Asymptotics of spectral gaps of 1D Dirac operator with two exponential terms potential

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Abstract

The one-dimensional Dirac operator

$$L = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad P, Q \in L^2([0,\pi]),$$

considered on $[0,\pi]$ with periodic and antiperiodic boundary conditions, has discrete spectra. For large enough $|n|$, $n \in \mathbb{Z}$, there are two eigenvalues λ_n^-, λ_n^+ such that $|\lambda_n^\pm - n| < 1/2$.
We study the asymptotics of spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ for

$$P(x) = ae^{-2ix} + Ae^{2ix}, \quad Q(x) = be^{-2ix} + Be^{2ix},$$

where a, A, b, B are nonzero complex numbers. We show, for large enough m, that $\gamma_{\pm 2m} = 0$ and

$$\gamma_{2m+1} = \pm 2 \frac{\sqrt{(Ab)^m (aB)^{m+1}}}{4^{2m}(m!)^2} \left[1 + O\left(\frac{\log^2 m}{m^2} \right) \right].$$

We consider one-dimensional Dirac operators of the form

$$L(v)y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v(x)y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

with π-periodic complex valued functions $P, Q \in L^2([0,\pi])$.

Boundary conditions:

$$\begin{aligned} Periodic, \quad & Per^+ : y(\pi) = y(0); \\ Antiperiodic, \quad & Per^- : y(\pi) = -y(0). \end{aligned}$$

If $v \equiv 0$, then $L_{Per^\pm}(0)$ is denoted by $L^0_{Per^\pm}$ and is called the free Dirac operator.

$$Dom(L^0_{Per^\pm}) = \{y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1 \text{ and } y_2 \text{ are absolutely continuous,}$$
$$y \text{ satisfies } Per^\pm, \text{ and } y_1', y_2' \in L^2([0,\pi])\}$$

$L^0_{Per^\pm}$ is a closed, densely defined operator and it has a discrete spectrum consisting only of its eigenvalues. Namely;

$$Sp(L^0_{Per^+}) = 2\mathbb{Z}, \quad Sp(L^0_{Per^-}) = 2\mathbb{Z} + 1.$$

Each eigenvalue n, both for Per^+ (if n is even), or Per^- (if n is odd) has multiplicity 2 and

$$e_n^1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-inx}, \quad e_n^2(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{inx}$$

are eigenfunctions corresponding to the eigenvalue of n.

Introduction

If the Hilbert space $\mathbb{H} = L^2[0,\pi] \times L^2[0,\pi]$ is equipped with the scalar product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \int_0^\pi (f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)})\, dx,$$

then each of the systems

$$\{e^1_{2k}, e^2_{2k}, \quad k \in \mathbb{Z}\}, \quad \{e^1_{2k+1}, e^2_{2k+1}, \quad k \in \mathbb{Z}\},$$

is an orthonormal basis in $\mathbb{H}$.

The operator $L(v)$ is symmetric if and only if $Q(x) = \overline{P(x)}$; then L gives rise to a self-adjoint operator in $L^2(\mathbb{R}, \mathbb{C}^2)$ whose spectrum has a band-gap structure, i.e., $Sp(L) = \mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}}(\lambda_n^-, \lambda_n^+)$. The points λ_n^-, λ_n^+ are eigenvalues of the same operator L subject to periodic $Per^{\pm}$ and $\gamma_n = \lambda_n^+ - \lambda_n^-$ is called a *spectral gap*.

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Localization Lemma

The spectra of $L_{Per^{\pm}}(v)$ are discrete. There is $N_0 = N_0(v)$ such that the union $\cup_{|n| > N_0} D_n$, where $D_n = \{z : |z - n| < \frac{1}{2}\}$, contains all but finitely many of the eigenvalues of $L_{Per^{\pm}}(v)$.

Moreover, each disc D_n, $|n| > N_0$, contains exactly two (counted with algebraic multiplicity) periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+.

Hill-Schrödinger operators $Ly = -y'' + v(x)y$

Hochstadt '63,'65

If (A) $v \in C^\infty$, then (B) $\gamma_n = o(1/n^p), \forall p$. Moreover, if a continuous function v is a finite-zone potential, i.e., $\gamma_n = 0$ for large enough n, then $v \in C^\infty$.

Marchenko, Ostrovskii, McKean, Trubowitz mid-70s

(B) $\Rightarrow$ (A), for real $L^2([0,\pi])$ potentials v .

Trubowitz '77

$L^2([0,\pi])$-potential v is analytic if and only if (γ_n) decays exponentially.

Hill-Schrödinger operators $Ly = -y'' + v(x)y$

In the case of Mathieu potential $v(x) = 2a\cos(2x)$, the spectral gaps $\gamma_n(v) \neq 0$.

Harrell, Avron, Simon '81

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}(1 + o(1/n^2)).$$

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$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}(1 + o(1/n^2)).$$

Anahtarci, Djakov '12 (JMAA)

$$\gamma_n = \pm\frac{8(|a|/4)^n}{[(n-1)!]^2}(1 - a^2/4n^3 + O(1/n^4)).$$

Dirac operators

In the case of potential $v(x) = \begin{pmatrix} 0 & 2a\cos(2x) \\ 2a\cos(2x) & 0 \end{pmatrix}$,

Djakov, Mityagin '03

$\gamma_{\pm 2m} = 0$ for $m \in \mathbb{Z}_+$ and

$$\gamma_{-(2m+1)} = \gamma_{2m+1} = 2|a| \frac{a^{2m}}{4^{2m}(m!)^2} \left[1 + O\left(\frac{\log m}{m} \right) \right].$$

Dirac operators

In the case of potential $v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}$ with

$$P(x) = ae^{-i2x} + Ae^{i2x}, \quad Q(x) = be^{-i2x} + Be^{i2x},$$

Anahtarci, Djakov '13

$\gamma_{\pm 2m} = 0$ for $m \in \mathbb{Z}_+$ and

$$\gamma_{2m+1} = \pm 2 \frac{\sqrt{(Ab)^m (aB)^{m+1}}}{4^{2m}(m!)^2} \left[1 + O\left(\frac{\log^2 m}{m^2} \right) \right],$$

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The operator $L = L^0 + V$ is considered on the Hilbert space $\mathbb{H}$. Let $E_n^0 = Span\{e_n^1, e_n^2\}$ be the eigenspace of L^0 corresponding to the eigenvalue n, and let P_n^0 be the orthogonal projection onto E_n^0. Set $Q_n^0 = 1 - P_n^0$, so $\mathbb{H} = E_n^0 \oplus H_n$, where H_n is the range of Q_n^0.

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Consider $(\lambda - L)f = 0$, where $\lambda = n + z$ with $|z| \leq 1/2$. With $f_1 = P_n^0 f$ and $f_2 = Q_n^0 f$, this equation is equivalent to the system:

$$P_n^0(\lambda - L^0 - V)(f_1 + f_2) = 0,$$
$$Q_n^0(\lambda - L^0 - V)(f_1 + f_2) = 0.$$

It turns out that $(z - S)f_1 = 0$, where $S : E_n^0 \to E_n^0$ and

$$S = P_n^0 V + P_n^0 V(1 - R_\lambda^0 Q_n^0 V)^{-1} R_\lambda^0 Q_n^0 V.$$

Denote the 2-dimensional operator $S := \begin{pmatrix} \alpha_n(z) & \beta_n^-(z) \\ \beta_n^+(z) & \alpha_n(z) \end{pmatrix}$.

$\lambda = n + z$ with $|z| \leq 1/2$ is an eigenvalue of $L_{Per^\pm}(v)$ if and only if z is a solution of $(z - \alpha_n(z))^2 = \beta_n^-(z)\beta_n^+(z)$.

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$$\alpha_n = \sum_{\nu=1}^{\infty} \sum_{j_1,\dots,j_{2\nu-1} \neq n} \frac{p(-n-j_1)q(j_1+j_2)\cdots p(-j_{2\nu-2}-j_{2\nu-1})q(j_{2\nu-1}+n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-2}+z)(n-j_{2\nu-1}+z)}$$

$$\beta_n^- = \sum_{\nu=1}^{\infty} \sum_{j_1,\dots,j_{2\nu} \neq n} \frac{p(-n-j_1)q(j_1+j_2)\cdots q(j_{2\nu-1}+j_{2\nu})p(-j_{2\nu}-n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-1}+z)(n-j_{2\nu}+z)}$$

$$\beta_n^+ = \sum_{\nu=1}^{\infty} \sum_{j_1,\dots,j_{2\nu} \neq n} \frac{q(n+j_1)p(-j_1-j_2)\cdots p(-j_{2\nu-1}-j_{2\nu})q(j_{2\nu}+n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-1}+z)(n-j_{2\nu}+z)}$$

Due to Localization Lemma, the basic equation

$$(z - \alpha_n(z))^2 = \beta_n^-(z)\beta_n^+(z)$$

has exactly two roots z_n^- and z_n^+ in the disc $|z| \leq 1/2$. Therefore, it splits into two equations

$$z_n^+ - \alpha_n(z_n^+) - \sqrt{\beta_n^-(z_n^+)\beta_n^+(z_n^+)} = 0,$$
$$z_n^- - \alpha_n(z_n^-) + \sqrt{\beta_n^-(z_n^-)\beta_n^+(z_n^-)} = 0.$$

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$$z_n^\pm = \lambda_n^\pm - n = \frac{Ab + aB}{2n} + \frac{aB - Ab}{2n^2} + O\left(1/n^3\right).$$

$$\alpha_n(z_n^+) - \alpha_n(z_n^-) = \gamma_n O\left(1/n^2\right).$$

$$\gamma_n \sim \pm 2\sqrt{\beta_n^-(z_n^\pm)\beta_n^+(z_n^\pm)}.$$

A walk x on the integer grid $\mathbb{Z}$ from a to b (where $a, b \in \mathbb{Z}$) is given by a finite sequence of integers $x = (x_t)_{t=1}^{\mu}$ with $x_1 + x_2 + \ldots + x_\mu = b - a$. The numbers

$$j_k = a + \sum_{t=1}^{k} x_t, \quad 1 \leq k < \mu$$

are known as *vertices* of the walk x.

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$$P(x) = ae^{-i2x} + Ae^{i2x}, \quad Q(x) = be^{-i2x} + Be^{i2x}$$

Fourier coefficients of P and Q:

$$p(-2) = a, \;\; p(2) = A; \quad q(-2) = b, \;\; q(2) = B,$$

$$p(m) = q(m) = 0 \;\; \text{for } m \neq \pm 2.$$

$$\beta_n^+ = \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_{2\nu}\neq n} \frac{q(n+j_1)p(-j_1-j_2)\cdots p(-j_{2\nu-1}-j_{2\nu})q(j_{2\nu}+n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-1}+z)(n-j_{2\nu}+z)}$$

There is one-to-one correspondence between the nonzero terms of $\beta_n^+(z)$ and the *admissible* walks $x = (x_t)_{t=1}^{2\nu+1}$ on $\mathbb{Z}$ from $-n$ to n with steps $x_t = \pm 2$ such that $j_1, j_3, \ldots, j_{2\nu-1} \neq n$ and $j_2, j_4, \ldots, j_{2\nu} \neq -n$.

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$$h^+(x,z) = \frac{q(x_1)p(x_2)q(x_3)\cdots p(x_{2\nu})q(x_{2\nu+1})}{(n-j_1+z)(n+j_2+z)\cdots(n-j_{2\nu-1}+z)(n+j_{2\nu}+z)}$$

Let $X_n(r),\ r = 0, 1, 2, \ldots$ denote the set of all admissible walks from $-n$ to n, with r negative steps. Then, every walk $x \in X_n(r)$ has totally $n + 2r$ steps and we have

$$\beta_n^+(z) = \sum_{r=0}^{\infty} \sigma_r^+(n,z) \quad \text{with} \quad \sigma_r^+(n,z) = \sum_{x\in X_n(r)} h^+(x,z).$$