Cross Ratio Problem On Some Subclasses of Univalent Functions

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Introduction

Let D be a simply connected domain in the closed complex plane $\mathbb{C}$. For given distinct poins $z_k(k = 1, ..., 4)$ in D, let $(z_1, z_2, z_{3,} z_4)$ denote their cross ratio. It is well known that if $w = h(z)$ is a Mobius transformation and $w_k = h(z_k)$, then $(z_1, z_2, z_{3,} z_4) = (w_1, w_2, w_{3,} w_4)$ but it may not be so if h is an arbitrary univalent function in D.

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If $w = f(z)$ is univalent in D then the quotient $(w_1, w_2, w_{3,} w_4)/(z_1, z_2, z_{3,} z_4)$ denoted by $Q(f) = Q_f(z_1, z_2, z_{3,} z_4)$ determines a **measure of deviation** of f from a Mobius transformation. So there arises the problem of finding the variability region of $Q(f)$ when f belons to a class of univalent functions which is compact in the space $H(D)$of holomorphic functions in D.

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In this talk we will point our a couple of results in this direction and give a concrete result for a small class "**univalent second degree polynomials**".

Let $f \in S$ and suppose that z, ζ, ϖ, τ are in $D = \{z \in \mathbb{C} : |z| < 1\}$, then

$$Q(f) = \frac{f'(z) - f(\varpi)}{f(z) - f(\tau)} \frac{f(\zeta) - f(\tau)}{f(\zeta) - f(\varpi)} \frac{z - \tau}{z - \varpi} \frac{\zeta - \varpi}{\zeta - \tau}.$$

Let us also define the function

$$G_f(z, \zeta) = \frac{f(z)}{z} \frac{f(\zeta)}{\zeta} \frac{\zeta - z}{f(\zeta) - f(z)}.$$

Observe that we have

$$Q(f) = \frac{G_f(z, \tau)}{G_f(z, \varpi)} \frac{G_f(\zeta, \varpi)}{G_f(\zeta, \tau)}.$$

Define $z^* = \varphi(z) = e^{i\gamma}\frac{z-\varpi}{1-\overline{\varpi}z}$ such that $\tau^* = \varphi(\tau) = p \in (0,1)$. Then the formula

$$\frac{f(z)-f(\varpi)}{f(z)-f(\tau)} = e^{-i\gamma}\frac{f'(\varpi)(1-|\varpi|^2)}{f(\tau)-f(\varpi)}F(e^{i\gamma}\frac{\varpi - z}{1-\overline{\varpi}z}) \tag{1}$$

defines a function $F : D \to \overline{\mathbb{C}}$ in $S(p)$; such that it is meromorphic and univalent with the normalization $F(0) = F'(0) - 1 = 0$ and $F(p) = \infty$. In this setting, it follows that

$$Q(f) = \frac{(p-z^*)F(z^*)}{pz^*}\frac{p\zeta^*}{(p-\zeta^*)F(\zeta^*)} = \frac{G_F(z^*,p)}{G_F(\zeta^*,p)} \tag{2}$$

Therefore, the variability region of

$$\frac{G_f(z,\tau)}{G_f(z,\varpi)}\frac{G_f(\zeta,\varpi)}{{}_f(\zeta,\tau)}$$

in the class S is the same of the variability region of $\frac{G_F(z^*,p)}{G_F(\zeta^*,p)}$ in the class $S(p)$.

2. Similarly, for each f in the class of univalent functions in S the function $g(z) = \dfrac{1}{f(\frac{1}{z})}$ is a function in the class of univalent functions in Σ in the exterior $\Delta = \{z \in \overline{\mathbb{C}} : |z| > 1\}$. Hence, it is easy to see that

$$Q(g) = \frac{g(z) - g(\varpi)}{g(z) - g(\tau)} \frac{g(\zeta) - g(\tau)}{g(\zeta) - g(\varpi)} \frac{z - \tau}{z - \varpi} \frac{\zeta - \varpi}{\zeta - \tau} = Q_{\frac{1}{f}}(\frac{1}{z}, \frac{1}{\zeta}, \frac{1}{\varpi}, \frac{1}{\tau}).$$

Therefore, the variability region of $Q(g)$ in the class Σ is the same as the variability region of $Q(f)$ in the class S.

3. It follows from the equality (1) that

$$G_F(z^*, p) = \frac{G_f(z, \tau)}{G_f(z, \varpi)} \frac{G_f(\varpi, \varpi)}{G_f(\varpi, \tau)}$$

from which the equality (2) follows. Let us try to maximize the quantity $\Phi = e^{i\gamma} \log \frac{G(a,p)}{G(b,p)}$ and suppose that $F \in S(p)$ maximizes Φ. Considering the competing function

$$F^* = F + \frac{\varepsilon F^2}{w(F - w)} + o(\varepsilon^2)$$

where $F \neq w$, it follows that

$$\Phi^* = \Phi + \frac{\varepsilon e^{i\gamma}(F(b) - F(a))}{w(w - F(a))(w - F(b))} + o(\varepsilon^2).$$

From the theory of univalent functions we get that the maximal function F satisfies the Schiffer differential equation

$$\left(\frac{zF'(z)}{F(z)}\right)^2 \frac{F(z)(F(b)-F(a))e^{i\gamma}}{(F(z)-F(a))(F(z)-F(b))} = \frac{Az(1-e^{-i\alpha}z)^2(1-e^{-i\beta}z)^2}{(p-z)(1-pz)(a-z)(1-\bar{a}z)(b-z)(1-\bar{b}z)}$$

for $|z|=1$. In other words, $Ae^{-i(\alpha+\beta)} \geq 0$ with $A = \frac{(F(b)-F(a))e^{i\gamma}}{F(a)F(b)}pab$ and

$$\left(\frac{zF'(z)}{F(z)}\right)^2 \frac{F(z)F(a)F(b)}{(F(z)-F(a))(F(z)-F(b))} = \frac{pabz(1-e^{-i\alpha}z)^2(1-e^{-i\beta}z)^2}{(p-z)(1-pz)(a-z)(1-\bar{a}z)(b-z)(1-\bar{b}z)}.$$

Preceeding differential equation is equivalent to the equality

$$\frac{G(z,p)G(z,a)G(z,b)}{G(z,z)^2} = \frac{(1-e^{-i\alpha}z)^2(1-e^{-i\beta}z)^2}{(1-pz)(1-\bar{a}z)(1-\bar{b}z)}.$$

4. If $|\alpha|^2 + |(\alpha-\beta)(\alpha-\gamma)| + \left|(\beta+\gamma)|\alpha|^2 - \overline{\alpha}\beta\gamma - \alpha\right| \leq 1$, then the functions $f(z) = \frac{z(1-\alpha z)}{(1-\beta z)(1-\gamma z)}$ are univalent Avci [2] and form a subset S_2 of the class S. For this class the variability region of

$$Q(f) = \frac{1-\alpha(z+\varpi)+(\alpha\beta+\alpha\gamma-\beta\gamma)z\varpi}{1-\alpha(z+\tau)+(\alpha\beta+\alpha\gamma-\beta\gamma)z\tau}\frac{1-\alpha(\zeta+\tau)+(\alpha\beta+\alpha\gamma-\beta\gamma)\zeta\tau}{1-\alpha(\zeta+\varpi)+(\alpha\beta+\alpha\gamma-\beta\gamma)\zeta\varpi}$$

over S_2 is obtained by the image of the subdomain $(\alpha,\beta,\gamma) \in \overline{D}\times\overline{D}\times\overline{D}$ satisfying the condition above. If $\alpha=\beta=0$ then S_2 reduces the class of second degree univalent polynomials P_2 and the variability region

$$Q(f) = \frac{1-\alpha(z+\varpi)}{1-\alpha(z+\tau)}\frac{1-\alpha(\zeta+\tau)}{1-\alpha(\zeta+\varpi)}$$

over P_2 is obtained by the image of the closed disc

$$\overline{D}_{\frac{1}{2}}(0) = \{\alpha \in \mathbb{C} : |\alpha| \leq \frac{1}{2}\}.$$

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Thank you very much!