Kolmogorov problem on the diameters
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Let $K$ be a compact set in an open set $D$ on a Stein manifold $\Omega$. By $A^D_K$ we denote the unit ball of the space $H^\infty(D)$ considered as a subset of the space $C(K)$ (we always suppose that the restriction operator $R : H^\infty(D) \to C(K)$ is one-to-one and treat it as the identical imbedding). For a pair of Banach spaces $X_1 \subset X_0$ with the linear continuous imbedding $d_i(X_1; X_0)$ denotes the Kolmogorov $i$-diameter (width) of the unit ball of $X_1$ with respect to the unit ball of $X_0$ (see, e.g., [16]),

Kolmogorov raised the problem ([7, 8]) about the strict asymptotics for $\varepsilon$-entropy of the set $A^D_K$, which is equivalent (see, e.g., [10]) to the problem on the strong asymptotics for the Kolmogorov diameters of this set:

$$\ln d_i(A^D_K) \sim -\sigma i^{1/n}, \; i \to \infty.$$ (1)

here $d_i(A^D_K) := d_i(H^\infty(D), AC(K))$ and $AC(K)$ is the closure of $H^\infty(D)$ in the space $C(K)$.

Kolmogorov explained that, for good enough pairs $(K,D)$, the weak asymptotics $-\ln d_i(A^D_K) \sim \sigma i^{1/n}$ holds with $n = \dim \Omega$ and conjectured that in the case $n = 1$ the asymptotics should hold with the constant $\sigma = \frac{1}{\tau(K,D)}$, where $\tau(K,D)$ is Green capacity of a condenser $(K,D)$.

Kolmogorov's hypothesis has been confirmed by efforts of many authors (see, e.g., [1, 3, 4, 19, 10, 11, 18, 23, 15, 5]). The following statement summarizes, to some extent, those one-dimensional results.

**Proposition 1** (see, e.g., [23]) Let $K$ be a regular compact subset of a regular relatively compact open set $D$ on an open Riemann surface $\Omega$, $K = \overline{K}_D$, and $D$ is an intersection of countable (i.e. finite or enumerable) decreasing sequence of open sets $D_s$ with each boundary $\partial D_s$ consisted of a finite set of closed Jordan curves. Then the asymptotics (1) holds with $n = 1$ and $\sigma = \frac{1}{\tau(K,D)}$.

In multidimensional case, for a long time, the asymptotics (1) was known only for some quite particular frames $(K,D)$ (see, e.g., [17, 20]). The author conjectured [21, 22] that, for proper pairs $K \subset D$ on a Stein manifold $\Omega$, $\dim \Omega = n$, the asymptotics (1) holds with $\sigma = 2\pi \left( \frac{n!}{C(K,D)} \right)^{1/n}$, where $C(K,D)$ is the pluricapacity of the "pluricondenser" $(K,D)$[2]; it was also shown in [21, 22] how to reduce the problem about the asymptotics (1) for $n \geq 2$ to the certain problem of pluripotential theory. We state this pluripotential problem below after some necessary definitions.

The Green pluripotential of a pluricondenser $(K,D)$ on a Stein manifold $\Omega$ is defined by the formula

$$\omega(z) = \omega(D,K;z) := \limsup_{\zeta \to z} \sup \{ u(\zeta) : u \in P(K,D) \},$$ (2)

where $P(K,D)$ is the set of all plurisubharmonic in $D$ functions $u$ such that $u \mid_K \leq 0$ and $u(\zeta) < 1$ in $D$. We say that $(K,D)$ is a pluriregular pair on $\Omega$.
provided the conditions: (a) $K$ is a compact subset of an open set $D \subset \Omega$ such that $K = \bar{K}_D$ and $D$ has no component disjoint with $K$; (b) $\omega(D, K; z) \equiv 0$ on $K$ and $\lim \omega(D, K; z_j) = 1$ for any discrete sequence $\{z_j\}$ in $D$. Given a finite subset $F = \{\zeta_1, \ldots, \zeta_m\} \subset D$ and $\alpha = (\alpha_\mu) \in \mathbb{R}^n_+$, the Green multipole plurisubharmonic function $g_D(F, \alpha; z)$ is defined ([21, 22], cf. [6, 9]) as a regularized upper envelope of the family of all functions $u \in Psh(D)$, negative in $D$ and satisfying the estimate $u(z) \leq \alpha_\mu \ln |t(\zeta_\mu) - t(z)| + \text{const}$ in some neighborhood $U_\mu$ of each point $\zeta_\mu$ (in some local coordinates $t: U_\mu \to \mathbb{C}^n$ at $\zeta_\mu$). The following problem is crucial for many problems in Complex Analysis.

**Problem 2** ([21, 22]) Given a pluriregular pair $(K, D)$ on a Stein manifold does there exist a sequence of multipole Green functions $g_D(F, \alpha; z)$ converging to $\omega(D, K; z) - 1$ uniformly on any compact subset of $D \setminus K$?

This problem has been solved recently by Poletsky [14] and Nivoche [12, 13]. Actually, we consider the Kolmogorov problem in the following more general setting. Denote by $A(D)$ the Fréchet space of all functions analytic in $D$ with the topology of uniform convergence on compact subsets and by $A(K)$ the locally convex space of all germs of analytic functions on $K$ with the usual inductive limit topology. We are concerned with the strict asymptotics of the sequence of Kolmogorov diameters $d_i(X_1, X_0)$ for couples of Banach spaces $X_0, X_1$ satisfying the linear continuous imbeddings:

$$X_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow X_0 \quad (3)$$

and closely related with the spaces $A(D)$ and $A(K)$ in the following sense.

**Definition 3** We say that a couple of Banach spaces $(X_0, X_1)$, satisfying the imbeddings (3) is admissible for a pair $(K, D)$ if for any other couple of Banach spaces $Y_0, Y_1$ satisfying the linear continuous imbeddings:

$$X_1 \hookrightarrow Y_1 \hookrightarrow A(D); \quad A(K) \hookrightarrow Y_0 \hookrightarrow X_0,$$

we have $\ln d_i(Y_1, Y_0) \sim \ln d_i(X_1, X_0)$ as $s \to \infty$.

For a pluriregular pair $(K, D)$, there exists an admissible couple $(X_0, X_1)$ and the asymptotic class of the sequence $\ln d_i(X_1, X_0)$ is rather a characteristic of the pair $(K, D)$, since it does not depend on an individual couple $X_0, X_1$ admissible with this pair.

**Problem 4** ([21, 22]) Let $(K, D)$ be a pluriregular pair on a Stein manifold $\Omega$, $\dim \Omega = n$. Does the strict asymptotics

$$\ln d_i(X_1, X_0) \sim -2\pi \left(\frac{n! \, i}{C(K, D)}\right)^{1/n}, \quad s \to \infty \quad (4)$$

hold for some (hence, for any) couple of Banach spaces $X_0, X_1$, admissible for $(K, D)$?
Combining our approach from [21, 22] with the above-mentioned result of Nivoche-Poletsky we establish the positive solution of this problem, namely

**Theorem 5** The asymptotics (4) holds for any couple of Banach spaces $X_0, X_1$, admissible for $(K, D)$ if $(K, D)$ is a pluriregular pair on a Stein manifold $\Omega$.

**Corollary 6** Given a pluriregular pair $(K, D)$ the asymptotics

$$\ln d_i \left(A_K^D\right) \sim -2\pi \left(\frac{n! \ i}{C(K, D)}\right)^{1/n}, \ i \to \infty. \quad (5)$$

holds if and only if the couple $(AC(K), H^\infty(D))$ is admissible for $(K, D)$.

Although it is not clear, in general, how to describe the admissibility of couples $(AC(K), H^\infty(D))$, there are some simple sufficient conditions provided, due to Corollary 6, the asymptotics (5).

**Theorem 7** Suppose that a pluriregular pair $(K, D)$ is such that $D \subset \Omega$ is strictly pluriregular, i.e. the function $\omega(z) = \omega(D, K; z)$ has a plurisubharmonic extension onto some open neighborhood of $\bar{D}$. Then the asymptotics (5) is true.

**References**


