Kolmogorov problem on the diameters

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Let K be a compact set in an open set D on a Stein manifold Ω. By A_K^D we denote the unit ball of the space $H^\infty(D)$ considered as a subset of the space $C(K)$ (we always suppose that the restriction operator $R : H^\infty(D) \to C(K)$ is one-to-one and treat it as the identical imbedding). For a pair of Banach spaces $X_1 \subset X_0$ with the linear continuous imbedding $d_i(X_1, X_0)$ denotes the Kolmogorov i-diameter (width) of the unit ball of X_1 with respect to the unit ball of X_0 (see, e.g., [16]).

Kolmogorov raised the problem ([7, 8]) about the strict asymptotics for ε-entropy of the set A_K^D, which is equivalent (see, e.g., [10]) to the problem on the strong asymptotics for the Kolmogorov diameters of this set:

$$\ln d_i\left(A_K^D\right) \sim -\sigma\, i^{1/n},\ i \to \infty. \tag{1}$$

here $d_i\left(A_K^D\right) := d_i\left(H^\infty(D), AC(K)\right)$ and $AC(K)$ is the closure of $H^\infty(D)$ in the space $C(K)$.

Kolmogorov explained that, for good enough pairs (K, D), the weak asymptotics $-\ln d_i\left(A_K^D\right) \asymp \sigma\, i^{1/n}$ holds with $n = \dim \Omega$ and conjectured that in the case $n = 1$ the asymptotics should hold with the constant $\sigma = \frac{1}{\tau(K,D)}$, where $\tau(K, D)$ is Green capacity of a condenser (K, D).

Kolmogorov's hypothesis has been confirmed by efforts of many authors (see, e.g., [1, 3, 4, 19, 10, 11, 18, 23, 15, 5]). The following statement summarizes, to some extent, those one-dimensional results.

Proposition 1 *(see, e.g., [23]) Let K be a regular compact subset of a regular relatively compact open set D on an open Riemann surface Ω, $K = \widehat{K}_D$, and D is an intersection of countable (i.e. finite or enumerable) decreasing sequence of open sets D_s with each boundary ∂D_s consisted of a finite set of closed Jordan curves. Then the asymptotics (1) holds with $n = 1$ and $\sigma = \frac{1}{\tau(K,D)}$.*

In multidimensional case, for a long time, the asymptotics (1) was known only for some quite particular frames (K, D) (see, e.g., [17, 20]).The author conjectured [21, 22] that, for proper pairs $K \subset D$ on a Stein manifold Ω, $\dim \Omega = n$, the asymptotics (1) holds with $\sigma = 2\pi \left(\frac{n!}{C(K,D)}\right)^{1/n}$, where $C(K, D)$ is the pluricapacity of the "pluricondenser" (K, D)[2]; it was also shown in [21, 22] how to reduce the problem about the asymptotics (1) for $n \geq 2$ to the certain problem of pluripotential theory. We state this pluripotential problem below after some necessary definitions.

The *Green pluripotential* of a *pluricondenser* (K, D) on a Stein manifold Ω is defined by the formula

$$\omega(z) = \omega(D, K; z) := \limsup_{\zeta \to z} \sup \left\{u(\zeta) : u \in P(K, D)\ \right\}, \tag{2}$$

where $P(K, D)$ is the set of all plurisubharmonic in D functions u such that $u|_K \leq 0$ and $u(\zeta) < 1$ in D. We say that (K, D) is a *pluriregular pair on* Ω

provided the conditions: (a) K is a compact subset of an open set $D \subset \Omega$ such that $K =.\widehat{K}_D$ and D has no component disjoint with K; (b) $\omega(D, K; z) \equiv 0$ on K and $\lim \omega(D, K; z_j) = 1$ for any discrete sequence $\{z_j\}$ in D. Given a finite subset $F = \left\{\zeta_1, \ldots, \zeta_\mu, \ldots, \zeta_m\right\} \subset D$ and $\alpha = (\alpha_\mu) \in \mathbb{R}^n_+$, the *Green multipole plurisubharmonic function* $g_D(F, \alpha; z)$ is defined ([21, 22], cf. [6, 9]) as a regularized upper envelope of the family of all functions $u \in Psh(D)$, negative in D and satisfying the estimate $u(z) \le \alpha_\mu \ln\left|t(\zeta_\mu) - t(z)\right| + const$ in some neighborhood U_μ of each point ζ_μ (in some local coordinates $t : U_\mu \to \mathbb{C}^n$ at ζ_μ). The following problem is crucial for many problems in Complex Analysis.

Problem 2 *([21, 22]) Given a pluriregular pair (K, D) on a Stein manifold does there exist a sequence of multipole Green functions $g_D\left(F^{(j)}, \alpha^{(j)}; z\right)$ converging to $\omega(D, K; z) - 1$ uniformly on any compact subset of $D \smallsetminus K$?*

This problem has been solved recently by Poletsky [14] and Nivoche [12, 13].

Actually, we consider the Kolmogorov problem in the following more general setting. Denote by $A(D)$ the Fréchet space of all functions analytic in D with the topology of uniform convergence on compact subsets and by $A(K)$ the locally convex space of all germs of analytic functions on K with the usual inductive limit topology. We are concerned with the strict asymptotics of the sequence of Kolmogorov diameters $d_i(X_1, X_0)$ for couples of Banach spaces $X_0,\ X_1$ satisfying the linear continuous imbeddings:

$$X_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow X_0 \tag{3}$$

and closely related with the spaces $A(D)$ and $A(K)$ in the following sense.

Definition 3 *We say that a couple of Banach spaces $(X_0,\ X_1)$, satisfying the imbeddings (3) is admissible for a pair (K, D) if for any other couple of Banach spaces $Y_0,\ Y_1$ satisfying the linear continuous imbeddings:*

$$X_1 \hookrightarrow Y_1 \hookrightarrow A(D);\ A(K) \hookrightarrow Y_0 \hookrightarrow X_0,$$

we have $\ln d_i(Y_1, Y_0) \sim \ln d_i(X_1, X_0)$ as $s \to \infty$.

For a pluriregular pair (K, D), there exists an admissible couple $(X_0,\ X_1)$ and the asymptotic class of the sequence $\ln d_i(X_1, X_0)$ is rather a characteristic of the pair (K, D), since it does not depend on an individual couple $X_0,\ X_1$ admissible with this pair.

Problem 4 *([21, 22]) Let (K, D) be a pluriregular pair on a Stein manifold Ω, $\dim \Omega = n$. Does the strict asymptotics*

$$\ln d_i(X_1, X_0) \sim -2\pi \left(\frac{n!\ i}{C(K, D)}\right)^{1/n},\ s \to \infty \tag{4}$$

hold for some (hence, for any) couple of Banach spaces $X_0,\ X_1$, admissible for (K, D)?

Combining our approach from [21, 22] with the above-mentioned result of Nivoche-Poletsky we establish the positive solution of this problem, namely

Theorem 5 *The asymptotics (4) holds for any couple of Banach spaces $X_0,\ X_1$, admissible for (K,D) if (K,D) is a pluriregular pair on a Stein manifold Ω.*

Corollary 6 *Given a pluriregular pair (K,D) the asymptotics*

$$\ln d_i\left(A_K^D\right) \sim -2\pi\left(\frac{n!\ i}{C\left(K,D\right)}\right)^{1/n},\ i\to\infty. \tag{5}$$

holds if and only if the couple $(AC\left(K\right), H^{\infty}\left(D\right))$ is admissible for (K,D).

Although it is not clear, in general, how to describe the admissibility of couples $(AC\left(K\right), H^{\infty}\left(D\right))$, there are some simple sufficient conditions provided, due to Corollary 6, the asymptotics (5).

Theorem 7 *Suppose that a pluriregular pair (K,D) is such that $D \Subset \Omega$ is strictly pluriregular, i.e. the function $\omega\left(z\right)=\omega\left(D,K;z\right)$ has a plurisubharmonic extension onto some open neighborhood of $\overline{D}$. Then the asymptotics (5) is true.*

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