \to f(z+a)$
$T$ is hypercyclic $\Leftrightarrow$ $a \neq 0$.

Ex: $D: H(\mathbb{C}) \to H(\mathbb{C})$
$f \to f'$
$D$ is hypercyclic. (Maclane 1952)

Proof: Let $U, V \subset X$ be nonempty open polynomials $p \in U$ and $q \in V$.

s.t $p(z) = \sum_{k=0}^{N} a_k z^k$ and $q(z) = \sum_{k=0}^{N} b_k z^k$

Define $r_n(z) = p(z) + \sum_{k=0}^{N} \frac{k!}{(k+n)!} b_k z^{k+n}$

$$D^n r_n(z) = q(z) \in V \qquad (n \geq N+1)$$

$$\sup_{|z| \leq R} |r_n(z) - p(z)| \leq \sum \frac{k! |b_k|}{(k+n)!} R^{k+n} \to 0.$$

For large enough $n$, $r_n(z) \in U$

so $D^n(U)\cap V\neq\emptyset$. $D$ is topolog. trans., $D$ is hypercyclic.

Ex: (Rolewicz 1965) $\ell^p$ – sequence space.
$c_0$

$\lambda B(x_1,x_2,x_3,\dots)=\lambda(x_2,x_3,x_4,\dots)$ then $\|B\|=1$.

$\lambda B$ is hypercyclic iff $|\lambda|>1$.

Ex: $2B:\ell^2\to\ell^2$ is hypercyclic.

- let $\{y^{(k)}\}_{k\ge1}$ be a countable basis for $\ell^2$. $m_k$ is the biggest index s.t $y^k_{m_k}\neq0$.
- $S:=\frac{1}{2}F$ $\quad F(x_1,x_2,\dots)=(0,x_1,x_2,\dots)$

then $2BS=I$. Choose by induction a sequence $(n_k)_k\subset\mathbb{N}^{\mathbb{N}}$ which is increasing s.t $n_k\ge m_j+n_j$ and $2^{n_k}\ge 2^{n_j+k}\|y^{(k)}\|$ $\quad k>j\ge1$.

Claim: $x=\sum_{k=1}^{\infty}S^{n_k}y^{(k)}$ is hypercyclic vector for $T$.

$$\|x\|^2\le\sum_{k=1}^{\infty}\|S^{n_k}y^{(k)}\|=\sum 2^{-n_k}\|y^{(k)}\|\le\sum_{k=1}^{\infty}2^{-k}<\infty.$$

$$T^{n_k}x=\underbrace{\sum_{j=1}^{k-1}T^{n_k}S^{n_j}y^{(j)}}+T^{n_k}S^{n_k}y^{(k)}+\sum_{j=k+1}^{\infty}T^{n_k}S^{n_j}y^{(j)}$$

$$=\underbrace{\sum_{j=1}^{k-1}2^{n_k-n_j}B^{n_k-n_j}y^{(j)}}_{=0}+y^{(k)}+\sum_{j=k+1}^{\infty}T^{n_k}S^{n_j}y^{(j)}$$

$$\|T^{n_k}x-y^{(k)}\|\le\sum_{j=k+1}^{\infty}2^{n_k-n_j}\|F^{n_j-n_k}y^{(j)}\|\le\sum 2^{n_k-n_j}\|y^{(j)}\|\le\sum_{j=k+1}^{\infty}2^{-j}=2^{-k}\to0.$$

$$\boxed{\{y^{(k)}\}=X_0=\text{dense set}\qquad \begin{array}{l}T^{n_k}\to0 \text{ ptw on } X_0\\ S^{n_k}\to0 \text{ " " "}\end{array}\qquad T^{n_k}S^{n_k}=0}$$
$\Rightarrow$ IDEA behind the proof.

Hypercyclicity Criterion: $(X,T)$ linear dynamical system. If $\exists$ dense subsets $X_0,Y_0$ of $X$, an incr. sequence $(n_k)_{k\ge0}$ and map $S_k:Y_0\to X$ s.t

(i) $T^{n_k}\to0$ ptw on $X_0$
(ii) $S^{n_k}\to0$ ptw on $Y_0$
(iii) $T^{n_k}S_k\to Id$ ptw on $Y_0$

} Then $T$ is hypercyclic

Proof: $\cdot x\in X_0$ $U$ $\quad$ $\cdot y\in Y_0$ $V$ $\quad z_k:=x+S_ky\to x$ ($S_ky\to0$)

$$T^{n_k}z_k=\underbrace{T^{n_k}x}_{\downarrow 0}+\underbrace{T^{n_k}S_ky}_{\downarrow y}\to y$$

$T^{n_k}(U)\cap V\neq\emptyset$ for large enough $k$.

<u>Corollary</u>: If $T$ satisfies H.C. then $T \oplus T$ satisfies H.C. $\Rightarrow T \oplus \cdots \oplus T$ is hypercyclic

<u>proof</u>: choose $X_0 \times X_0$, $Y_0 \times Y_0$ and $S_k \oplus S_k$

(Bès & Peris 1999) $T$ is weak mixing $\Leftrightarrow T$ satisfies H.C

(C. Read, M. De la Rosa 2009): $\exists T: \ell^1 \to \ell^1$ s.t $T$ is hypercyclic but not weakly mixing that is $T$ does not satisfy H.C.

<u>Prop</u>: $(X,T)$ linear dynamical system $T$ is chaotic $\Rightarrow T$ is weakly mixing

<u>Proof</u>: $U_1, U_2, V_1, V_2 \subset X$ non-empty open. $(U \times V)$ is basis for topology of $X \times X$

Want to show $\exists n \in \mathbb{N}$ s.t $\begin{cases} T^n(U_1) \cap V_1 \neq \emptyset \\ T^n(U_2) \cap V_2 \neq \emptyset \end{cases}$

Choose $m$ s.t $T^m(U_1) \cap V_1 \neq \emptyset$, we can find $u_1 \in \mathrm{Per}\,T$ s.t $u_1 \in U_1$, $T^m u_1 \in V_1$

(Theorem of Ansari: $T$ is hypercyclic $\Rightarrow T^n$ is hypercyclic $\forall n \geq 1$) and since $u_1 \in \mathrm{Per}\,T$ $T^p u_1 = u_1 \in U_1$. Then $\exists r \in \mathbb{N}$ s.t $T^{rp}(U_2) \cap T^{-m}(V_2) \neq \emptyset$.

define $n = rp + m$ so we have $T^n(U_2) \cap V_2 \neq \emptyset$ and $T^n u_1 = T^{rp+m} u_1$
$= T^m(T^{rp} u_1) \in V_1$

so $T^n(U_1) \cap V_1 \neq \emptyset$.

<u>Prop</u>: If $T: X \to X$ is hypercyclic then $T^*$ can't have an eigenvalue.

$X^*$ is the set of all cts functionals $x^*(x) = \langle x, x^* \rangle$

$T^*: X^* \to X^*$ $T^*(x^*)x = x^*(Tx) \Leftrightarrow \langle x, T^* x^* \rangle = \langle Tx, x^* \rangle$

Assume $T^* x^* = \lambda x^*$, let $x$ be a hypercyclic vector for $T$

$\langle T^n x, x^* \rangle = \langle x, (T^*)^n x^* \rangle = \langle x, \lambda^n x^* \rangle = \lambda^n \langle x, x^* \rangle$ can't be dense in $\mathbb{C}$
Contradiction.