Polynomials on S^*-parabolic manifolds

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A Stein manifold X is called parabolic, in case it does not possess a non-constant bounded above plurisubharmonic function.

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A Stein manifold X is called S-parabolic, if there exit exhaustion function $\rho \in psh(X)$ that is maximal outside a compact subset of X. If in addition we can choose ρ to be **continuous** then we will say that X is S^*-parabolic.

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For open Riemann surfaces the notions of S-parabolicity, S^*-parabolicity and parabolicity coincide. This is a consequence of the existence of Evans-Selberg potentials (subharmonic exhaustion functions that are harmonic outside a given point) on parabolic Riemann surface.

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It turns out that the paraboliticity of a Stein manifold X and certain linear topological properties of the Fréchet space of analytic functions on X are connected.

As usual, the topology on the space of analytic functions on a complex manifold X, $O(X)$,is the topology of uniform convergence on compact subsets of X, which makes $O(X)$ a nuclear Fréchet space. We start by recalling the *DN* condition of Vogt from the structure theory of Fréchet spaces;

Definition

A Fréchet space Y has the property *DN* in case for a system $(\|*\|_k)$ of seminorms generating the topology of Y one has:

$$\exists\, k_0 \text{ such that } \forall p \exists q C > 0 : \|x\|_p \leq C \|x\|_{k_0}^{\frac{1}{2}} \|x\|_q^{\frac{1}{2}} \ \forall x \epsilon Y$$

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The first result we will state is an adaptation of a result from (A-Krone-Terzioğlu) part of which was proved by D.Vogt, V.Zaharyuta, and A. independently.

Theorem

For a Stein manifold X of dimension n, the following conditions are equivalent:

1. *X is parabolic*
2. *$O(X)$ has the property DN*
3. *$O(X)$ is isomorphic as Fréchet spaces to $O(C^n)$.*

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Mitiagin and Henkin, in their seminal paper initiated a program which they called "linearization of the basic theorems of complex analysis". One of the problems they considered (in connection with Remmerd's theorem) was the possibility of finding continuous linear right inverse operators to the restriction operator for analytic functions defined on closed complex submanifolds of C^N.

In other words for a closed complex submanifold V of some C^N,denoting by R the restriction operator from $O\left(C^N\right)$ onto $O\left(V\right)$ the query was to find a continuous linear (extension) operator $E : O\left(V\right) \rightarrow O\left(C^N\right)$ such that $R \circ E = \mathit{Identity}$ on $O\left(V\right)$.

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A Stein manifold is parabolic if and only if whenever it is embedded into a Stein manifold as a closed submanifold it admits a continuous linear extension operator.

We now wish to pass to a more refined category of Fréchet spaces. Recall that a graded Fréchet space is a tuple $\left(Y, ^{\|*\|_s}\right)$ where Y is a Fréchet space and $(\|*\|_s)$ is a fixed system of seminorms on Y defining the topology. The morphisms in this category are tame linear operators

Definition

A continuous linear operator T between two graded Fréchet spaces $\left(Y, ^{\|*\|_s}\right)$ and $\left(Z, ^{|*|_k}\right)$ is said to be tame in case:

$$\exists A > 0 \forall k \exists C > 0 : |T(x)|_k \leq C \|x\|_{k+A} .$$

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Two graded Fréchet spaces are called tamely isomorphic in case there is a one to one tame linear operator from one onto the other whose inverse is also tame.

On a Stein manifold X, each exhaustion $(K_n)_{n=1}^{\infty}$ of holomorphically convex compact sets with $K_n \subset\subset int\,(K_{n+1})$, $n = 1, 2, ..,$ induces a grading $\{\|*\|_{K_n}\}_n$ on $O(X)$ by considering the sup norms on these compact sets.

Theorem

(A-Sadullaev) A Stein manifold of dimension n is $\mathbb{S}^*$-parabolic if and only if there exits an exhaustion $(K_n)_{n=1}^{\infty}$ of X such that the graded spaces $\left(O(X), \|*\|_{K_s}\right)$ and $\left(O(C^n), \|*\|_{P_s}\right)$ are tamely isomorphic where $P_s \doteq (z \epsilon C^n : \|z\| \leq e^s)$, $s = 1, 2, \ldots$.

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Let X be a Stein manifold. We will say that $O(X)$ is tamely isomorphic to $O(C^n)$ in case there exits an exhaustion $(K_n)_{n=1}^{\infty}$of X by holomorphically convex compact sets such that the graded spaces $\left(O(X), ^{\|*\|_{K_s}}\right)$ and $\left(O(C^n), ^{\|*\|_{P_s}}\right)$ are tamely isomorphic where, as usual $P_s \doteq (z \epsilon C^n : \|z\| \leq e^s)$, $s = 1, 2, ..$.

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Conjecture (Burns conjecture, 1982)

Let M be a complex manifold of dimension n . If there exits a strictly plurisubharmonic exhaustion $\tau : M \to [0, \infty]$ with

$$(dd^c \log \tau)^n = 0$$

outside a compact subset of M, then M is biholomorphic to an algebraic submanifold of some $\mathbb{C}^N$.

Theorem (Demailly, 1985)

Let M be a complex manifold of dimension n. If there a C^4 strictly plurisubharmonic exhaustion $\varphi : M \to [0, \infty]$ such that;

- $\int (dd^c\varphi)^n < \infty$
- *critical points of φ is finite*
- *there exits a continuous function $\psi : M \to \mathbb{R}$ with $\exists\ a > 0$ and $b > 0$: $\psi \leq a\varphi + b$*
- *and $Ricci\,(dd^c\,(e^{\varphi})) \geq -\frac{1}{2}dd^c\psi$,*

then M is biholomorhic to an affine algebraic submanifold of some $\mathbb{C}^N$.

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Let (X, p) be a $\mathbb{S}$-parabolic manifold. A holomorphic function $f \in \mathbb{O}(X)$ is called a *polynomial* on X in case for some integer d and $c > 0$, f satisfies the growth estimate

$$\ln|f(z)| \leq d \cdot p^{+}(z) + c \quad \forall z \in X.$$

The minimal such d will be called the *degree* of f and the set of all polynomials on X with degree less than or equal to d will be denoted by $\mathcal{P}_p^d$

Theorem

Let (X, ρ) be an S-parabolic Stein manifold. The space $\mathcal{P}_\rho^d$ is a finite dimensional complex vector space and there exists a $C = C(X) > 0$ such that $\dim \mathcal{P}_p^d \leq Cd^n$, where n is the dimension of X.

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In the case of algebraic affine manifolds of dimension n with canonical special exhaustion function, we actually have that the sequences $\{\dim \mathcal{P}_\rho^d\}_d$ and $\{d^n\}_d$ are weakly asymptotic i.e. $\exists C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leq \underline{\lim}_{d\to\infty} \frac{\dim \mathcal{P}_p^d}{d^n} \leq \overline{\lim}_{d\to\infty} \frac{\dim \mathcal{P}_\rho^d}{d^n} \leq C_2.$$

For more information on these matters we refer to the reader to (Zeriahi) and (A-Sadullaev 1)

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Consider the segment $[0, 1]$, and denote it as $K_0 = [a_{01}, b_{01}]$, the length of K_0 is 1.

Next we proceed as in the construction of Cantor sets: fix $\delta = 1/4$ and the sequence $t_m = 4^{m-1}$, $m = 1, 2, ...$ From (a_{01}, b_{01}) we take off the interval $[a_{01} + \delta, b_{01} - \delta]$. We get the union of two segments, $K_1 = [a_{01}, a_{01} + \delta] \cup [a_{02} - \delta, a_{02}]$.

Introducing a new notation we rewrite this set as;

$$K_1 = [a_{01}, a_{01} + \delta] \cup [b_{01} - \delta, b_{01}] = [a_{11}, b_{11}] \cup [a_{12}, b_{12}]$$

Distances between knot-points $a_{11}, b_{11}, a_{12}, b_{12}$ are:

$$|b_{1j} - a_{1j}| = \delta, \; j = 1, 2, \quad |b_{11} - a_{12}| = 1 - 2\delta.$$

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Then with each of these segments we do the same procedure, changing δ to δ^{t_2}: we get 4 segments,4 segments,

$$\begin{aligned} K_2 &= [a_{11}, a_{11} + \delta^{t_2}] \cup [b_{11} - \delta^{t_2}, b_{11}] \cup [a_{12}, a_{12} + \delta^{t_2}] \\ &\quad \cup [b_{12} - \delta^{t_2}, b_{12}] \\ &= [a_{21}, b_{21}] \cup [a_{22}, b_{22}] \cup [a_{23}, b_{23}] \cup [b_{24}, b_{24}], \end{aligned}$$

with length δ^{t_2}, and with distances between knot points:

$$\begin{aligned} &|b_{2j} - a_{2j}| = \delta^{t_2}, j = 1, 2, 3, 4, \\ &|b_{21} - a_{22}| = \delta - 2\delta^{t_2}, |b_{22} - a_{23}| = 1 - 2\delta, \\ &|b_{23} - a_{24}| = \delta - 2\delta^{t_2}. \end{aligned}$$

In the m-th step we get union of 2^m segments

$$K_m = [a_{m1}, b_{m1}] \cup [a_{m2}, b_{m2}] \cup ... \cup [a_{m2^m}, b_{m2^m}],$$

with length δ^{t_m}. Note, $K_0 \supset K_1 \supset ... \supset K_m ..., l(K_m) = 2^m \delta^{t_m}$.

The Hausdorff measure of K_m with respect to kernel $h(s) = \ln^{-1} \frac{1}{s}$ is equal to

$$H^h(K_m) = 2^m h(\delta^{t_m}/2) = 2^m \ln^{-1} \frac{1}{\delta^{t_m}/2} = \frac{2^m}{t_m} \ln^{-1} \frac{2^{1/t_m}}{\delta}.$$

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Put:

$$K = \bigcap_{m=1}^{\infty} K_m.$$

If $\frac{2^m}{t_m} \leq C < \infty$, $m = 1, 2, ...,$ then $H^h(K) < \infty$ and by the well-known property of the logarithm capacity, $C(K) = 0$.

In our case $t_m = 4^{m-1}$, so the compact set K is polar

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Now we will select a measure μ on K. To this end for,

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we put

$$\mu_m = \frac{\delta(a_{m1}) + ... + \delta(a_{m2^m}) + \delta(b_{m1}) + ...\delta(b_{m2^m})}{2 \cdot 2^m},$$

where $\delta(c)$-discrete probably measure, supported in c.

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Let

$$U^{\mu}(z) = \int \ln|z - w| d\mu(w)$$

be the potential corresponding to μ.

Theorem

The potential U^{μ} is subharmonic on C, harmonic off K , $U^{\mu}(z) = -\infty$ on K and

$$\lim_{z \to K} \frac{U^{\mu}(z)}{\ln dist(z, K)} = 0.$$

THE EXAMPLE

Now we consider the manifold $X = \overline{C} \setminus K$. As special exhaustive function we put $\phi(z) = -U^{\mu}(z)$. Then $\phi(z)$-harmonic on $X \setminus \{\infty\}$, $\phi(\infty) = -\infty$ and $\phi(z) \to \infty$ as $z \to K$. Therefore, $(X,^{\phi})$ becomes a S^*-parabolic manifold.

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$$\lim_{z \to K} \frac{U^{\mu}(z)}{\ln dist(z, K)} = 0.$$

THE EXAMPLE

Now we consider the manifold $X = \overline{C} \setminus K$. As special exhaustive function we put $\phi(z) = -U^{\mu}(z)$. Then $\phi(z)$-harmonic on $X \setminus \{\infty\}$, $\phi(\infty) = -\infty$ and $\phi(z) \to \infty$ as $z \to K$. Therefore, $(X,^{\phi})$ becomes a S^*-parabolic manifold.

Choose a polynomial $f \in O(X)$. Say $\ln|f| \leq C + k\phi$ for some positive integers C and k.

First we fix an m and take a closed curve $(\text{union of rectangles})$, $\gamma = \gamma_m$, circumscribing $K \subset K_m = [a_{m1}, b_{m1}] \cup [a_{m2}, b_{m2}] \cup ... \cup [a_{m2^m}, b_{m2^m}]$

γ is bounded above by a part of $\{\mathrm{Im} z = r\}, r > 0$, below by $\{\mathrm{Im} z = -r\}$ and from the sides by a part $\{\mathrm{Re} z = a_{mj} - r\}$ and $\{\mathrm{Re} z = b_{mj} + r\}$.

The length of γ is equal

$$l(\gamma) = 2 \cdot 2^m(\delta^{t_m} + 2r) + 2 \cdot 2^m r = 3 \cdot 2^{m+1} r + 2^{m+1}\delta^{t_m}.$$

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In view of the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\xi|=2} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\xi)}{\xi - z} d\xi, \quad z \in B(0,2) \setminus \hat{\gamma}, \quad (1)$$

where $\hat{\gamma}$ is the polynomial convex hull of γ.

For the second integral of (1) we have:

$$\left| \int\limits_{\gamma} \frac{f(\xi)}{\xi - z} d\xi \right| \leq \frac{||f||_{\gamma} l(\gamma)}{dist(z,\gamma)}$$

$$\leq \frac{e^{C+k||\phi||_{\gamma}}}{dis(z,\gamma)} (3 \cdot 2^{m+1} r + 2^{m+1} \delta^{t_m}) \leq C_1 e^{k||\phi||_{\gamma}} (2^{m+3} r + 2^{m+1} \delta^{t_m}).$$

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According to the estimate of the theorem above, we have for a given $\epsilon > 0$ and r, m large:

$$\left| \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi \right| \leq C_2 r^{-\epsilon k} 2^m (r + \delta^{t_m}).$$

Now we choose $\epsilon = \frac{1}{2k}$ and $r = \frac{1}{2^{4m}}$. Then $r^{-\epsilon k} 2^m (r + \delta^{t_m}) = \frac{1}{2^m} + 2^{3m} \delta^{t_m} \to 0$ as $m \to \infty$. We see that, the second integral in (1) tends zero, which means the function

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=2} \frac{f(\xi)}{\xi - z} d\xi$$

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As we have seen not every parabolic manifold has a large supply of polynomials. On the other hand most important examples of parabolic manifolds like affine algebraic submanifolds (with their canonical special exhaustion function), complements of zero sets of Weierstrass polynomials do have a rich class of polynomials, namely in these examples polynomials are dense in the corresponding spaces of analytic functions.

Example

Algebraic sets $X \subset \mathbb{C}^N$, $\dim A = n$. In this case by the well-known theorem of W. Rudin we can assume, that (after an appropriate transformation)

$$X \subset \{w = (w', w'') = (w_1, ..., w_n, w_{n+1}, ..., w_N) : ||w''|| < A(1 + ||w'||^B\},$$

where A, B are constants.

Then the restriction $\rho|_X$ of the function $\rho(w) = \ln ||w'||$ is a special exhaustion function on X. It is clear, that polynomials on X with respect to this exhaustion function are simply restrictions to X of polynomials $p(w', w'')$. Therefore, $\mathcal{P}_\rho(X)$ is dense in $O(X)$.

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Example (Complements of Weierstrass algebroid sets)

Let

$$A = \{z = ('z, z_n) = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : F('z, z_n) = z_n^k + f_1('z) z_n^{k-1} + ... + f_k('z) = 0\}$$

be a Weierstrass polynomial set, where $f_j \in O(\mathbb{C}^{n-1})$ are entire functions, $j = 1, 2, ..., k$, $k > 1$.

Then $X = \mathbb{C}^n \setminus A$ with exhaustion function $\rho(z) = -\ln|F(z)| + \ln(|'z| + |F(z) - 1|^2)$ is S^*-parabolic (Aytuna- Sadullaev). If $p(z, \tau)$ is a polynomial in $\mathbb{C}^{n+1}$, then $p(z, 1/F(z))$ is a polynomial on $X = \mathbb{C}^n \setminus A$.

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Motivated by these examples, we give the following definition:

Definition

S^*-parabolic manifold (X, ρ) is called *regular* in case the space of all polynomials $\mathcal{P}_\rho(X)$ is dense in $O(X)$.

Our next example shows that non triviality of the polynomial space $\mathcal{P}_\rho(X)$ does not always guarantee the regularity of X.

Example

We add to compact K, from example 4.2 one more point: $E = K \cup \{z^0\}$, $z^0 \notin K$. The manifold $X = \overline{\mathbb{C}} \setminus E$ with exhaustive function $\rho(z) = -U^\mu(z) - \ln|z - z^0|$ be S^*-parabolic. On X there are polynomials, an example, $f(z) = (z - z^0)^m$, but the space of all polynomials $\mathcal{P}_\rho$ is not dense in $O(X)$: the function $f(z) = \frac{1}{z - z'}$, where $z' \in K$, cannot be approximated by polynomials.

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In search for more examples of S-parabolic manifolds one may look at closed complex submanifolds of regular S^*-parabolic manifolds. Since such manifolds are in particular parabolic, there exits, continuous linear extension operators for analytic functions on this submanifold to the ambient space. However the mere existence of continuous extension operators will not, in general give regularity as the example, in the previous section shows.

Recall that for S^*-parabolic manifolds (X, ρ) we will always consider, unless otherwise stated, the canonical grading on $O(X)$ given by ρ, and for a closed complex submanifolds V of X we will provide $O(V)$ with the induced grading, i.e. the grading coming from the sup norms on $V \cap (z : \rho(z) \leq k)$, $k = 1, 2, \ldots$. With this convention we have:

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Theorem

Let (X, ρ) be a regular S^*-parabolic Stein manifold and let V be a closed complex submanifold of X. If there exits a tame linear extension operator from $O(V)$ into $O(X)$ then V becomes a regular S^*-parabolic manifold

Remark (1)

The existence of a tame linear extension operator as above is of course related to the tame splitting of tame short exact sequence:

$$0 \to I \to O(X) \stackrel{R}{\to} O(V) \to 0,$$

where R is the restriction operator and I is the ideal sheaf of V with the subspace grading induced from $O(X)$. Tame splitting of short exact sequences in the category of graded Fréchet spaces were studied by various authors. We refer to (Popenberg) for a survey and for structural conditions on the underlying Fréchet nuclear spaces which ensure that short exact sequences in this category split.

Remark (2)

It was shown in (A-) that in $\mathbb{C}^N$ closed complex submanifolds that admit tame extension operators are precisely the affine algebraic submanifolds of $\mathbb{C}^N$. Since there are non algebraic regular S^*-parabolic Stein manifolds of $\mathbb{C}^N$, the statement of the Proposition is not an if and only if statement.

For a given S^*- parabolic Stein manifold $\left(X,^{\Phi}\right)$, as usual we will assume that the special exhaustion function Φ is maximal outside a compact set that lies in $(z : \Phi(z) < 0)$ and equip the Freĉhet space $O(X)$ with the grading $(\|\|_k)_{k=1}^{\infty}$, where for $k = 1, 2, .., \|f\|_k \doteq \sup_{z \epsilon D_k} |f(z)|$, where $D_k \doteq (z : \Phi(z) \leq k)$, k =1,2,.... .This graded Freĉhet space is denoted by $\left(O(X),^{\Phi}\right)$.

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The notation σ will be reserved for the usual bijection between $\mathbb{N}$ and $\mathbb{N}^n$ satisfying $|\sigma(n)| \leq |\sigma(n+1)|, \forall n\epsilon\mathbb{N}$. Observe that the identity operator gives a tame isomorphism between $O(C^n)$ with the usual grading and $\left(O(C^n), ^{|*|_k}\right)$ where

$$|f|_k \doteq \sum_n |x_n| \; e^{k|\sigma(n)|}, \;\; \forall f = \sum_s x_s z^{\sigma(s)} \epsilon O(C^n)$$

in view of the Caucy estimates.

Let X be a S^*- parabolic Stein manifold. We have seen that with a suitable special exhaustion function Φ, $\left(O(X), ^{\Phi}\right)$ is tamely isomorphic to $O(C^n)$ with the usual grading.

If we denote by

$$L_\Phi = (u\epsilon\, PSH(X) \ : \exists\, c > 0, \text{ such that } \ u \leq \Phi + c)$$

then there is a basis $(f_n)_n$of $O(X)$, obtained from monomials via this tame linear isomorphism with the property :

$$\limsup_{w \to z} \limsup_{n} \frac{\ln |f_n(w)|}{|\sigma(n)|} \epsilon\, L_\Phi$$

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Unfortunately tame isomorphisms between S^*- parabolic Stein manifolds do not necessarily map polynomials into polynomials even when the spaces are regular as the multiplication operator with the exponential function on $O(C)$ shows.

However our next result states that for a regular S^*- parabolic Stein manifold $(X,^{\Phi})$ there exits a positive constant C and a tame isomophism T from $O(C^n)$, $n = \dim X$, onto $(O(X),^{C\Phi})$ that maps polynomials into $\Phi-$polynomials.

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Theorem

Let $\left(X,^{\Phi}\right)$ be a regular S^*- parabolic Stein manifold. There exits a polynomial basis $\{p_n\}_n$ for $O\left(M\right)$ and a $C > 0$, such that the linear transformation T defined through $T\left(p_n\right) = z^{\sigma(n)}$,$n = 1, 2, ...$gives a tame isomorphism between $\left(O\left(X\right),^{C\Phi}\right)$ and $O\left(C^n\right)$ with the usual grading.

Remark

1. The proof shows something more , namely that the polynomial basis found also constitute bases for the Fréchet spaces $O\left(\left(z : \Phi\left(z\right) < s\right)\right)$, for large s large.

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Remark

If we only assume that the Stein manifold X is $S-$ parabolic then we can choose a Fréchet space isomorphism S from $O(C^n)$, $n = \dim X$, onto $O(X)$. The general argument given in the first part of the proof of the above theorem is valid in this set up so as a corollary of the proof of the theorem we have:

Corollary

Let (X, Φ) be a regular $S-$ parabolic Stein manifold of dimension n. There exits an isomorphism from $O(C^n)$ onto $O(X)$ that maps polynomials into $\Phi-$polynomials. In particular $O(X)$ has a basis consisting of $\Phi-$polynomials.

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Thank you for your attention...

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