Global Eigenvalues of Cyclically Compact Operators in Kaplansky–Hilbert Modules

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Cyclically compact sets and operators in lattice-normed spaces were introduced by A. G. Kusraev and a preliminary study of this notions was initiated. Cyclical compactness is the Boolean-valued interpretation of compactness and it also deserves an independent study. In [2] a general form of cyclically compact operators in Kaplansky–Hilbert modules, which is similar to the Schmidt representation of compact operators on Hilbert spaces, as well as a variant of the Fredholm alternative for cyclically compact operators, were also given with Boolean-valued techniques. Thus, the natural problem arises to investigate the class of cyclically compact operators in more details.

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- **Definition.** Let Λ be a Stone algebra and X be a Λ-module. The mapping $\langle\cdot \mid \cdot\rangle : X \times X \longrightarrow \Lambda$ is a Λ-*valued inner product*, if for all x, y, z in X and a in Λ the following are satisfied:
  (1) $\langle x \mid x\rangle \geq 0$; $\langle x \mid x\rangle = 0 \Leftrightarrow x = 0$;
  (2) $\langle x \mid y\rangle = \langle y \mid x\rangle^*$;
  (3) $\langle ax \mid y\rangle = a\langle x \mid y\rangle$;
  (4) $\langle x + y \mid z\rangle = \langle x \mid z\rangle + \langle y \mid z\rangle$.
- We can introduce the norms in X by the formulas

$$|x| := \sqrt{\langle x \mid x\rangle}, \qquad |||x||| := \|\langle x \mid x\rangle\|^{\frac{1}{2}}.$$

- the Cauchy–Bunyakovskiĭ–Schwarz inequality is valid

$$|\langle x \mid y\rangle| \leq |x|\,|y|.$$

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- We call X a C^*-*module* over Λ if it is complete with respect to the mixed norm $|||\cdot|||$.
- A *Kaplansky–Hilbert module* or an AW^*-*module* over Λ is a C^*-module satisfing the following two properties:
  (1) let x be an arbitrary element in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$ with $e_\xi x = 0$ for all $\xi \in \Xi$; then $x = 0$;
  (2) let $(x_\xi)_{\xi\in\Xi}$ be a norm-bounded family in X, and let $(e_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathfrak{P}(\Lambda)$; then there exists an element $x \in X$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$.
- It follows from (1) that the element x of (2) is unique; we shall write $x = \text{mix}_{\xi\in\Xi}(e_\xi x_\xi)$.
- Throughout the rest of this talk, the letters X and Y denote Kaplansky–Hilbert modules over Λ.

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Definition. A subset $\mathscr{E}$ of X is said to be *orthonormal* (*projection orthonormal*) if

(1) $\langle x \mid y \rangle = 0$ for all distinct $x, y \in \mathscr{E}$;

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An orthonormal (projection orthonormal) set $\mathscr{E} \subset X$ is a *basis* (*projection basis*) for X provided that

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- X is said to be λ-*homogeneous* if λ is a cardinal and X has a basis of cardinality λ.
- X is called *homogeneous* if it is λ-homogeneous for some cardinal λ.
- For $0 \neq b \in \mathfrak{P}(\Lambda)$, denote by $\varkappa(b)$ the least cardinal γ such that a Kaplansky–Hilbert module bX over $b\Lambda$ is γ-homogeneous. If X is homogeneous then $\varkappa(b)$ is defined for all $0 \neq b \in \mathfrak{P}(\Lambda)$. It is convenient to assume that $\varkappa(0) = 0$. We shall say that X is *strictly* γ-*homogeneous* if X is homogeneous and $\gamma = \varkappa(b)$ for all nonzero $b \in \mathfrak{P}(\Lambda)$.
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- **Theorem.**[2, 7.4.7.(2)] *There exists a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(\Lambda)$ such that $b_\xi X$ is a strictly $\varkappa(b_\xi)$-homogeneous Kaplansky–Hilbert module over $b_\xi\Lambda$.*
- Not every Kaplansky–Hilbert module has a basis, but we can split it into strictly homogeneous parts. Thus, every Kaplansky–Hilbert module has a projection basis.

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Suppose that Q is an extremal compact space. Let $C_{\#}(Q,E)$ be the set of cosets of bounded continuous vector-functions u that act from comeager subsets $\operatorname{dom} u \subset Q$ into some normed space E. Vector-functions u and v are equivalent if $u(t) = v(t)$ whenever $t \in \operatorname{dom} u \cap \operatorname{dom} v$. Note that each bounded continuous function $u : \operatorname{dom} u \longrightarrow \mathbb{R}$ admits a unique continuous extention $\overline{u} : Q \longrightarrow \mathbb{R}$, where it follows from $u \sim v$ that $\overline{u} = \overline{v}$. The set $C_{\#}(Q,E)$ is endowed, in a natural way, with the structure of a module over $C(Q)$. Moreover, we can introduce the vector norm $|\cdot| : C_{\#}(Q,E) \longrightarrow C(Q)$ by the formula

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Kaplansky–Hilbert Modules

Let H be a Hilbert space and $\langle\cdot,\cdot\rangle$ be the inner product of H. Then we can introduce some $C(Q)$-valued inner product in $C_{\#}(Q,H)$ as follows. Take continuous vector-functions $u : \operatorname{dom} u \longrightarrow H$ and $v : \operatorname{dom} v \longrightarrow H$. The function $q \mapsto \langle u(q), v(q)\rangle$ $(q \in \operatorname{dom} u \cap \operatorname{dom} v)$ is continuous and admits a unique continuation $z \in C(Q)$ to the whole of Q. If x and y are the cosets containing vector-functions u and v then assign $\langle x \mid y\rangle := z$. $\langle\cdot \mid \cdot\rangle$ is a $C(Q)$-valued inner product and $\big|x\big| = \big|\sqrt{\langle x \mid x\rangle}\big|$ $(x \in C_{\#}(Q,H))$.

Theorem. [2, 7.4.8.(1)] Suppose that Q is an extremal compact space, and H is a Hilbert space of dimension λ. The space $C_{\#}(Q,H)$ is a λ-homogeneous Kaplansky–Hilbert module over the algebra $C(Q)$.

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Theorem. [2, 7.4.8.(1)] Suppose that Q is an extremal compact space, and H is a Hilbert space of dimension λ. The space $C_{\#}(Q,H)$ is a λ-homogeneous Kaplansky–Hilbert module over the algebra $C(Q)$.

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Theorem. [2, 7.4.8.(1)] Suppose that Q is an extremal compact space, and H is a Hilbert space of dimension λ. The space $C_{\#}(Q, H)$ is a λ-homogeneous Kaplansky–Hilbert module over the algebra $C(Q)$.

Kaplansky–Hilbert Modules

- Let $(A_\xi)_{\xi\in\Xi}$ be a family of Stone algebras. If

$$A := \sum_{\xi\in\Xi}^{\oplus} A_\xi := \left\{ a = (a_\xi)_{\xi\in\Xi} \in \prod_{\xi\in\Xi} A_\xi : \sup_{\xi\in\Xi} \{\|a_\xi\|\} < \infty \right\}$$

  is equipped with the coordinatewise $*$-algebra operations, and the norm $\|a\| := \sup_{\xi\in\Xi} \{\|a_\xi\|\}$, then A is a Stone algebra and $\mathfrak{P}(A) = \prod_{\xi\in\Xi} \mathfrak{P}(A_\xi)$.

- Let Y_ξ be a Kaplansky–Hilbert modules over A_ξ. Then

$$Y := \sum_{\xi\in\Xi}^{\oplus} Y_\xi := \left\{ x = (x_\xi)_{\xi\in\Xi} \in \prod_{\xi\in\Xi} Y_\xi : \sup_{\xi\in\Xi} \{\|\|x_\xi\|\|\} < \infty \right\}$$

  equipped with the coordinatewise module operations over A and the inner product $\langle x \mid y\rangle := (\langle x_\xi \mid y_\xi\rangle)_{\xi\in\Xi}$, is a Kaplansky–Hilbert module over A.

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The following result on functional representation of Kaplansky–Hilbert modules is one of the main tools of our investigation.

Theorem.[2, Theorem 7.4.12.] *To each Kaplansky–Hilbert module X over Λ there is a family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals, such that Q_γ is γ-stable for all $\gamma \in \Gamma$ and the following unitary equivalence holds:*

$$X \simeq \sum_{\gamma\in\Gamma}^{\oplus} C_{\#}\left(Q_\gamma, \ell_2(\gamma)\right).$$

If some family $(P_\delta)_{\delta\in\Delta}$ of extremal compact spaces satisfies the above conditions then $\Gamma = \Delta$ and P_γ is homeomorphic with Q_γ for all $\gamma \in \Gamma$.

The unitary equivalence means that there are an isomorphism

$$\Psi : X \longrightarrow \sum_{\gamma\in\Gamma}^{\oplus} C_{\#}\left(Q_{\gamma}, \ell_2(\gamma)\right) \tag{1}$$

and a $*$-isomorphism

$$\Phi : \Lambda \longrightarrow \sum_{\gamma\in\Gamma}^{\oplus} C(Q_{\gamma}) \tag{2}$$

such that

$$\Phi \langle x \mid y \rangle = \langle \Psi(x) \mid \Psi(y) \rangle \quad (x, y \in X).$$

So, for every $x, y \in X$, $a \in \Lambda$ and $\pi \in \mathfrak{P}(\Lambda)$ the followings hold:

(i) $\Psi(ax) = \Phi(a)\Psi(x)$;

(ii) $\Phi|x| = |\Psi(x)|$ and $|||x||| = |||\Psi(x)||| = \sup_{\gamma\in\Gamma}\{\|(\Psi(x))_{\gamma}\|\}$.

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Let B be a complete Boolean algebra. Denote by $\mathrm{Prt}(B)$ the set of sequences $\nu : \mathbb{N} \longrightarrow B$ which are partitions of unity in B. For $\nu_1, \nu_2 \in \mathrm{Prt}(B)$, the formula $\nu_1 \ll \nu_2$ abbreviates the following assertion:

$$\text{if } m, n \in \mathbb{N} \text{ and } \nu_1(m) \wedge \nu_2(n) \neq 0_B \text{ then } m < n.$$

Given a mix-complete subset $K \subset X$, a sequence $s : \mathbb{N} \longrightarrow K$, and a partition $\nu \in \mathrm{Prt}(B)$, put $s_\nu := \mathrm{mix}_{n\in\mathbb{N}}(\nu(n)s(n))$. A *cyclic subsequence* of $s : \mathbb{N} \longrightarrow K$ is any sequence of the form $(s_{\nu_k})_{k\in\mathbb{N}}$, where $(\nu_k)_{k\in\mathbb{N}} \subset \mathrm{Prt}(B)$ and $(\forall k \in \mathbb{N})\ \nu_k \ll \nu_{k+1}$. A subset $C \subset X$ is said to be *cyclically compact* if C is mix-complete and every sequence in C has a cyclic subsequence that converges (in norm) to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set.

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Cyclically Compact Sets in Kaplansky–Hilbert Modules

Let B be a complete Boolean algebra. Denote by $\mathrm{Prt}(B)$ the set of sequences $\nu : \mathbb{N} \longrightarrow B$ which are partitions of unity in B. For $\nu_1, \nu_2 \in \mathrm{Prt}(B)$, the formula $\nu_1 \ll \nu_2$ abbreviates the following assertion:

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- Define $K(q) := \{x(q) : \widetilde{x} \in K, q \in \mathrm{dom}(x)\}$ $(q \in Q, K \subset C_{\#}(Q, H))$.
- **Proposition.** *Let K be a cyclically compact subset of $C_{\#}(Q, H)$. Then for each $q \in Q$, $K(q)$ is a closed set in H.*
- **Lemma.** *Let K be a relatively cyclically compact subset of $C_{\#}(Q, H)$. Then there exists a comeager set $Q_0 \subset Q$ such that $K(q)$ is precompact in H for all $q \in Q_0$.*

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- **Proposition.** *Let K be a mix-complete subset of $C_{\#}(Q,H)$ and $Q_0 \subset Q$ be a comeager subset. If K_q is a compact set in H for $q \in Q_0$ and*

$$K \subset \{\widetilde{x} \in C_{\#}(Q,H) : x(q) \in K_q \, (\forall q \in \mathrm{dom}(x) \cap Q_0)\},$$

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- As a corollary of the above proposition, the closed unit ball of a Stone algebra is a cyclically compact set.

Operators on Kaplansky–Hilbert Modules

- Let $B_\Lambda(X, Y)$ denote the set of all continuous Λ-linear operators from X into Y. For brevity, $B_\Lambda(X, X)$ will be denoted by $B_\Lambda(X)$.
- We call a Λ-linear operator $T^* : Y \longrightarrow X$ the *adjoint* of Λ-linear operator $T : X \longrightarrow Y$ if $\langle Tx \mid y \rangle = \langle x \mid T^* y \rangle$ for all x and y.
- I. Kaplansky showed that a Λ-linear operator $T : X \longrightarrow Y$ is continuous if and only if T has an adjoint [1, Theorem 6]. Moreover, he also showed that $B_\Lambda(X)$ is an AW^*-algebra of type I with center isomorphic to Λ [1, Theorem 7].

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Denote by $SC_{\#}(Q, B(H))$ the set of all cosets $\widetilde{u}$ such that operator-function $u : \operatorname{dom} u \longrightarrow B(H)$ is defined on the comeager set $\operatorname{dom} u \subset Q$ and continuous in the strong operator topology and the set $\left\{ \left|\widetilde{u}h\right| : \|h\| \le 1 \right\}$ is bounded in $C(Q)$ where $\widetilde{u}h := \widetilde{uh} \in C_{\#}(Q, H)$ $(h \in H)$. Since $\left|\widetilde{u}h\right|$ agrees with the function $q \mapsto \|u(q)h\|$ $(q \in \operatorname{dom} u)$, the inclusion $\widetilde{u} \in SC_{\#}(Q, B(H))$ means that the function $q \mapsto \|u(q)\|$ $(q \in \operatorname{dom} u)$ is bounded and continuous on a comeager set. Hence, there are an element $\left|\widetilde{u}\right| \in C(Q)$ and a comeager set $Q_0 \subset Q$ satisfying $\left|\widetilde{u}\right|(q) = \|u(q)\|$ $(q \in Q_0)$.

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Moreover, $SC_{\#}(Q, B(H))$ can be equipped with the structure of a $*$-algebra and a unitary $C(Q)$-module.

If $\widetilde{u} \in SC_{\#}(Q, B(H))$ and the element $\widetilde{x} \in C_{\#}(Q, H)$ is determined by a continuous vector-function $x : \operatorname{dom} x \longrightarrow H$ then we may define $\widetilde{u}\widetilde{x} := \widetilde{ux} \in C_{\#}(Q, H)$, with $ux : q \mapsto u(q)x(q)$ $(q \in \operatorname{dom} u \cap \operatorname{dom} x)$. We also have

$$\left|\widetilde{u}x\right| \leq \left|\widetilde{u}\right| |x| \quad (x \in C_{\#}(Q, H)).$$

Denote the operator $x \mapsto \widetilde{u}x$ by $S_{\widetilde{u}}$.

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Theorem.[2, Theorem 7.5.10.] *To each operator $U \in \mathrm{End}(C_{\#}(Q,H))$ there is a unique element $u \in SC_{\#}(Q,B(H))$ satisfying $U = S_u$. The mapping $U \mapsto u$ is a $*$-B-isomorphism of $\mathrm{End}(C_{\#}(Q,H))$ onto $SC_{\#}(Q,B(H))$.*

Corollary. *To each Kaplansky–Hilbert module X over Λ there is a family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals and the following holds:*

$$B_\Lambda(X) \simeq \sum_{\gamma\in\Gamma}^{\oplus} SC_{\#}\left(Q_\gamma, B(\ell_2(\gamma))\right).$$

Cyclically Compact Operators on Kaplansky–Hilbert Modules

- An operator $T \in B_\Lambda(X, Y)$ is called *cyclically compact* if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact in Y.
- The set of all cyclically compact operators is denoted by $\mathscr{K}(X, Y)$.
- **Proposition.** *Let $U = S_{\tilde{u}}$ be in* $\mathrm{End}\,(C_\#(Q, H))$*. Then the following statements are equivalent:*
  - (i) *$U = S_{\tilde{u}}$ is a cyclically compact operator on $C_\#(Q, H)$;*
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Theorem.[2, Theorem 8.5.6.] *Let T in $\mathscr{K}(X, Y)$. There are orthonormal families $(e_k)_{k\in\mathbb{N}}$ in X, $(f_k)_{k\in\mathbb{N}}$ in Y, and a family $(\mu_k)_{k\in\mathbb{N}}$ in Λ such that the following hold:*

(1) $\mu_{k+1} \leq \mu_k$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim_{k\longrightarrow\infty} \mu_k = 0$;

(2) *there exists a projection π_∞ in Λ such that $\pi_\infty \mu_k$ is a weak order-unity in $\pi_\infty \Lambda$ for all $k \in \mathbb{N}$;*

(3) *there exists a partition $(\pi_k)_{k=0}^{\infty}$ of the projection $\pi_\infty^\perp$ such that $\pi_0\mu_1 = 0$, $\pi_k \leq [\mu_k]$, and $\pi_k\mu_{k+1} = 0$, $k \in \mathbb{N}$;*

(4) *the representation is valid*

$$T = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} \mu_k e_k^\sharp \otimes f_k + bo\text{-}\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} \mu_k e_k^\sharp \otimes f_k.$$

Theorem. *Let T be a cyclically compact operator from X to Y. There exist sequences $(e_k)_{k\in\mathbb{N}}$ in X and $(f_k)_{k\in\mathbb{N}}$ in Y and a sequence $(s_k(T))_{k\in\mathbb{N}}$ of positive elements in Λ such that*

(1) $\langle e_k \mid e_l\rangle = \langle f_k \mid f_l\rangle = 0$ $(k \neq l)$ *and* $[s_k(T)] = \left|e_k\right| = \left|f_k\right|$;

(2) $s_{k+1}(T) \leq s_k(T)$ *and* $o\text{-}\lim s_k(T) = \inf_{k\in\mathbb{N}} s_k(T) = 0$;

(3) *there exists a projection* π_∞ *in* $\mathfrak{P}(\Lambda)$ *such that* $\pi_\infty s_k(T)$ *is a weak order- unity in* $\pi_\infty\Lambda$ *for all* $k \in \mathbb{N}$;

(4) *there exists a partition* $(\pi_k)_{k=0}^{\infty}$ *of the projection* $\pi_\infty^{\perp}$ *such that* $\pi_0 s_1(T) = 0$, $\pi_k \leq [s_k(T)]$, *and* $\pi_k s_{k+1}(T) = 0$, $k \in \mathbb{N}$;

(5) *for each* x *the following equality is valid*

$$Tx = \pi_\infty bo\text{-}\sum_{k=1}^{\infty} s_k(T)\langle x \mid e_k\rangle f_k + bo\text{-}\sum_{n=1}^{\infty}\pi_n\sum_{k=1}^{n} s_k(T)\langle x \mid e_k\rangle f_k$$
$$= bo\text{-}\sum_{k\in\mathbb{N}} s_k(T)\langle x \mid e_k\rangle f_k.$$

Theorem. (The Rayleigh–Ritz minimax formula) *Let T be a cyclically compact operator from X to Y. Then*

$$s_n(T) = \inf\left\{\sup\left\{\left|Tx\right| : \left|x\right| \leq \mathbf{1},\, x \in J^{\perp}\right\}\right\}$$

where the infmum is taken over all projection orthonormal subset J of X such that $\mathrm{card}(J) < n$, *and the infimum is achieved.*

The Multiplicity of Global Eigenvalues

- Denote $[\lambda] = \inf\{\pi \in \mathfrak{P}(\Lambda) : \pi\lambda = \lambda\}$, the *support* of λ in Λ.
- Let T be an operator on X. A scalar $\lambda \in \Lambda$ is said to be an *eigenvalue* if there exists nonzero $x \in X$ such that $Tx = \lambda x$.
- A nonzero eigenvalue λ is called a *global eigenvalue* if for any nonzero projection $\pi \in \Lambda$ with $\pi \leq [\lambda]$ there exists a nonzero $x \in \pi X$ such that $Tx = \lambda x$.
- **Proposition.** *Let T be a continuous Λ-linear operator on X and λ be a nonzero scalar. Then the following statements are equivalent.*
  - (i) *The scalar $\lambda \in \Lambda$ is a global eigenvalue of T.*
  - (ii) *There is $x \in X$ such that $Tx = \lambda x$ and $[\,|x|\,] \geq [\lambda]$.*
  - (iii) *There is $x \in X$ such that $Tx = \lambda x$ and $|x| \in \mathfrak{P}(\Lambda)$ with $|x| \geq [\lambda]$*

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The Multiplicity of Global Eigenvalues

- **Lemma.** *Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. If π is a projection with $0 < \pi \leq [\lambda]$, then there exist a projection μ with $0 < \mu \leq \pi$ and $n \in \mathbb{N}$ such that $\mu \mathrm{Ker}\,(T - \lambda I)^n = \mu \mathrm{Ker}\,(T - \lambda I)^{n+1}$, i. e., $\mu N_\lambda := \mu \mathrm{Ker}\,(T - \lambda I)^n$.*
- Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. Define $\rho_N(\lambda) := \sup\left\{\pi \in \mathfrak{P}(\Lambda) : \pi N_\lambda = \pi \mathrm{Ker}\,(T - \lambda I)^N,\ \pi \leq [\lambda]\right\}$ for each $N \in \mathbb{N}$.

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Corollary. *Let T be a cyclically compact operator on X and λ be a global eigenvalue of T. The following conditions are satisfied:*

(1) $\rho_N(\lambda) \le \rho_{N+1}(\lambda)$.

(2) $\rho_N(\lambda)\mathrm{Ker}\,(T - \lambda I)^N = \rho_N(\lambda)\mathrm{Ker}\,(T - \lambda I)^{N+1}$.

(3) $\rho_N(\lambda) N_\lambda = \rho_N(\lambda)\mathrm{Ker}\,(T - \lambda I)^N$.

(4) $[\lambda] = \bigvee_{N\in\mathbb{N}} \rho_N(\lambda) = \sup\{\rho_N(\lambda) : N \in \mathbb{N}\}$.

(5) $\rho_N(\lambda) N_\lambda$ *is a Kaplansky–Hilbert module over* $\rho_N(\lambda)\Lambda$.

The Multiplicity of Global Eigenvalues

For each $N \in \mathbb{N}$, there exists a partition of $\rho_N(\lambda)$, $(\tau_{\lambda,N}(n))_{n\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)$ such that $\tau_{\lambda,N}(n)N_\lambda$ is n-homogeneous Kaplansky−Hilbert module over $\tau_{\lambda,N}(n)\Lambda$. So, there is a unique sequence $(\tau_{\lambda,l})_{l\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)^{\mathbb{N}}$ such that $\tau_{\lambda,l} := (\tau_{\lambda,l}(n))_{n\in\mathbb{N}}$ is a partition of $\rho_l(\lambda)$ and $\tau_{\lambda,l}(n)N_\lambda = \tau_{\lambda,l}(n)\mathrm{Ker}\,(T - \lambda I)^l$ is n-homogeneous Kaplansky−Hilbert module over $\tau_{\lambda,l}(n)\Lambda$. Moreover, $\tau_{\lambda,l}(n) \leq \tau_{\lambda,l+1}(n)$ and $\tau_{\lambda,l}(n) \wedge \tau_{\lambda,k}(m) = 0$ are satisfied for all $k, l, m, n \in \mathbb{N}$ with $n \neq m$. So, $(\tau_\lambda(n))_{n\in\mathbb{N}}$ is a partition of $[\lambda]$ where $\tau_\lambda(n) := \sup\{\tau_{\lambda,l}(n) : l \in \mathbb{N}\}$.

Definition. Let λ be a global eigenvalue of T. We call

$$\overline{\tau}_\lambda := o\text{-}\sum_{n\in\mathbb{N}} n\tau_\lambda(n) \in (\mathrm{Re}\Lambda)^\infty$$

the *multiplicity* of global eigenvalue λ of T.

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Global Eigenvalues of Cyclically Compact Operators on $C_{\#}(Q,H)$

- A_λ will denote the clopen set corresponding to the projection $[\lambda]$ $(\lambda \in \Lambda)$.
- **Lemma.** *Let $U = S_{\tilde{u}}$ be in* $\mathrm{End}(C_{\#}(Q,H))$ *and the function λ be a global eigenvalue of U. Then there is a meager subset B_0 such that $\lambda(q)$ is a nonzero eigenvalue of $u(q)$ for all $q \in A_\lambda \setminus B_0$.*
- Denote by $\mathrm{Sp}^*(u(q)) := \mathrm{Sp}(u(q)) \setminus \{0\}$ the set of nonzero elements of the spectrum of $u(q)$.

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Lemma. *Let* $U = S_{\widetilde{u}}$ *be a cyclically compact operator on* $C_{\#}(Q, H)$ *and let* Σ *be a finite subset of* $C(Q)$ *and the set*

$$A_u \subset \{q \in \mathrm{dom}(u) : \mathrm{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\} \neq \varnothing\}$$

be not meager in Q. *If* λ_q *is in* $\mathrm{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\}$ *for each* $q \in A_u$, *then there is a global eigenvalue* λ *of* U *and a comeager set* Q_0 *that satisfy the following conditions:*

(1) $[\lambda] = \bigvee_{N \in \mathbb{N}} \pi_N$ *where* π_N *is the projection corresponding to clopen set* $U_N := \mathrm{int}\,(\mathrm{cl}\,(A_N))$ *with*

$$A_N := \{q \in A_u : (\forall \sigma \in \Sigma)|\sigma(q) - \lambda_q| \geq 1/N \text{ and } |\lambda_q| \geq 1/N\};$$

Global Eigenvalues of Cyclically Compact Operators on $C_{\#}(Q, H)$

(2) $\pi_N|\lambda| \geq \frac{1}{N}\pi_N$ *and* $\pi_N|\sigma - \lambda| \geq \frac{1}{2N}\pi_N$ $(N \in \mathbb{N}, \sigma \in \Sigma)$*;*

(3) *if q is in $A_N \cap Q_0$, then $|\lambda(q)| \geq \frac{1}{N}$ and $|\sigma(q) - \lambda(q)| \geq \frac{1}{2N}$ hold for each $\sigma \in \Sigma$;*

(4) *if $\lambda(q) \neq 0$ holds for some $q \in Q_0$, then $\lambda(q) \in \mathrm{Sp}^*(u(q)) \setminus \{\sigma(q) : \sigma \in \Sigma\}$;*

(5) *if $\lambda(q) = 0$ holds for some $q \in Q_0$, then $q \notin A_u$.*

Global Eigenvalues of Cyclically Compact Operators on $C_{\#}(Q, H)$

Theorem. *Let T be a cyclically compact operator on X. Then there exist a sequence $(\lambda_k)_{k\in\mathbb{N}}$ consisting of global eigenvalues or zeros in Λ with the following properties:*

1. $|\lambda_k| \leq |\!|T|\!|$, $[\lambda_k] \geq [\lambda_{k+1}]$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim \lambda_k = 0$;
2. *there exists a projection* π_∞ *in* Λ *such that* $\pi_\infty|\lambda_k|$ *is a weak order-unity in* $\pi_\infty\Lambda$ *for all* $k \in \mathbb{N}$;
3. *there exists a partition* (π_k) *of the projection* $\pi_\infty^\perp$ *such that* $\pi_0\lambda_1 = 0$, $\pi_k \leq [\lambda_k]$, *and* $\pi_k\lambda_{k+m} = 0$, $m, k \in \mathbb{N}$;
4. $\pi\lambda_{k+m} \neq \pi\lambda_k$ *for every nonzero projection* $\pi \leq \pi_\infty + \pi_k$ *and for all* $m, k \in \mathbb{N}$;
5. *every global eigenvalue* λ *of* T *is of the form* $\lambda = \text{mix}_{k\in\mathbb{N}}(p_k\lambda_k)$, *where* $(p_k)_{k\in\mathbb{N}}$ *is a partition of* $[\lambda]$.

Definition. The sequence $(\lambda_k(T))_{k\in\mathbb{N}}$, where $\lambda_k(T) := \lambda_k$ is taken from the theorem above is called a *global eigenvalue sequence* of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$ where $\overline{\tau}_k(T) := \overline{\tau}_{\lambda_k}$.

- **Definition.** The *trace class* $\mathscr{S}_1(X, Y)$ consists of cyclically compact operators T such that $(\mu_k)_{k\in\mathbb{N}}$ is o-summable in Λ. We put
$$v_1(T) := o\text{-}\sum_{k\in\mathbb{N}} \mu_k.$$
The operators of class $\mathscr{S}_1(X, Y)$ are called *trace class operators*.
- **Definition.** For $T \in \mathscr{S}_1(X)$ define the *trace* of T by
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Theorem. *Let T be a cyclically compact operator on X and $(\lambda_k(T))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$. Then the following properties hold:*

(1) (Weyl-inequality) *if $(\pi s_k(T))_{k\in\mathbb{N}}$ is o-summable in Λ for some projection π, then the following inequality holds*

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(T)|\lambda_k(T)| \leq o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(T);$$

(2) (Horn-inequality) *Suppose that T_k is a cyclically compact operator on X for $1 \leq k \leq K$. Then*

$$\prod_{i=1}^{N} s_i(T_K \cdots T_1) \leq \prod_{k=1}^{K} \prod_{i=1}^{N} s_i(T_k) \quad (N \in \mathbb{N}).$$

Weyl- and Horn-type Inequalities and Lidskiĭ Trace Formula

(3) (Lidskiĭ trace formula) *if* $T \in \mathscr{S}_1(X)$, *then the following equality holds*

$$\mathrm{tr}(T) = o\text{-}\sum_{k \in \mathbb{N}} \overline{\tau}_k(T)\lambda_k(T).$$

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