RESIDUAL QUOTIENTS

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efficacious in every way

QUOTIENTS

$$AJ + JA \subseteq J$$

$$J \in LI(A) \cap RI(A)$$

$$A/J = \{a + J : a \in A\}$$

$$(a' + J) + (a + J) = a' + a + J$$

$$(a' + J)(a + J) = a'a + J$$

RESIDUAL QUOTIENTS

$$K : H \text{ ?}$$

$$KH = \{xy : x \in K, y \in H\} \subseteq A$$

$$K^{-1}H = \{x \in A : Kx \subseteq H\}$$

$$HK^{-1} = \{x \in A : xK \subseteq H\}$$

INTEGERS

$$A = \mathbf{Z}$$

$$[n] = \mathbf{Z}n$$

$$k = \mathsf{hcf}(m, n) \Longrightarrow [m]^{-1}[n] = [k^{-1}n]$$

$$H' \subseteq H, K \subseteq K' \Longrightarrow K'^{-1}H' \subseteq K^{-1}H$$

$$KL \subseteq H \Longleftrightarrow L \subseteq K^{-1}H \Longleftrightarrow K \subseteq LH^{-1}$$

$$L(KL)^{-1}H \subseteq K^{-1}H$$

$$(K^{-1}H)L \subseteq K^{-1}(HL)$$

$$K^{-1}H + K^{-1}H' \subseteq K^{-1}(H + H')$$

$$J \in LI(A) \iff AJ \subseteq J \iff J \subseteq A^{-1}J$$

$$H \subseteq K \Longrightarrow (K^{-1}H)(K^{-1}H) \subseteq K^{-1}H$$

$$K \subseteq H \Longrightarrow 1 \in K^{-1}H$$

$$1 \in K \Longrightarrow K^{-1}H \subseteq H$$

PRIMITIVE IDEALS

$$J \in MLI(A), J \subseteq J' \in LI(A) \Longrightarrow J' \in \{J, A\}$$

$$P = JA^{-1} \in MLI(A)A^{-1}$$

$$P \in LI(A) \cap RI(A)$$

Primitive ideals are prime:

$$\{H, H'\} \subseteq LI(A) \Longrightarrow$$

$$HH' \subseteq P \Longrightarrow (H \subseteq P \text{ or } H' \subseteq P)$$

$$\bigcap MLI(A) = \mathsf{Rad}(A) \equiv \{a \in A : \mathsf{1} - Aa \subseteq A^{-1}\}$$

$$\bigcap\{JA^{-1} : J \in MLI(A)\} = \bigcap\{J : J \in MLI(A)\}$$

INVERTIBILITY

$$A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$$

$$A^{-1}_{left} = \{a \in A : 1 \in Aa\}$$

$$A^{-1}_{right} = \{a \in A : 1 \in aA\}$$

JOINT INVERTIBILITY

$$A^{-n}_{left} = \{a \in A^n : 1 \in \sum_{j=1}^{n} A a_j\}$$

$$A^{-n}_{right} = \{a \in A^n : 1 \in \sum_{j=1}^{n} a_j A\}$$

$$a = (a_x)_{x \in X} \in A^X$$

$$A_{left}^{-X} = \{a \in A^X : 1 \in \sum_{x \in X} A a_x\}$$

$$X \subseteq A \ , \ a_x = x \ (x \in X)$$

$$X \mapsto N = \sum_{x \in X} A a_x$$

SPECTRAL THEORY

$$a \in A \Longrightarrow \sigma(a) = \sigma^{left}(a) \cup \sigma^{right}(a)$$

$$\sigma^{left}(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{left}\}$$

$$\sigma^{right}(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{right}\}$$

$$a \in A^n \Longrightarrow \sigma^{left}(a) = \{\lambda \in \mathbf{C}^n : a - \lambda \notin A_{left}^{-n}\}$$

$$a \in A^n \Longrightarrow \sigma^{right}(a) = \{\lambda \in \mathbf{C}^n : a - \lambda \notin A_{right}^{-n}\}$$

$$a \in A^X \Longrightarrow \sigma^{left}(a) = \{\lambda \in \mathbf{C}^X : a - \lambda \notin A^{-X}_{left}\}$$

$$a \in A^X \Longrightarrow \sigma^{right}(a) = \{\lambda \in \mathbf{C}^X : a - \lambda \notin A^{-X}_{right}\}$$

POLYNOMIALS

$$\mathsf{Poly}_X = \mathsf{Alg}(z_x)_{x \in X}$$

$$p \in \mathsf{Poly}_X : A^X \to A$$

$$z_x : a \mapsto a_x$$

$$p = (p_y)_{y \in Y} \in \mathsf{Poly}_X^Y : A^X \to A^Y$$

Remainder theorem

$$p(a) - p(\lambda) \in \sum_{x \in X} A(a_x - \lambda_x)$$

$$p = 1 \; ; \; p = z_x \; ; \; p = q + r \; ; \; p = qr$$

SPECTRAL MAPPING THEOREM ?

$$p\sigma^{left}(a) \subseteq \sigma^{left}p(a)$$

$$\cdots$$

$$a = (a_1, a_2) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$$

$$\sigma^{left}(a) \subseteq \sigma^{left}(a_1) \times \sigma^{left}(a_2) = \{(0.0)\}$$

$$a_2 a_1 + a_1 a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow (0,0) \notin \sigma^{left}(a) \cup \sigma^{right}(a)$$

$$p = z_2 z_1 + z_1 z_2 \Longrightarrow p\sigma^{left}(a) = \emptyset \neq \{1\} = \sigma^{left}p(a)$$

$$q \circ p \equiv z \equiv p \circ q \Longrightarrow \sigma^{left} p(a) \subseteq p \sigma^{left}(a)$$

$$\mu \in \sigma^{left} p(a) \Longrightarrow q(\mu) \in q \sigma^{left} p(a) \subseteq \sigma^{left}(a)$$

$$r(z,w) \equiv z \Longrightarrow r \circ (z,p) \equiv z \equiv (z,p) \circ r$$

$$s(z,w) \equiv w - p(z) \Longrightarrow s \circ (z,p) \equiv 0$$

$$\sigma^{left}(a,b) \subseteq \sigma^{left}(a) \times \sigma^{left}(b)$$

$$\sigma^{left}(a,p(a)) = \{(\lambda,p(\lambda)) : \lambda \in \sigma^{left}(a)\}$$

$$(\lambda,\mu) \in \sigma^{left}(a,p(a)) \Longrightarrow \mu = p(\lambda) \ :$$

$$q(z,w) = w - p(z) \Longrightarrow q(a,p(a)) = 0$$

$$\Longrightarrow q(\lambda,\mu) \in \sigma^{left} q(a,p(a)) = \sigma^{left}(0) = \{0\}$$

COMMUTING SYSTEMS

$a \in A^X$ commutative $\Longrightarrow$ $(a, p(a))$ commutative

$$a \in \text{comm}(a, b) \Longrightarrow :$$

$$\mu \in \sigma^{left}(b) \Longrightarrow \exists \lambda \ : \ (\lambda, \mu) \in \sigma^{left}(a, b)$$

PROJECTION PROPERTY

$$\mu \in \sigma^{left}(b) \Longrightarrow 1 \notin N = \sum_{y \in Y} A(b_y - \mu_y)$$

$$1 \notin N \Longrightarrow 1 \notin \mathsf{cl}(N)$$

$$N \mapsto \mathsf{cl} \sum_{y \in Y} A(b_y - \mu_y) \ \ :$$

Residual quotient:

$$M = N^{-1}N \Longrightarrow N \in LI(M) \cap RI(M)$$

$$c \in M = N^{-1}N$$

$$[c]_N \in B = M/N \Longrightarrow \sigma_B[c]_N \neq \emptyset \Longrightarrow \partial\sigma_B[c]_N \neq \emptyset$$

$$\lambda \in \partial\sigma[c]_N \Longrightarrow \lambda \in \tau_B^{left}[c]_N$$

$$\lambda \in \tau_B^{left}[c]_N \Longrightarrow 1 \notin N + A(c-\lambda) \ :$$

$$1 \in N + c'_\lambda(c-\lambda) \Longrightarrow \mathsf{AND}_{x \in M} \ :$$

$$\mathsf{dist}(x, N) \leq \|c'_\lambda\| \mathsf{dist}((c-\lambda)x, N)$$

GELFAND/MAZUR

$$A \text{ commutative} \Longrightarrow LI(A) = RI(A)$$

$$N \in LI(A) \Longrightarrow N^{-1}N = A$$

$$1 \notin N \in LI(A) \Longrightarrow N \subseteq J \in MLI(A)$$

$$J \in MLI(A) \Longrightarrow J = \mathrm{cl}(J)$$

$$J \in MLI(A) \ \ :$$

$$B = A/J \Longrightarrow B = B^{-1} \cup \{0\} \Longrightarrow \dim(B) = 1$$

$$J \in MLI(A) \Longrightarrow A/J \cong \mathbf{C}$$

$$a \mapsto [a]_J \mapsto \varphi(a) \in \mathbf{C}$$

$$a \in A \Longrightarrow \sigma_B[a]_J = \{\varphi(a)\}$$

GELFAND

$$A \leftrightarrow a = (a_x)_{x \in A} = I : A \to A$$

$a = I$ continuous linear multiplicative unital

$$\lambda \in \sigma^{left}(a) \Longrightarrow \lambda : A \to \mathbf{C}$$

continuous linear multiplicative unital

$$\sigma^{left}(A) = \text{ Gelfand characters}$$

EXACTNESS

$$(b,a) \in A^2 \text{ splitting exact } \iff 1 \in Ab + aA$$

$$(b,a) \in A^{-2}_{left,right} \subseteq A^2 \iff 1 \in Ab + aA$$

$$a \in A \text{ self exact } \iff 1 \in Aa + aA$$

$$A^{-1}_{left,right} = \{a \in A \ : \ (a,a) \in A^{-2}_{left,right}\}$$

$$a \in A^{-1}_{left,right} \subseteq A \iff 1 \in Aa + aA$$

TAYLOR INVERTIBILITY

$$a \in A^{-1} \Longleftrightarrow (\Lambda_a, \Lambda_a) \text{ self exact}$$

$$a \in A \Longrightarrow \Lambda_a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \Lambda_A^{(1)} = A^{2\times 2}$$

$$a = (b, c) \Longrightarrow \Lambda_a = \begin{pmatrix} \Lambda_b & 0 \\ \triangle_c & -\Lambda_b \end{pmatrix} \in D^{2\times 2}$$

$$a = (b, c) \Longrightarrow \Lambda_a^2 = \begin{pmatrix} \Lambda_b^2 & 0 \\ \triangle_c\Lambda_b - \Lambda_b\triangle_c & \Lambda_b^2 \end{pmatrix}$$

$$D = \Lambda_A^{(n)} \Longrightarrow D^{2\times 2} = \Lambda_A^{(n+1)}$$

$$1 \in \Lambda_a D^{2\times 2} + D^{2\times 2}\Lambda_a \Longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in$$

$$\begin{pmatrix} \Lambda_b & 0 \\ \triangle_c & -\Lambda_b \end{pmatrix} \begin{pmatrix} D & D \\ D & D \end{pmatrix} + \begin{pmatrix} D & D \\ D & D \end{pmatrix} \begin{pmatrix} \Lambda_b & 0 \\ \triangle_c & -\Lambda_b \end{pmatrix}$$

$$N = aA + Aa \subseteq A \Longrightarrow$$

$$N : N = (L_N + R_N)^{-1}(N) = \{c \in A : Nc + cN \subseteq N\}$$

$$M = N : N \Longrightarrow N \in LI(M) \cap RI(M)$$

$$N \cdot N \subseteq N \Longleftrightarrow N \subseteq N : N$$

$$(\Lambda_b D + D\Lambda_b)(\Lambda_b D + D\Lambda_b) \subseteq \Lambda_b D + D\Lambda_b$$

$$\Longrightarrow$$

$$(\Lambda_a D^{2\times 2} + D^{2\times 2}\Lambda_a)(\Lambda_a D^{2\times 2} + D^{2\times 2}\Lambda_a)$$

$$\subseteq \Lambda_a D^{2\times 2} + D^{2\times 2}\Lambda_a$$

TAYLOR (SPLIT) SPECTRUM

$$\sigma^{left,right}(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{left,right}\}$$

$$\sigma^{Taylor}(a) = \{\lambda \in \mathbf{C}^n : \Lambda_{a-\lambda} \notin (\Lambda^{(n)}_A)^{-1}_{left,right}\}$$

POLYNOMIALS

$$\Lambda_p \Lambda_a \ :$$

$$\Lambda_p \Lambda_{b,c} = \begin{pmatrix} \Lambda_p \Lambda_b & 0 \\ \triangle_{p(c)} & -\Lambda_p \Lambda_b \end{pmatrix}$$

$$p \in \mathrm{Poly}_n^m \Longrightarrow p\sigma^{Taylor}(a) \subseteq \sigma^{Taylor} p(a) \ ?$$

IF

$$p(0) = 0 \ \& \ 1 \in \Lambda_p \Lambda_b D + D \Lambda_p \Lambda_b$$

THEN ?

$$p(0) = 0 \ \& \ 1 \in \Lambda_p \Lambda_a D^{2\times 2} + D^{2\times 2} \Lambda_p \Lambda_a$$

$$\Longrightarrow 1 \in \Lambda_a D^{2\times 2} + D^{2\times 2} \Lambda_a$$

PROJECTION PROPERTY

$$0 \in \sigma^{left,right}(b) \Longrightarrow \exists\lambda \; : \; 0 \in \sigma^{left,right}(b, c-\lambda) \; ?$$

$$1 \notin N = \mathrm{cl}(\Lambda_b D^{2\times 2} + D^{2\times 2}\Lambda_b) \Longrightarrow$$

$$\exists\lambda \; : \; N + \triangle_{c-\lambda} D^{2\times 2} \; ?$$