Introduction to Holomorphic Dynamics
Holomorphic Explosion

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Outline

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- Introduction to Holomorphic Dynamics

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- Introduction to Holomorphic Dynamics
- Studying a Specific Family

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- Studying a Specific Family
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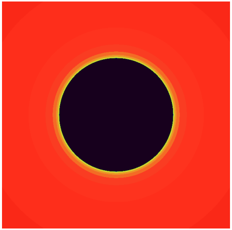
- Holomorphic Explosion
- Application

INTRODUCTION TO HOLOMORPHIC DYNAMICS



Basic words in Holomorphic Dynamics language

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- domain $\to \mathbb{C}$

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- function $\rightarrow$ holomorphic

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  - * attracting
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- singular values
  - critical value
  - asymptotic value
- entire function
  - polynomial
  - transcendental

Quadratic family in one slide

$$Q_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

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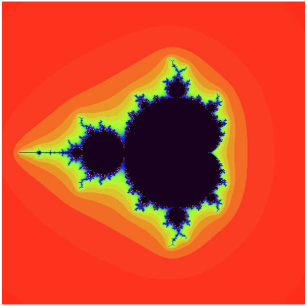

Mandelbrot Set

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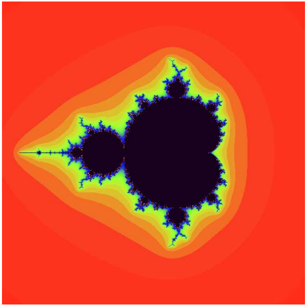

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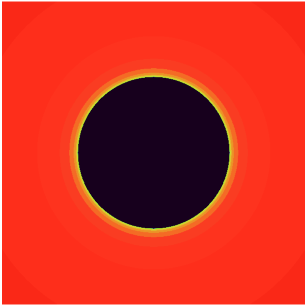

Dynamical plane for Q_0

More examples...

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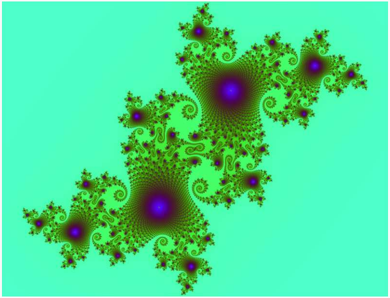

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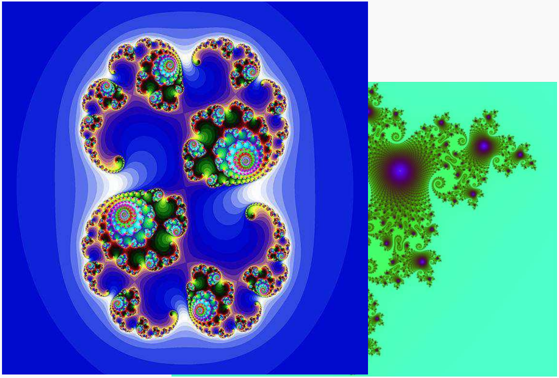

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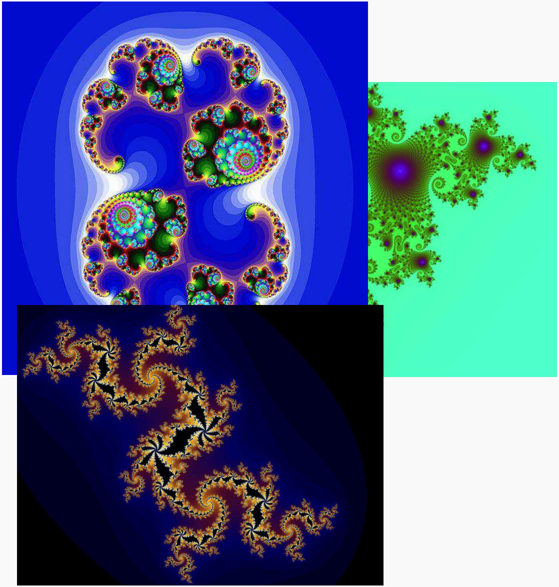

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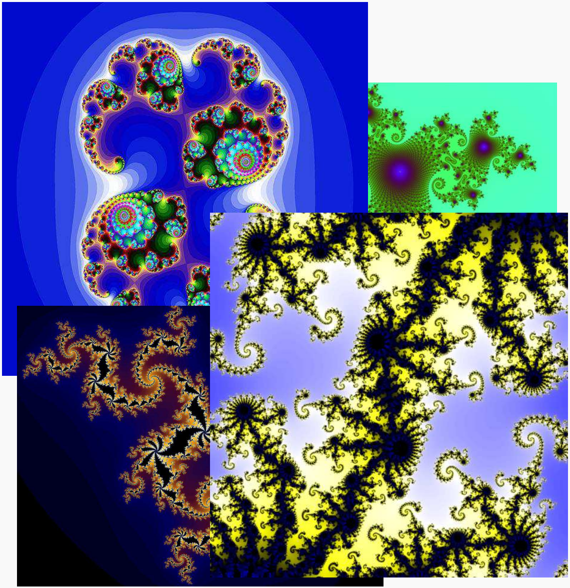

STUDYING A SPECIFIC FAMILY

Motivation

Motivation

Polynomial

Transcendental

Motivation

Polynomial

Transcendental

$Q_c(z) = z^2 + c$

Motivation

Polynomial

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$$Q_c(z) = z^2 + c \qquad \longrightarrow \qquad E_\kappa(z) = e^z + \kappa$$

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Polynomial | Transcendental

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$P_{a,b}(z) = z^3 - 3a^2 z + b$

Motivation

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Motivation

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$Q_c(z) = z^2 + c$	$\longrightarrow$	$E_\kappa(z) = e^z + \kappa$
$P_{a,b}(z) = z^3 - 3a^2z + b$	$\longrightarrow$	$f_{a,\lambda}(z) = a\lambda(e^{\frac{z}{a}}(z - a + 1) + a - 1)$
$P_a(z) = az^2(z-1)$	$\longrightarrow$	$f_a(z) = a(e^z(z-1) + 1)$

Theorem

Any entire transcendental map of finite order with

i. *one asymptotic value with only one finite preimage, and*

ii. *one simple critical point which is fixed (normalized at 0)*

is affine conjugate to one of the form

$$f_a(z) = a(e^z(z-1)+1), \quad a \in \mathbb{C}^*.$$

On singular values

There are available jobs for singular values...

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The function f_a has

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  - * 0 is a superattracting fixed point so f_a has a superattracting basin,

On singular values

There are available jobs for singular values...

The function f_a has

- a fixed critical value at $z = 0 \longrightarrow$ the critical value has a permanent job
  - * 0 is a superattracting fixed point so f_a has a superattracting basin,
- a free asymptotic value at $z = a \longrightarrow$ the asymptotic value is free to choose!
  - * dynamical properties depend on the behavior of the asymptotic value.

Parameter plane - a close look

Parameter planes explain the possible jobs a singular value may have...

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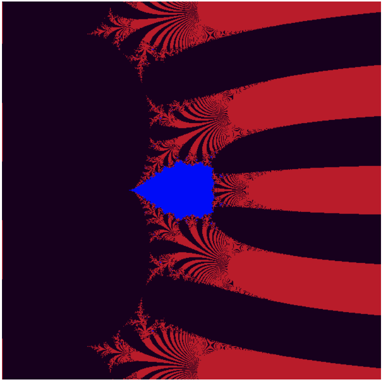

parameter plane for f_a

Parameter plane - a close look

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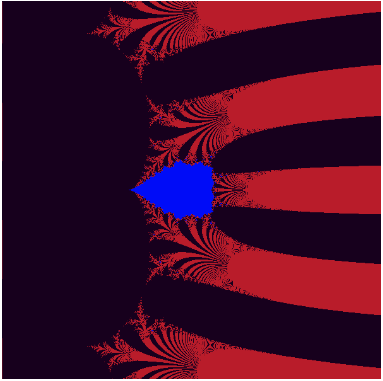

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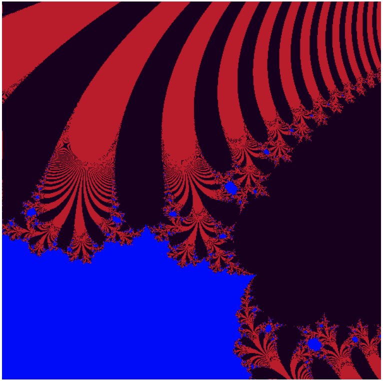
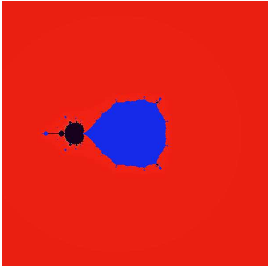
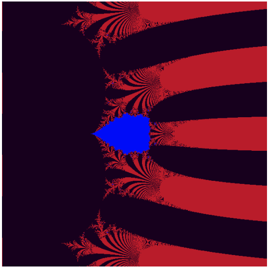


parameter plane for f_a zoom in

Parameter plane - a comparison



parameter plane for $P_a(z) = az^2(1 - z)$



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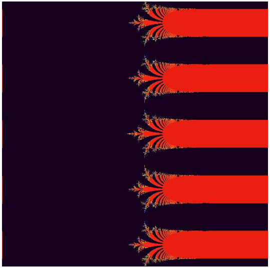

parameter plane for $E_a(z) = e^z + a$

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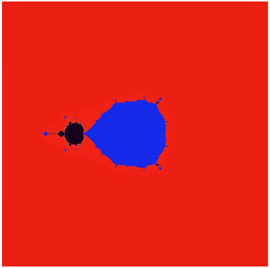

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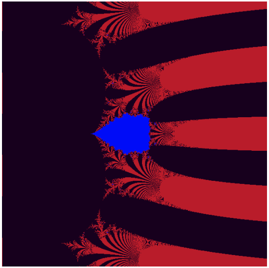

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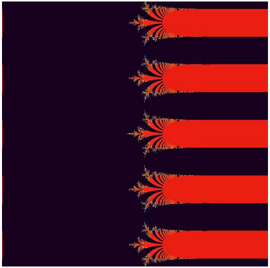

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Main hyperbolic component

$$\mathcal{C}^0 = \{a; \quad \mathrm{a} \in \mathrm{A}_{\mathrm{a}}^0\}$$

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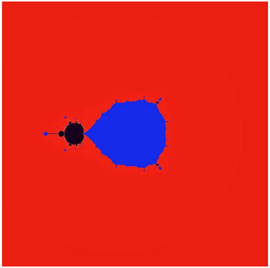

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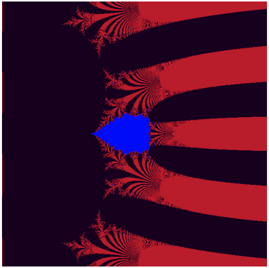

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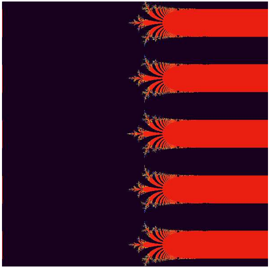

parameter plane for $E_a(z) = e^z + a$

Main hyperbolic component

$$\mathcal{C}^0 = \{a; \quad \mathrm{a} \in \mathrm{A}^0_\mathrm{a}\}$$

Theorem

$\mathcal{C}^0$ is bounded, connected and $\mathcal{C}^0 \cup \{0\}$ is simply connected.

Some results related to $\mathcal{C}^0$

Proposition

$a \in \mathcal{C}^0$ if and only if A_a consists of an unbounded, totally invariant component.

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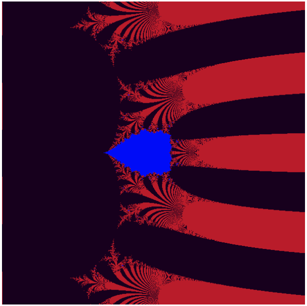

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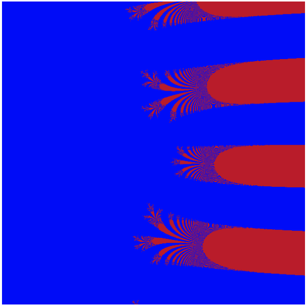

dynamical plane for $a = 0.4 + 0.6i$

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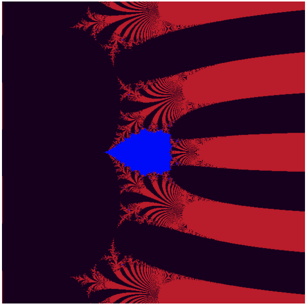

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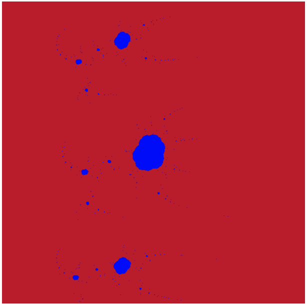

dynamical plane for $a = 1.9 + 0.7i$

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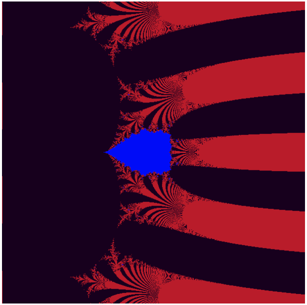

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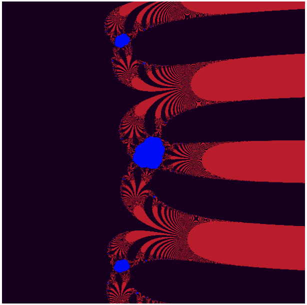

dynamical plane for $a = 1.7 + 1.4i$

HOLOMORPHIC EXPLOSION: AN EXTENSION OF HOLOMORPHIC MOTION

Definition (Holomorphic Motion)

Let $\Lambda \subset \widehat{\mathbb{C}}$ be a domain and $E \subset \widehat{\mathbb{C}}$. A function $H : \Lambda \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion of E parametrized over Λ with base point λ', if and only if

i. $H(\lambda', z) = z$ for all $z \in E$,

ii. for every fixed $\lambda \in \Lambda$, $z \mapsto H(\lambda, z)$ is injective, and

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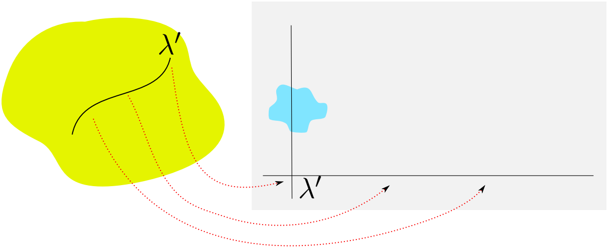

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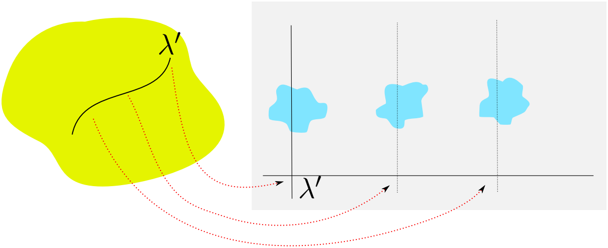

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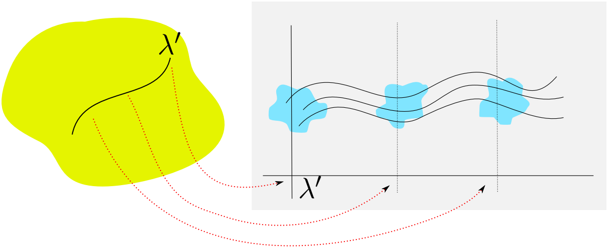

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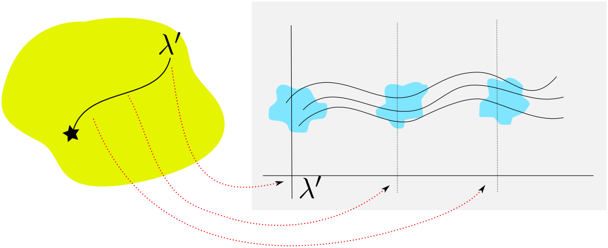

Extension in the parameter set

Definition (Holomorphic Explosion)

Let $\Lambda \subset \mathbb{C}$ be a domain and $E \subset \widehat{\mathbb{C}}$. A continuous function $H(\lambda, z) : \Lambda \times E \to \widehat{\mathbb{C}}$ is called a holomorphic explosion from (λ_0, z_0), for $\lambda_0 \in \Lambda$, $z_0 \in \mathbb{C}$ parametrized by Λ, if $H(\lambda_0, E) = z_0$, and for $\Lambda' = \Lambda \backslash \{\lambda_0\}$, the restriction $H|_{\Lambda' \times E}$ is a holomorphic motion.

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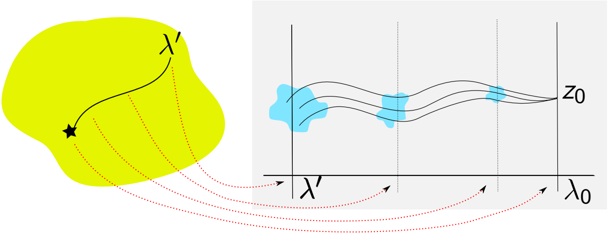

Proposition (Form of a holomorphic explosion)

Let $H : \Lambda \times E \to \widehat{\mathbb{C}}$ be a holomorphic explosion from (λ_0, z_0), $z_0 \in \mathbb{C}$, $E \subset \widehat{\mathbb{C}}$ is a connected set. In a neighborhood of $\lambda_0 \in \Lambda$, H can be expressed as:

$$H(\lambda, z) = P(\lambda - \lambda_0) + (\lambda - \lambda_0)^n \widehat{H}(\lambda, z),$$

where $\widehat{H}$ is a holomorphic motion parametrized over Λ and P is a complex polynomial.

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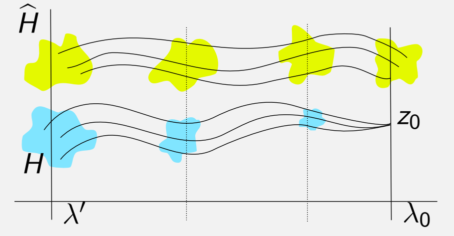

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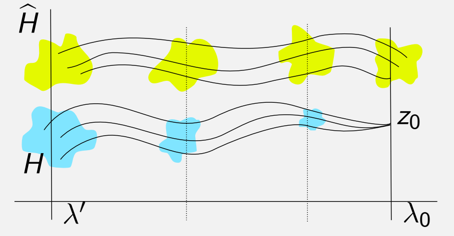
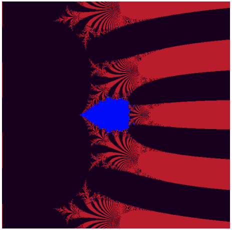


holomorphic explosion = scaling and translation of a holomorphic motion...

APPLICATION

Application: $f_a(z) = a(e^z(z-1)+1)$ - revisited

- The main hyperbolic component $\mathcal{C}^0 = \{a, \ \ a \in A_a^0\}$.



parameter plane

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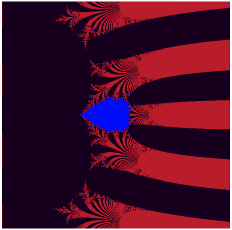

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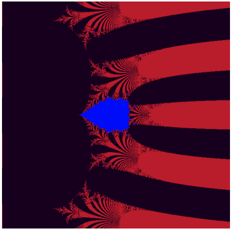
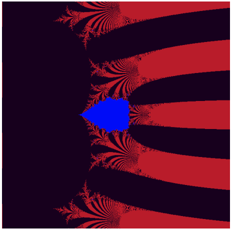


parameter plane

Let $\mathcal{U}$ be the unbounded connected component of $\mathbb{C} \backslash \overline{\mathcal{C}^0}$.

Recall for any $a_0 \in \mathcal{U}$

- The main hyperbolic component $\mathcal{C}^0 = \{a, \;\; a \in A_a^0\}$.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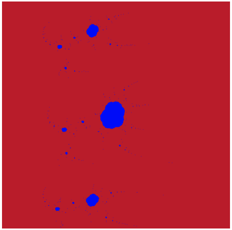
dynamical plane for $a_0 \in \mathcal{U}$

Application: $f_a(z) = a(e^z(z-1)+1)$ - revisited

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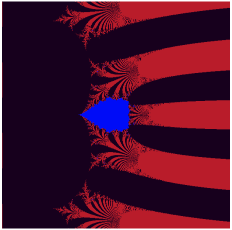

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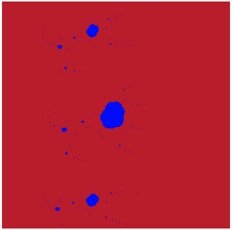


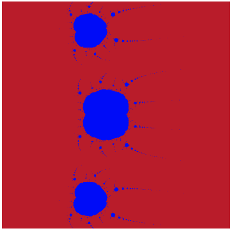
dynamical plane for $a_0 \in \mathcal{U}$

For $a_0 \in \mathcal{U}$, there exists a holomorphic motion

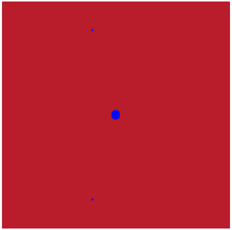
$$H : \mathcal{U} \times A_{a_0}^0 \rightarrow \widehat{\mathbb{C}}$$

with base point a_0.

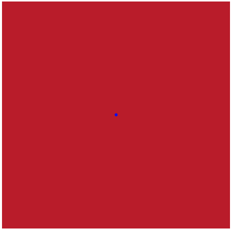
What do you feel?



$a = 1.2$

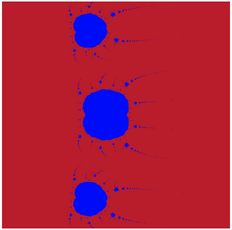


$a = 5$



$a = 15$

What do you feel?



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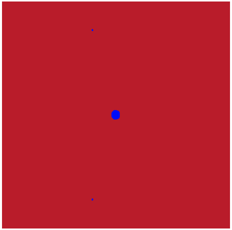

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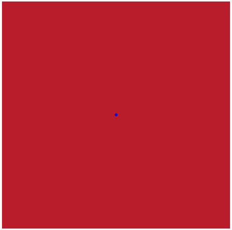

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Proposition

For $a_0 \in \mathcal{U}$, the holomorphic motion $H : \mathcal{U} \times A^0_{a_0} \to \widehat{\mathbb{C}}$ extends to a holomorphic explosion

$$\widetilde{H} : (\mathcal{U} \cup \{\infty\}) \times A^0_{a_0} \to \widehat{\mathbb{C}}$$

from $(\infty, 0)$.

Rescaling the dynamics

By $z \mapsto \frac{a}{2}z$, f_a conjugates to

$$\tau_a(z) = \frac{a^2}{2}(e^{\frac{2}{a}z}(\frac{2}{a}z - 1) + 1) = z^2 + \frac{2}{3a}z^3 + O(\frac{1}{a^3}z^4).$$

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- As $a \to \infty$, $\tau_a(z) \rightrightarrows z^2$ on compact sets of $\mathbb{C}$. Extend G to $\widetilde{G} : \mathcal{U} \cup \{\infty\} \times B^0_{a_0} \to \widehat{\mathbb{C}}$.

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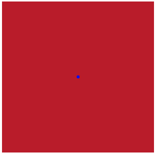
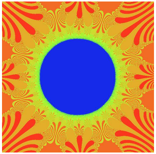


The immediate basin of 0 for f_a, $a = 50$ in $[-2, 2] \times [-2, 2]$



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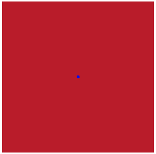

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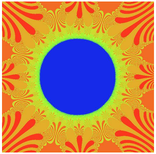

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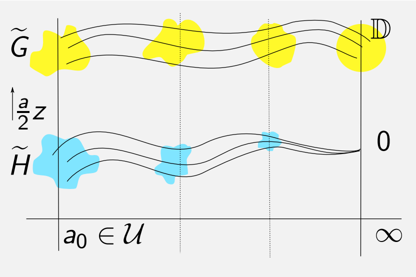

What we get by rescaling the dynamics?

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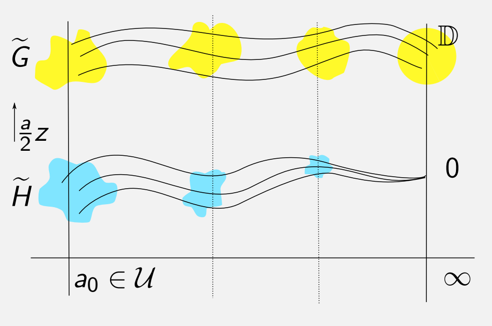

THANK YOU FOR YOUR ATTENTION!