Extension Properties of Pluricomplex Green Function

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- Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $a \in \Omega$ and g be a plurisubharmonic function in a neighborhood of a. If

$$g(z) - \log \|z - a\| = O(1) \text{ as } z \to a,$$

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g is said to have a logarithmic pole at a.

Let $\mathcal{G}(\Omega, a)$ denote the family of all negative plurisubharmonic functions on Ω which have a logarithmic pole at a.

- The extremal function

$$g_\Omega(z, a) = \sup\{g(z) : g \in \mathcal{G}(\Omega, a)\}$$

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It follows that if $f : \Omega \to \Omega'$ is holomorphic, then

$$g_{\Omega'}(f(z), f(w)) \leq g_\Omega(z, w)\ ,\ z, w \in \Omega.$$

- For smooth plurisubharmonic functions, the Monge-Ampére operator is defined by

$$(dd^c v)^n := \underbrace{dd^c v \wedge \cdots \wedge dd^c v}_{n \text{ times}}$$

where $d = \partial + \bar{\partial}$ is the exterior differentiation, $d^c = i(\bar{\partial} - \partial)$, and so $dd^c = 2i\partial\bar{\partial}$.

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It is proved by Demailly that for Ω nice, g_Ω solves the following Dirichlet problem:

$$\begin{cases} u \in \mathcal{C}(\Omega \setminus \{a\}) \cap \mathcal{G}(\Omega, a), \\ (dd^c u)^n = (2\pi)^n \delta_a \text{ in } \Omega, \\ u(z) \to 0 \text{ as } z \to \partial\Omega, \end{cases}$$

where δ_a is the Dirac mass at a.

Problem *Let* $\Omega^0 \subset \mathbb{C}^n$ *be a bounded pseudoconvex domain with* $0 \in \Omega^0$. *Let* $E \not\ni 0$ *be a compact subset of* Ω^0 *and* $\Omega = \Omega^0 \setminus E$.

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1. *For what kind of set* E *does there exist* $\tilde{\Omega}$ *with* $\Omega \subsetneq \tilde{\Omega} \subset \Omega^0$ *such that*

$$g_{\Omega}(z, 0) = g_{\tilde{\Omega}}(z, 0) \text{ for all } z \in \Omega?$$

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2. *What is the largest* $\tilde{\Omega}$ *that satisfies (1)?*

- $\Omega \subset \mathbb{C}^n$ is called a Reinhardt domain if $(z_1, \ldots, z_n) \in \Omega$ implies $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \Omega$ for every $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in (\partial\Delta)^n$.

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  It is called a complete Reinhardt domain if $(z_1, \ldots, z_n) \in \Omega$ implies $(r_1 e^{i\theta_1} z_1, \ldots, r_n e^{i\theta_n} z_n) \in \Omega$ for every $0 \leq r_j \leq 1$ and $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in (\partial\Delta)^n$.

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- Let Ω be a Reinhardt domain, g a function defined on it such that

$$g(z_1, \ldots, z_n) = g(|z_1|, \ldots, |z_n|), \text{ for any } z \in \Omega,$$

then g is called polyradial.

- If $\Omega \ni 0$ is Reinhardt domain, then the pluricomplex Green function of Ω with pole at 0 is polyradial as it is invariant under biholomorphic transformations

$$(z_1, \ldots, z_n) \to (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n),$$

where $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in (\partial\Delta)^n$.

- Let Ω^0 be a complete Reinhardt domain. Then

$$\Omega^0 = \{z \in \mathbb{C}^n : h_{\Omega^0}(z) < 1\}$$

where h_{Ω^0} is the Minkowski function of Ω^0, defined by

$$h_{\Omega^0}(z) := \inf\left\{t > 0 : \frac{z}{t} \in \Omega^0\right\}, z \in \mathbb{C}^n.$$

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Proposition *If Ω^0 is a bounded pseudoconvex complete Reinhardt domain with Minkowski function h_{Ω^0},*

$$g_{\Omega^0}(z, 0) = \log h_{\Omega^0}(z), \; z \in \Omega^0.$$

- Define functions

$$\ell(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|),\ (z_1, \ldots, z_n) \in \mathbb{C}^n$$

and

$$\mathfrak{e}(x_1, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_n}),\ (x_1, \ldots, x_n) \in [-\infty, \infty)^n$$

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Proposition *If g is a negative polyradial function in Ω, then $g \in \mathcal{PSH}(\Omega)$ if and only if the restriction of $g \circ \mathfrak{e}$ to ω is convex.*

- A function $u : \omega \to \mathbb{R}$ is said to be a function with normalized growth at $-\infty$ if for any $a = (a_1, \dots, a_n) \in \omega$, there exists C_a such that

$$u(a_1 + t, \dots, a_n + t) \leq t + C_a, \text{ for any } t \leq 0.$$

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We are going to call this function the convex envelope of ω with normalized growth at $-\infty$.

Theorem *(Klimek) If $\Omega \subset \mathbb{C}^n$ is a bounded Reinhardt domain, then*

$$u_\omega(x) = g_\Omega(\mathfrak{e}(x), 0), \ x \in \omega.$$

Theorem *Let* $\Omega^0 \ni 0$ *be a pseudoconvex, bounded, complete Reinhardt domain in* $\mathbb{C}^2$*. Let* $E \not\ni 0$ *be a closed, strictly logaritmically convex subset of* Ω^0 *and* $\Omega = \Omega^0 \setminus E$*. Then, there exists a Reinhardt domain* $\tilde{\Omega}$ *such that* $\Omega \subsetneq \tilde{\Omega} \subset \Omega^0$ *and*

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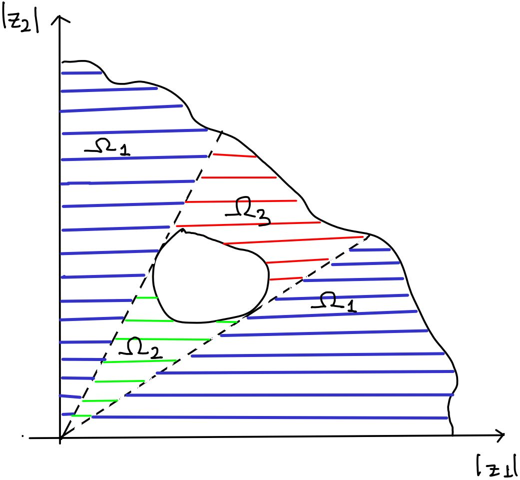
Strictly logarithmically convex means that $\ell(E)$ is strictly convex.



$|z_2|$
Ω_1
Ω_3
Ω_1
Ω_2
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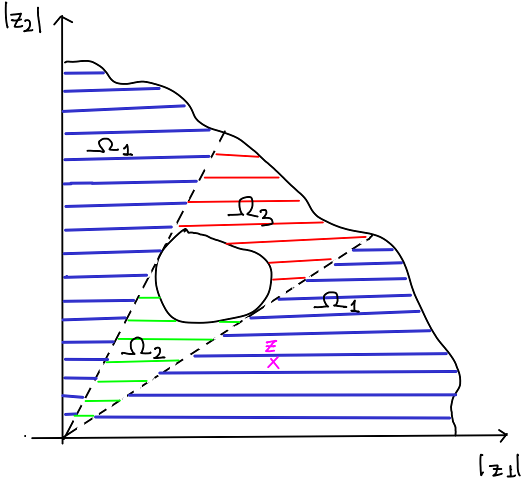
$|z_2|$
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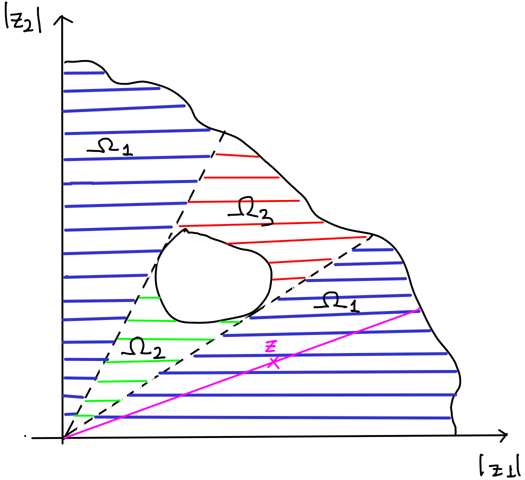

|z2|
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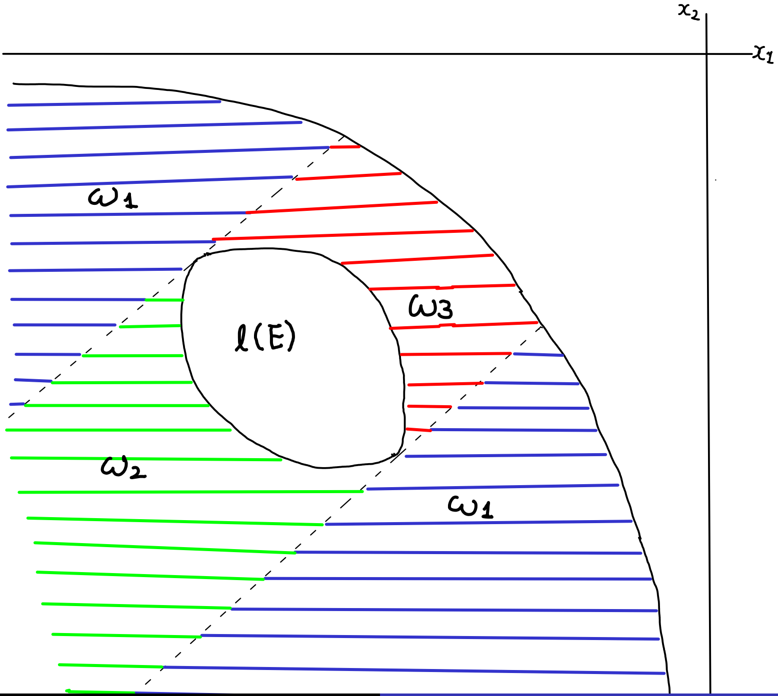
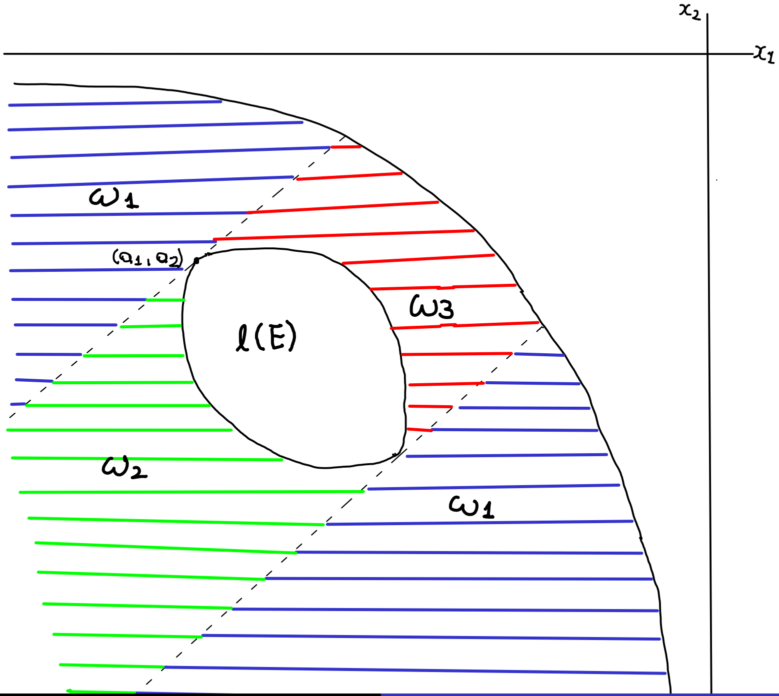
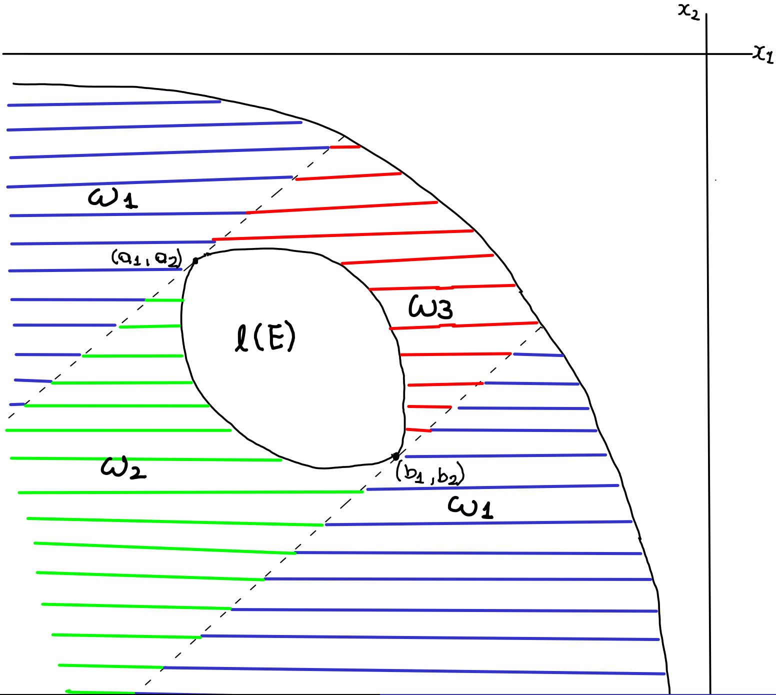


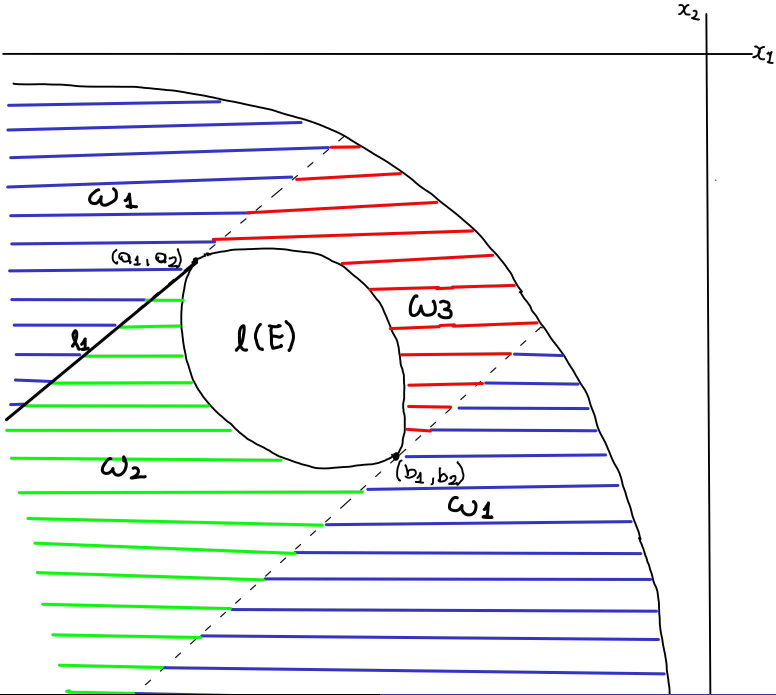
x_2
x_1
ω_1
$\ell(E)$
ω_3
ω_2
ω_1



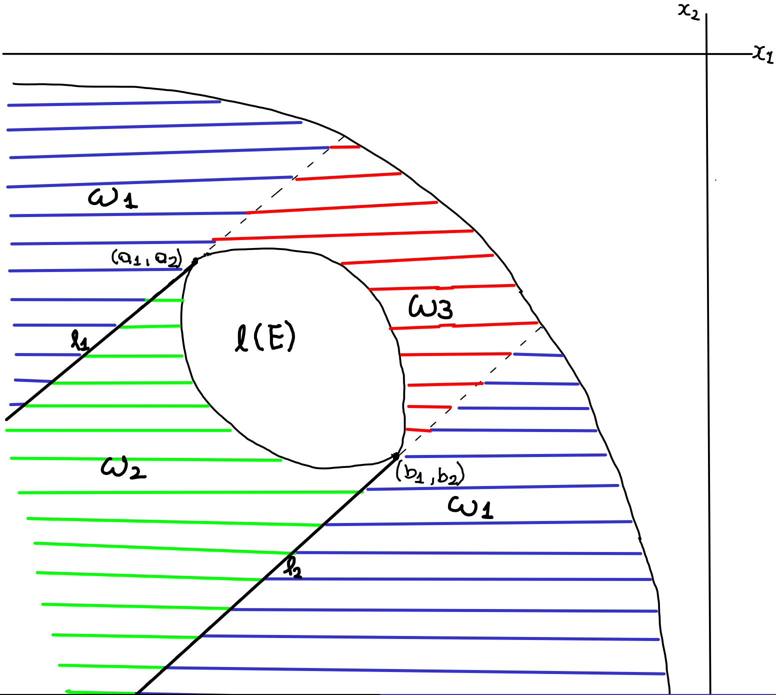
x_2
x_1
ω_1
(a_1, a_2)
$l(E)$
ω_3
ω_2
ω_1



x_2
x_1
ω_1
(a_1, a_2)
$l(E)$
ω_3
ω_2
(b_1, b_2)
ω_1



x2
x1
ω1
(a1,a2)
l1
l(E)
ω3
ω2
(b1,b2)
ω1



x_2
x_1
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(a_1, a_2)
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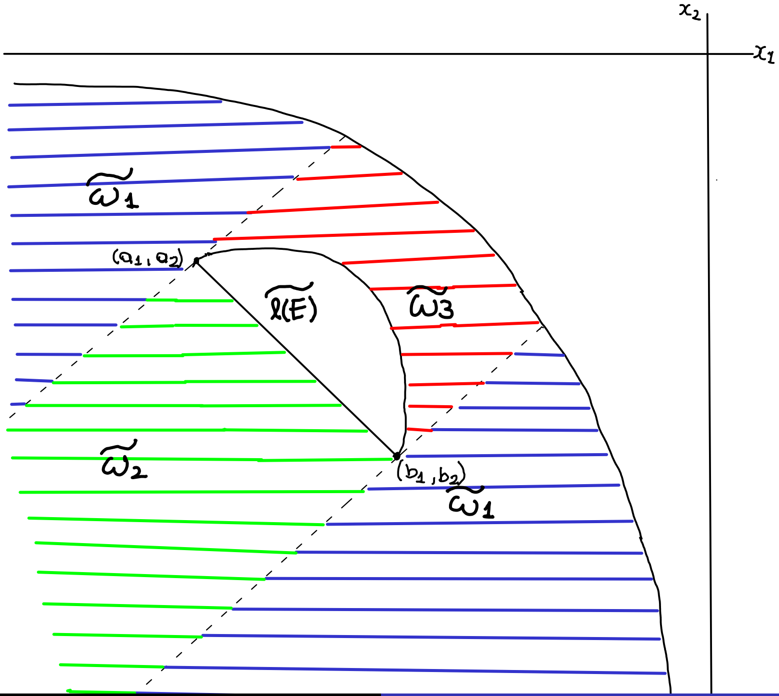
$l(E)$
ω_3
ω_2
(b_1, b_2)
ω_1
l_2


x_2
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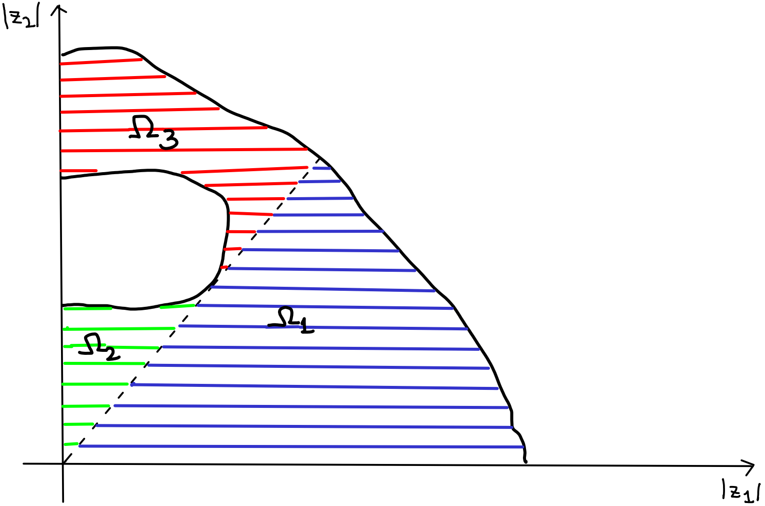
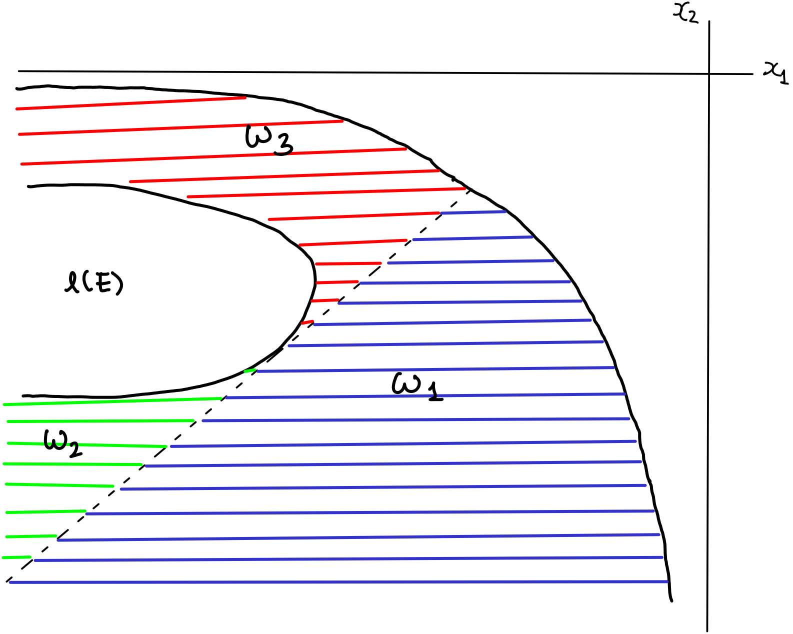
$\widetilde{\omega}_3$
$\widetilde{\omega}_2$
(b_1, b_2)
$\widetilde{\omega}_1$



$|z_2|$
Ω_3
Ω_1
Ω_2
$|z_1|$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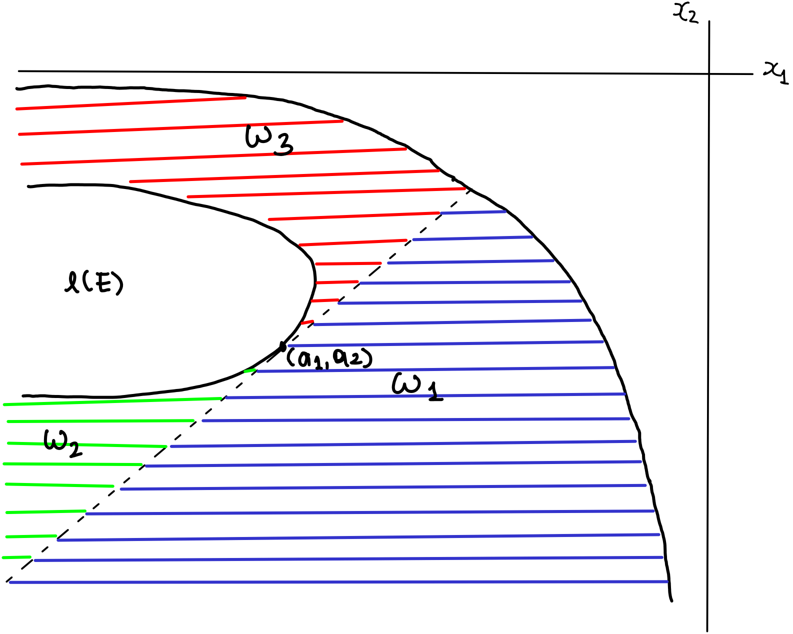
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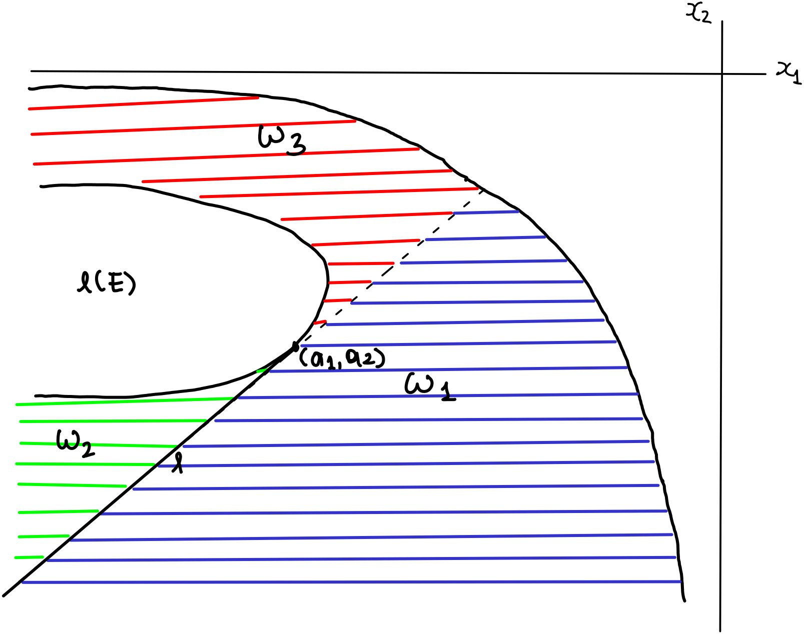

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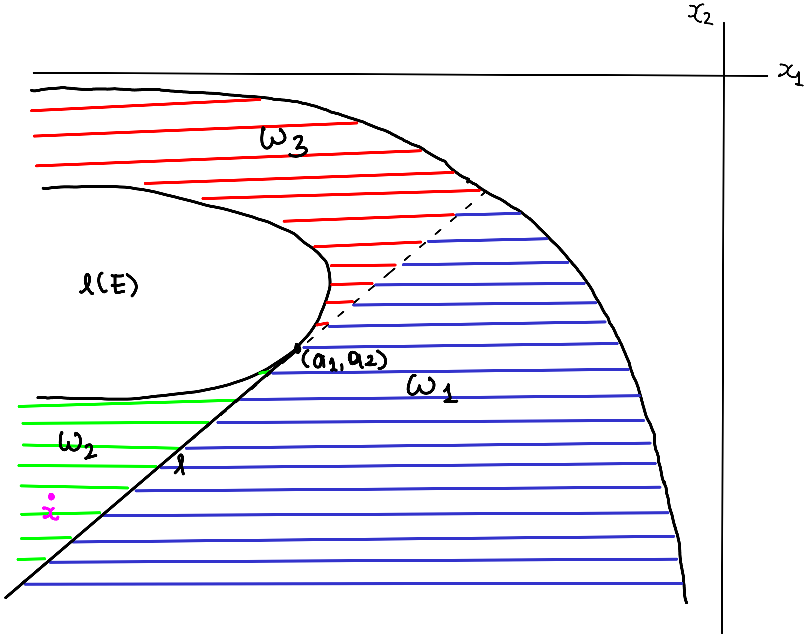

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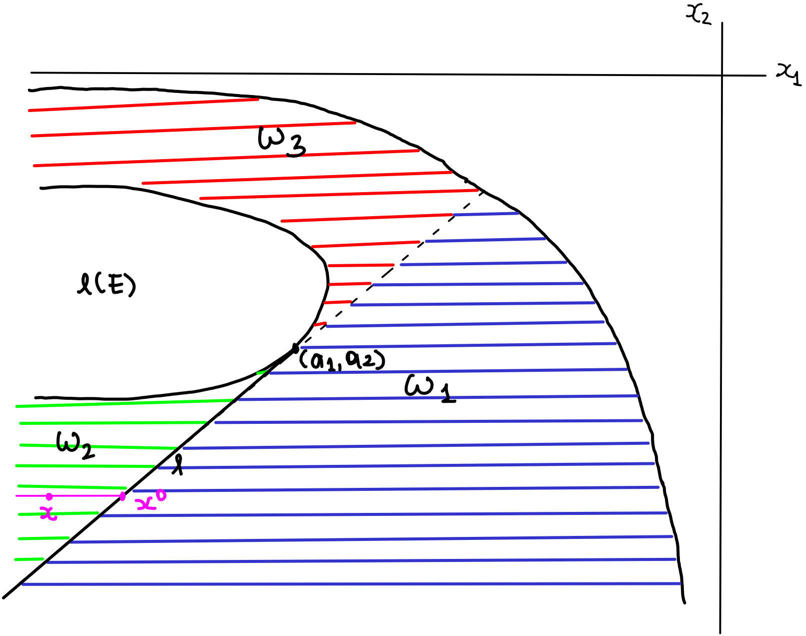

x_2
x_1
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$l(E)$
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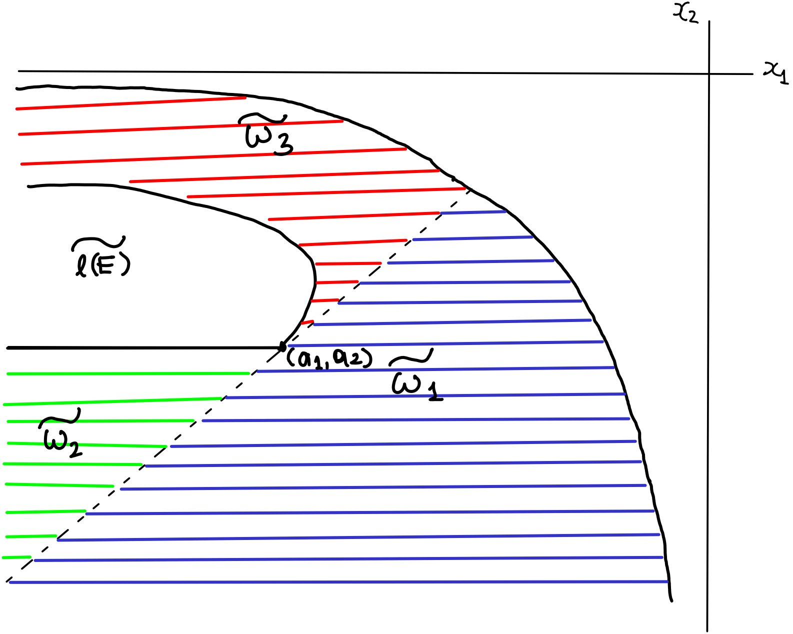
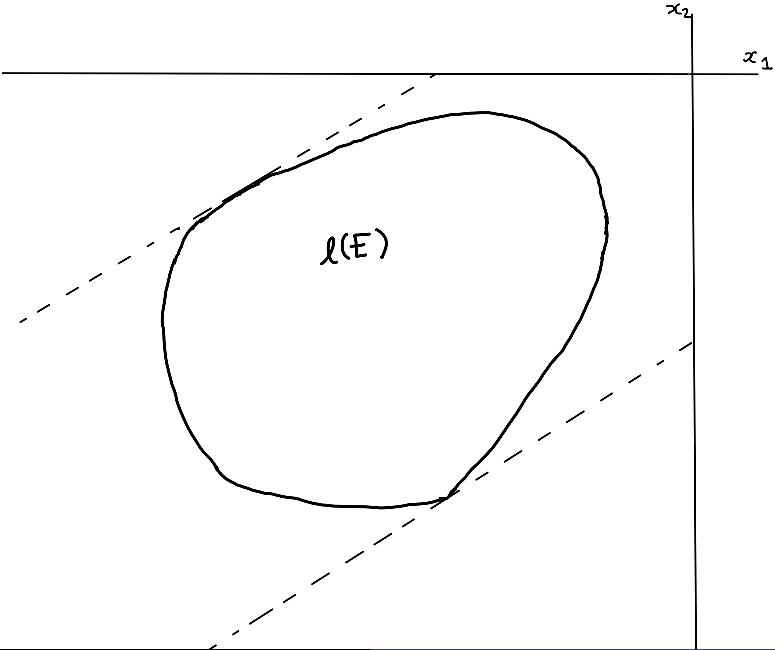
x^0



x_2
x_1
$\widetilde{\omega_3}$
$\widetilde{l(E)}$
(a_1, a_2)
$\widetilde{\omega_1}$
$\widetilde{\omega_2}$

Case $\Omega^0 = \Delta^2$



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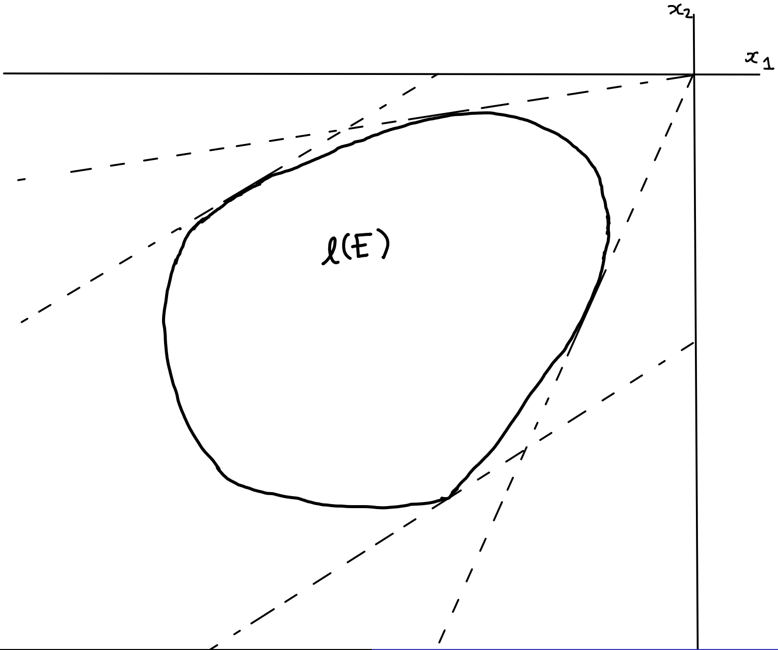

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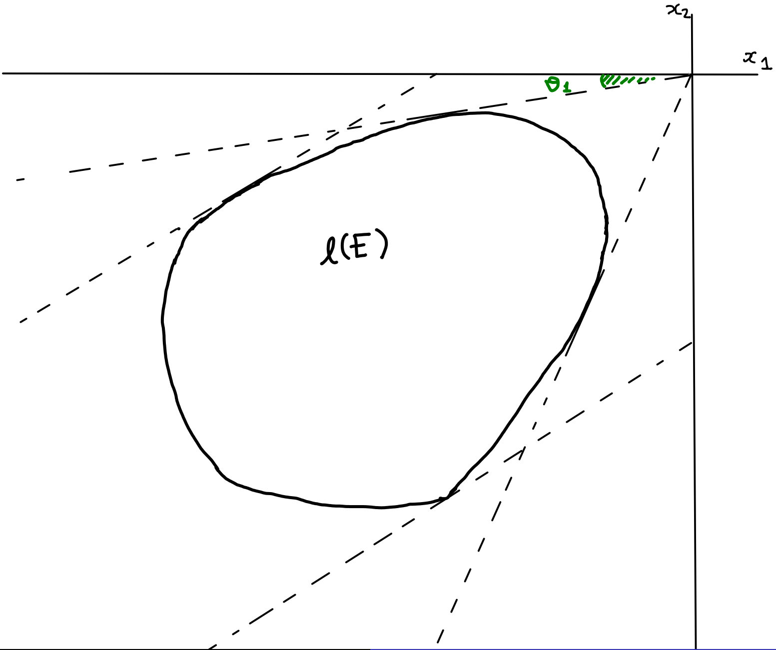

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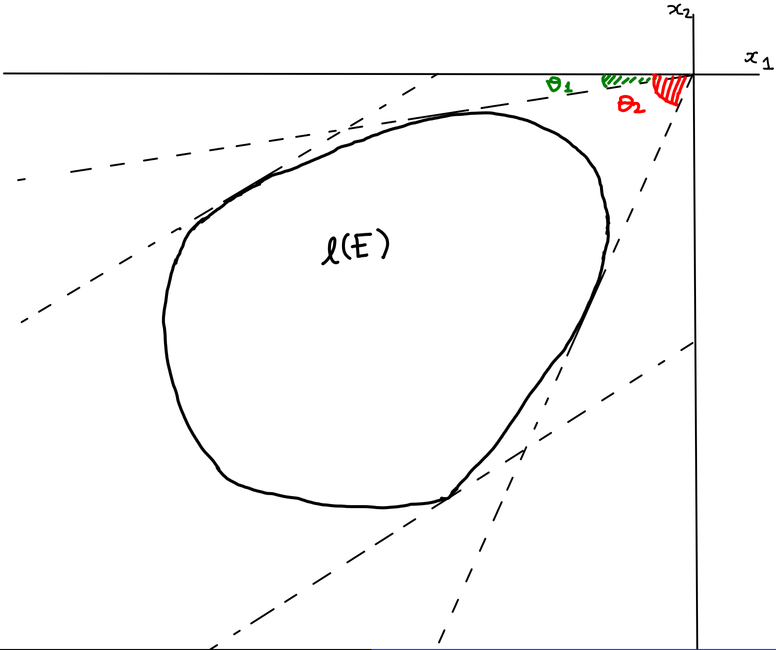
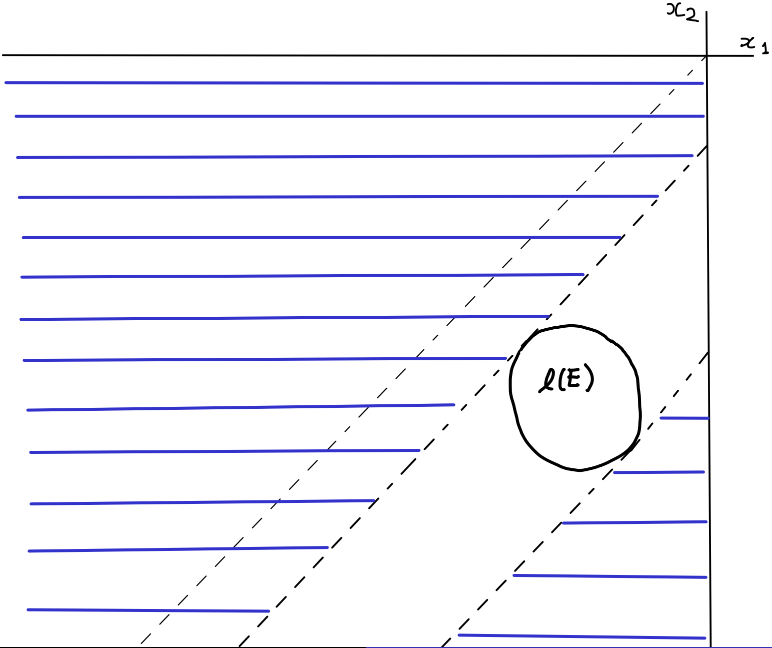


Case 1 If $\theta_1, \theta_2 \leq \frac{\pi}{4}$ or $\theta_1, \theta_2 \geq \frac{\pi}{4}$, then $\tilde{\Omega} = \Delta^2$.



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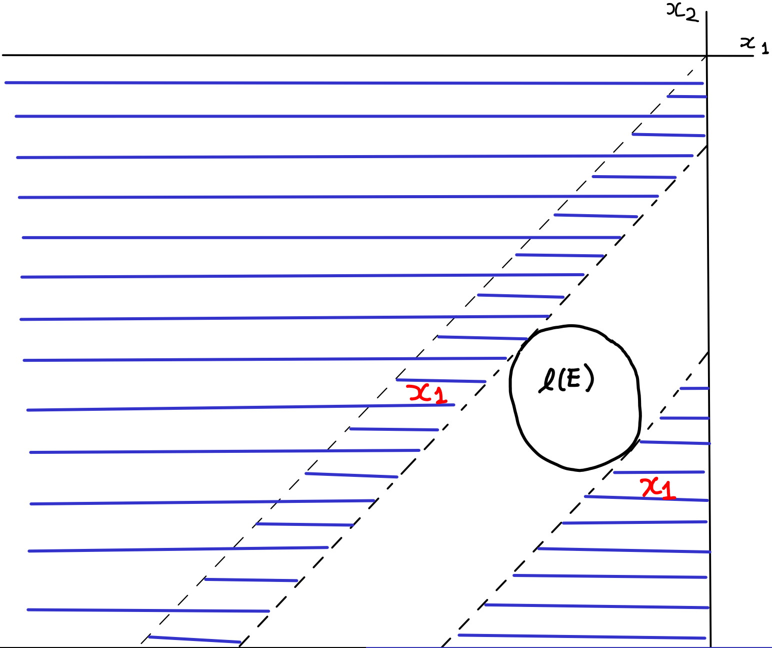

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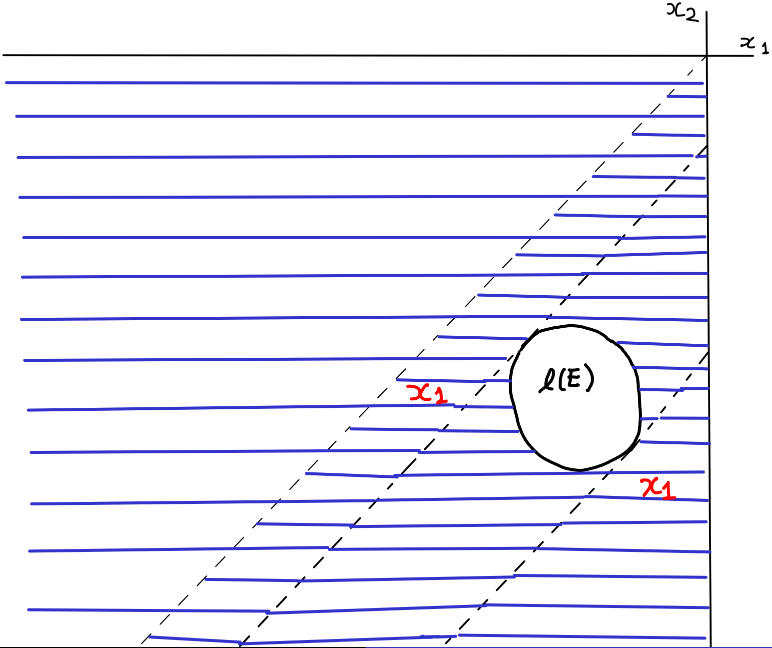
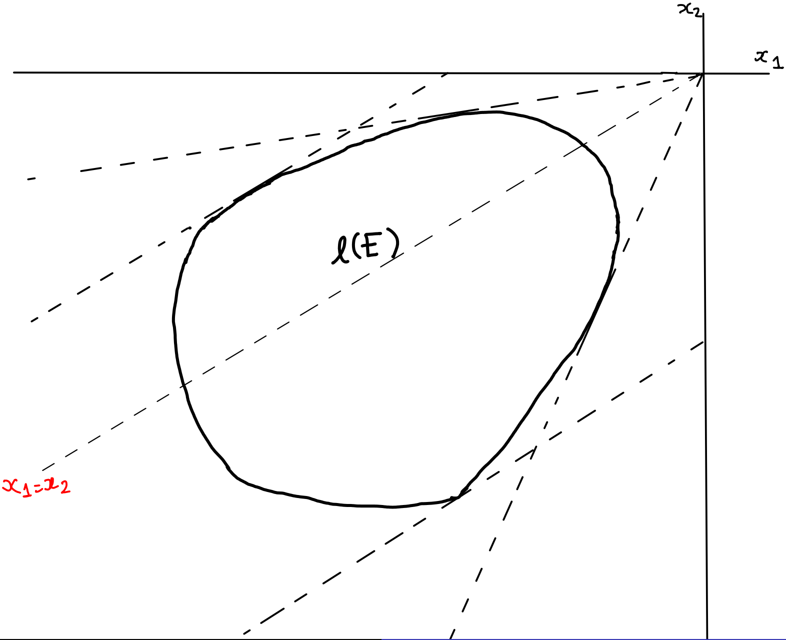


Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



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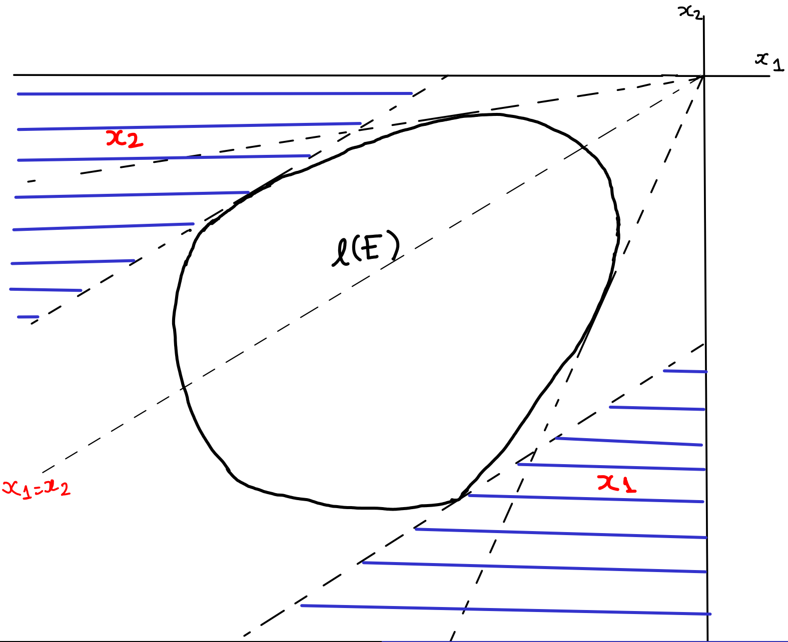

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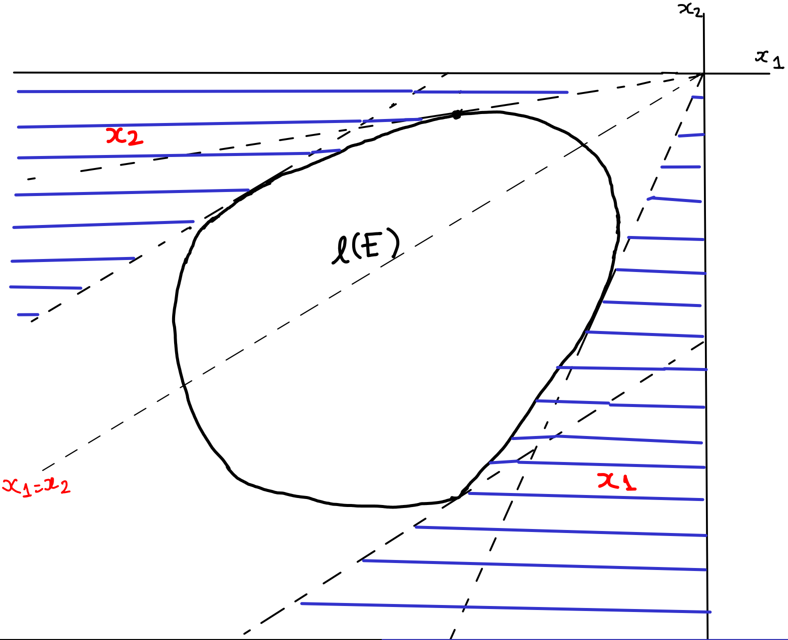

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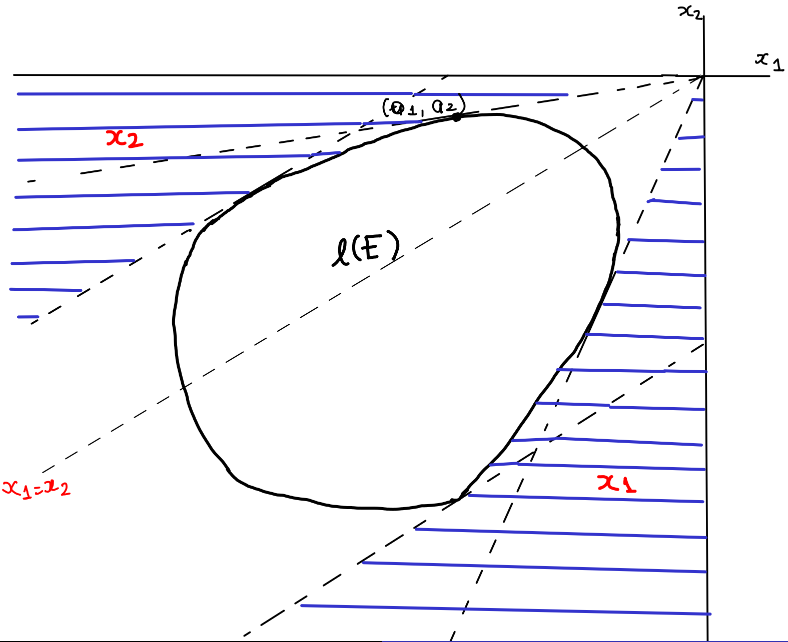

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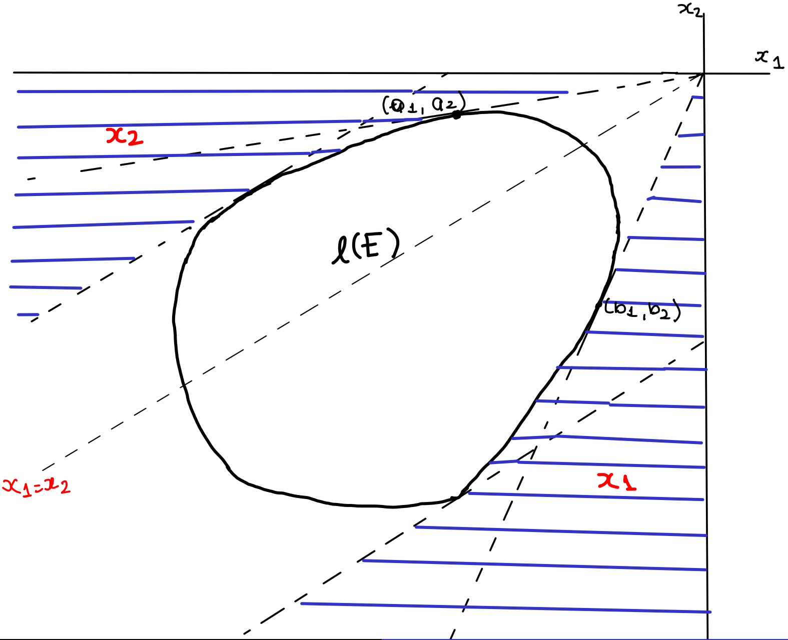

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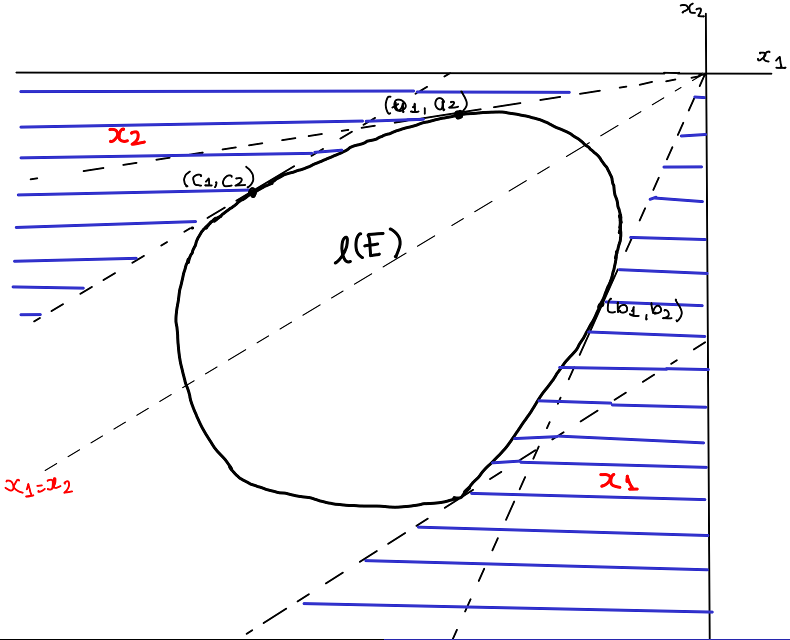

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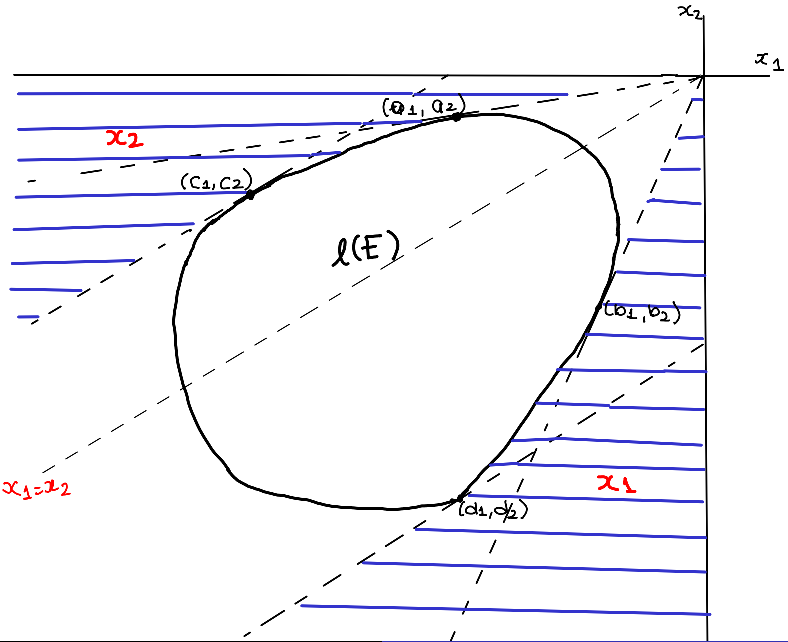

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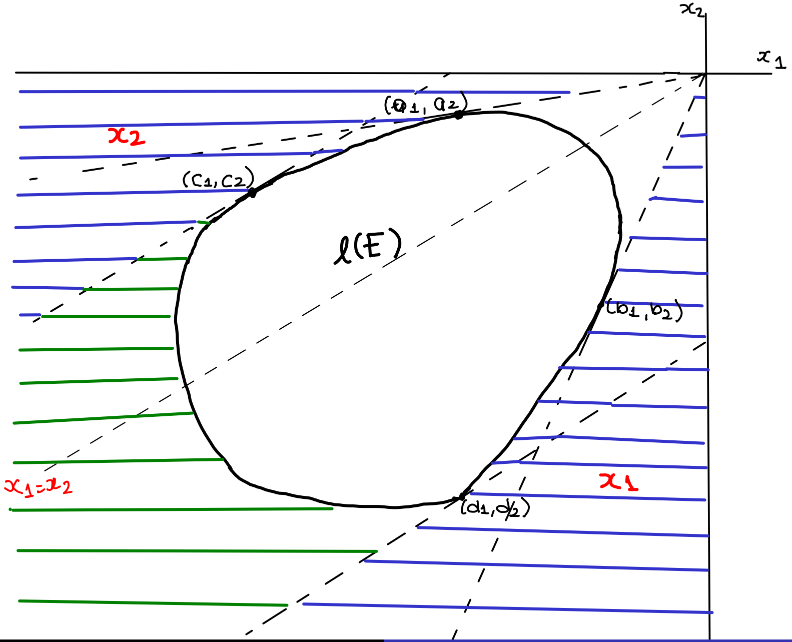

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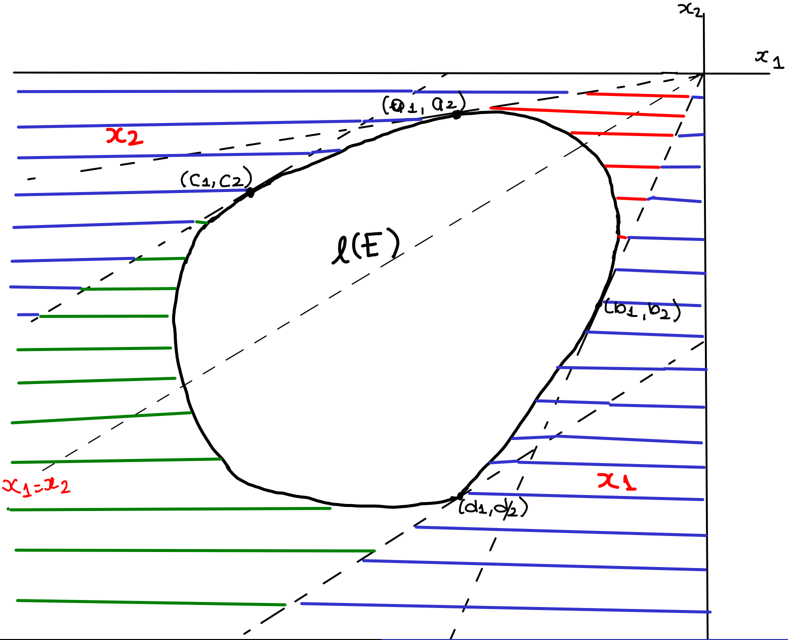
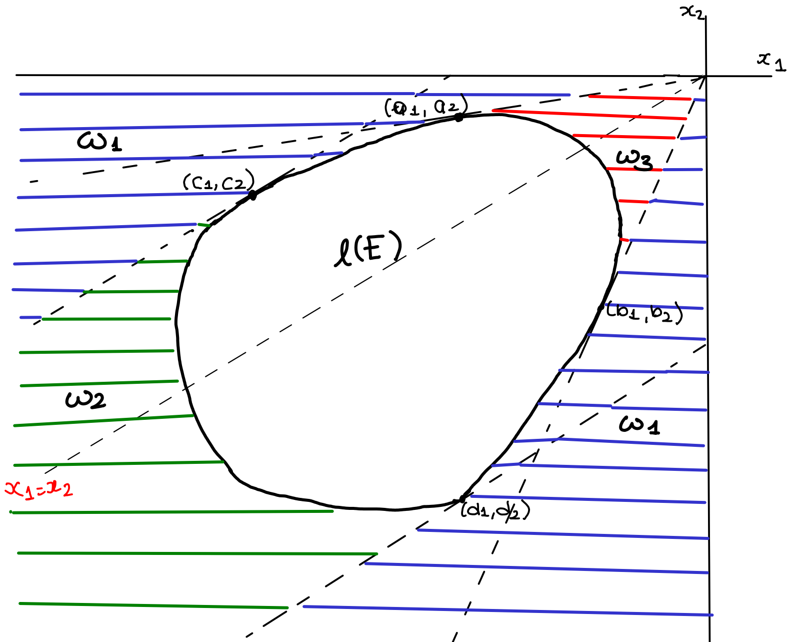
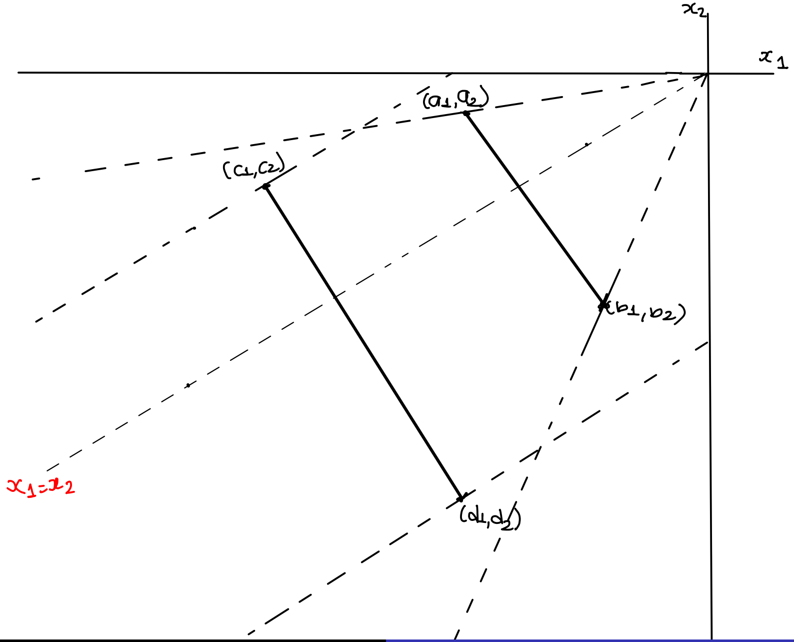


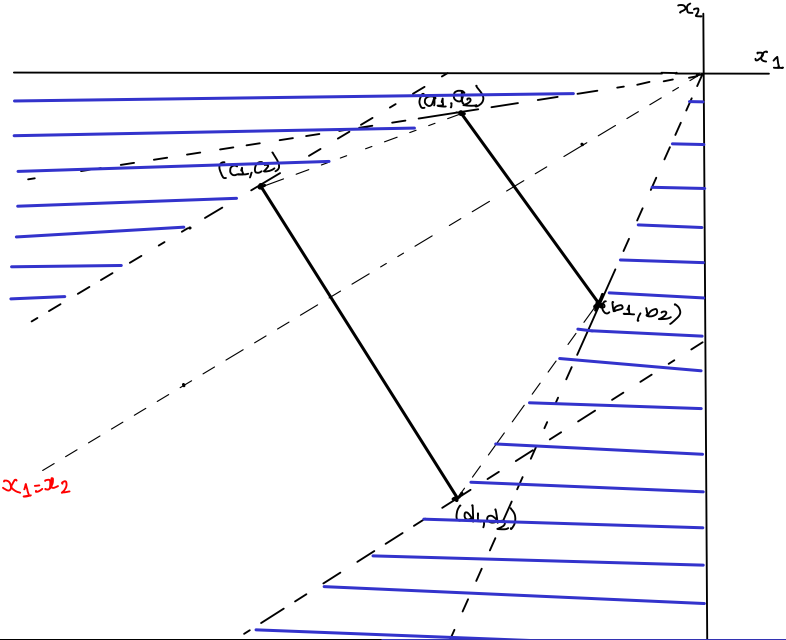
Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



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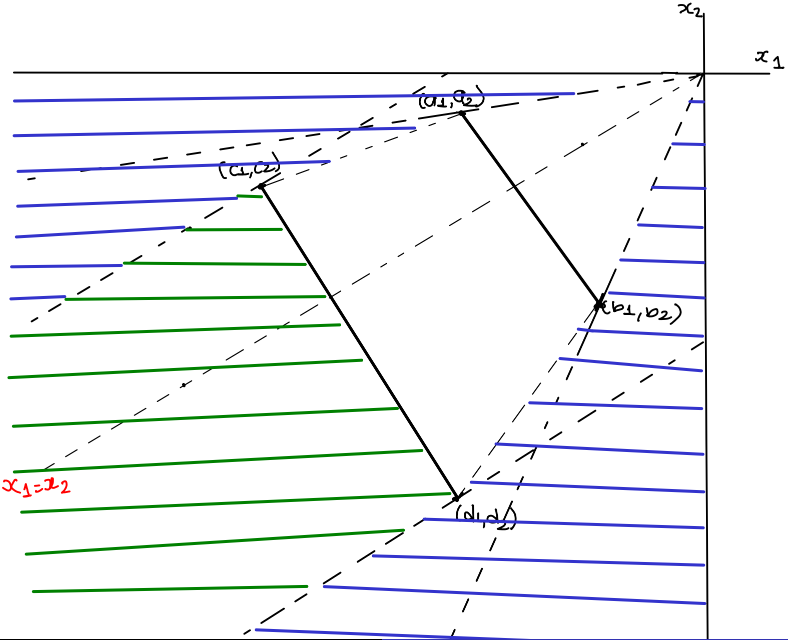
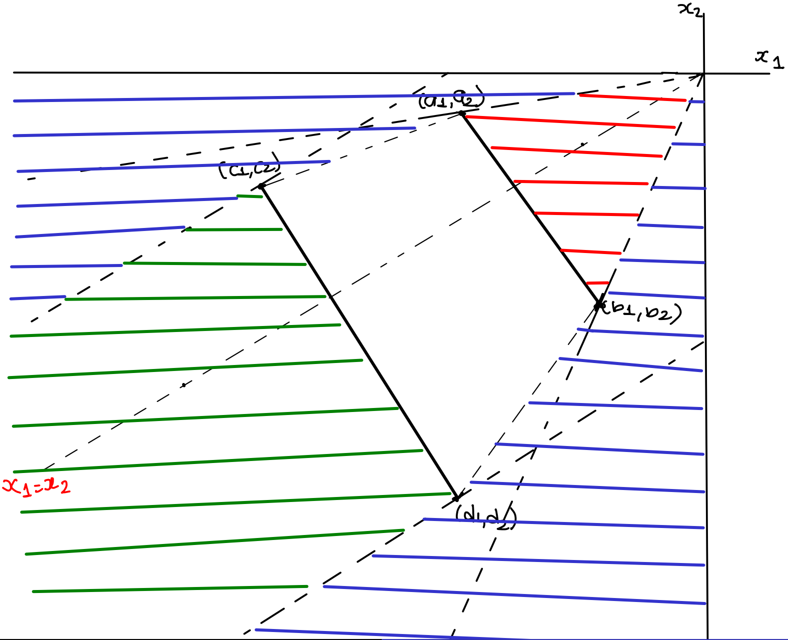
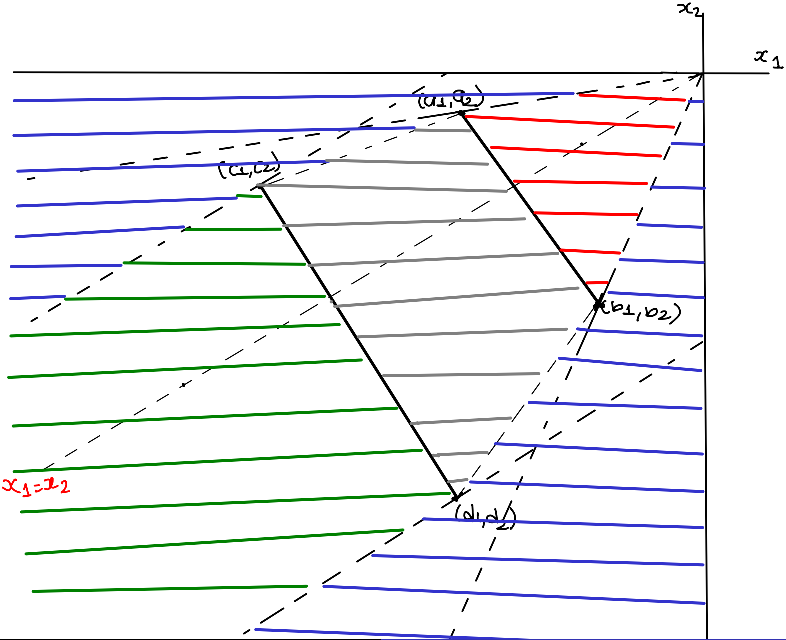


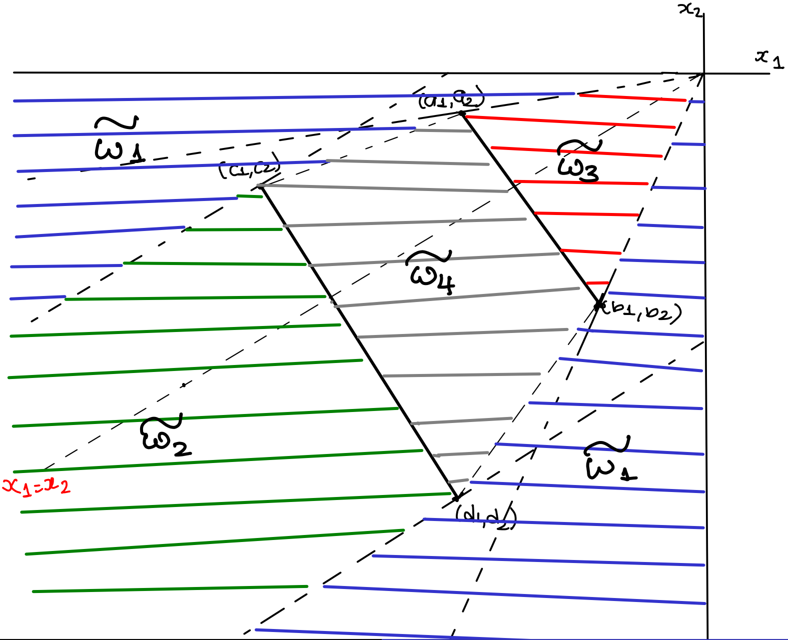
Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



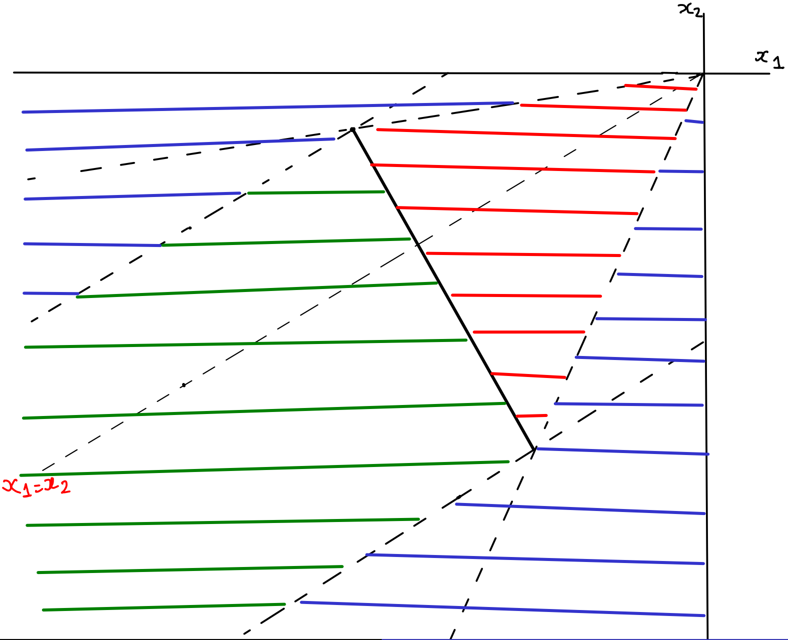
Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



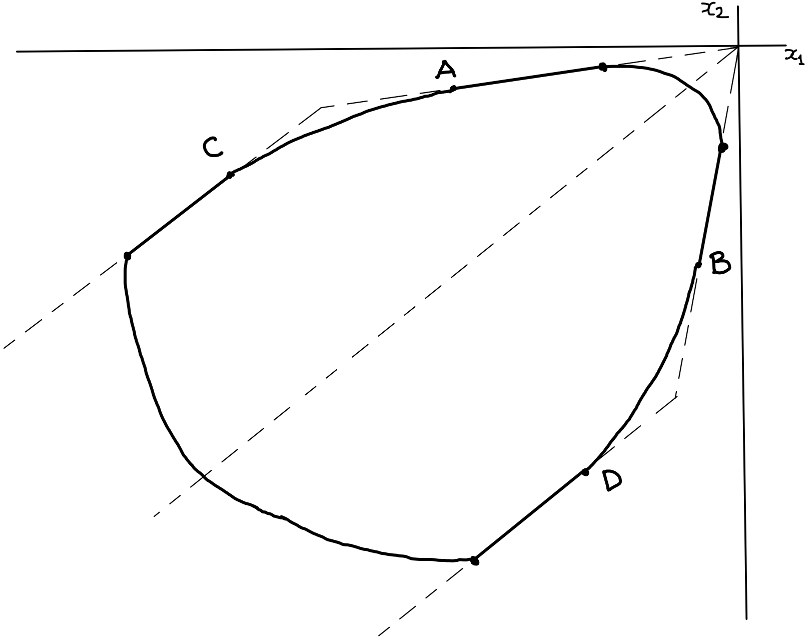
Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



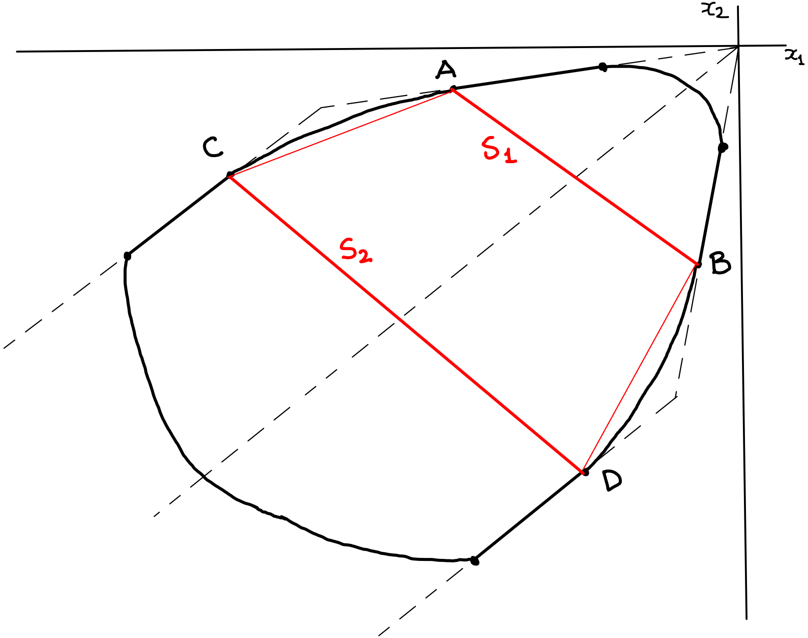
Case 2 If $\theta_1 < \frac{\pi}{4}$ and $\theta_2 > \frac{\pi}{4}$:



When $\ell(E)$ is not strictly convex:



When $\ell(E)$ is not strictly convex:



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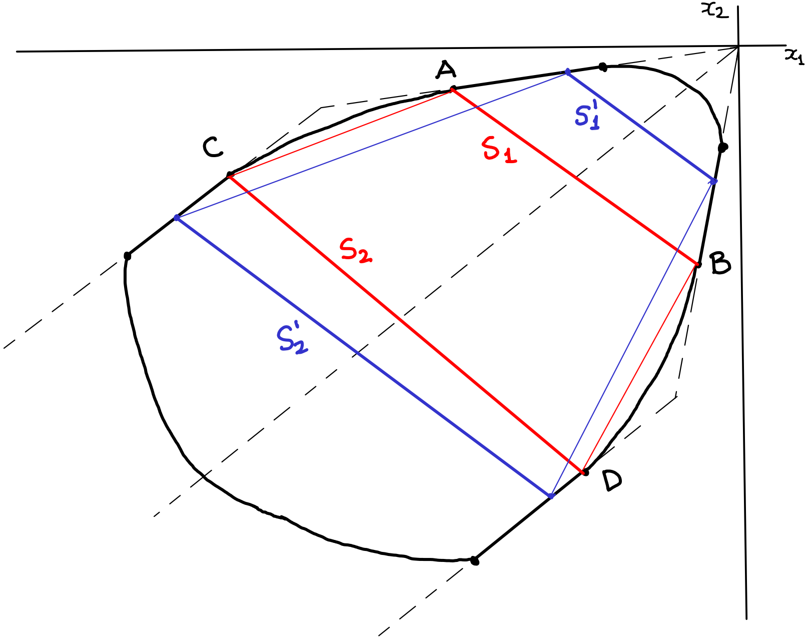

Thank you!