Disjoint Hypercyclic Operators

Özgür Martin

Miami University

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A Surprising Approximation Theorem

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- That is, for any entire $g(z)$, $\epsilon > 0$, and compact $K \subset \mathbb{C}$, we can find an integer n such that

$$\sup_{z \in K} |f(z+n) - g(z)| < \epsilon.$$

Dynamics of Linear Operators

Let T be a continuous linear operator on a topological vector space X. If there is a vector $f \in X$ such that

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Such a vector f is said to be a hypercyclic, supercyclic, or cyclic vector for T, respectively.

Invariant Subspace (Subset) Problem

- Given a linear continuous operator $T : X \to X$, is it possible to find a non-trivial (not X or $\{0\}$) closed subspace (subset) $F \subset X$ for which $T(F) \subset F$?

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- Both were solved in the negative for Banach spaces (Subspace by P. Enflo, 1987 and Subset by C. Reed, 1988).

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- Both were solved in the negative for Banach spaces (Subspace by P. Enflo, 1987 and Subset by C. Reed, 1988).
- Problems are still open for Hilbert spaces.

Linear Dynamics is Complicated

(Feldman, 2001) There exists a hypercyclic operator T acting on a separable Hilbert space $\mathcal{H}$ which has the following property. For any compact metrizable space K and any continuous map $f : K \to K$, there exists a T-invariant compact set $L \subset \mathcal{H}$ such that f and $T|_L$ are topologically conjugate.

Differentiation Operator

(MacLane, 1952)

- Let $H(\mathbb{C})$ denote the space of entire functions endowed with the topology of locally uniform convergence.

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- Let $H(\mathbb{C})$ denote the space of entire functions endowed with the topology of locally uniform convergence.
- The differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $D(f) = f'$ is hypercyclic.

Backward Shift Operator

(Rolewicz, 1969)

- Let $\ell_p(\mathbb{N}) = \{(x_0, x_1, x_2, \ldots) : \sum_{n=0}^{\infty} |x_n|^p < \infty\}$ for $p \geq 1$.

Backward Shift Operator

(Rolewicz, 1969)

- Let $\ell_p(\mathbb{N}) = \{(x_0, x_1, x_2, \ldots) : \sum_{n=0}^{\infty} |x_n|^p < \infty\}$ for $p \geq 1$.
- Let $B : \ell_p(\mathbb{N}) \to \ell_p(\mathbb{N})$ be the backward shift operator defined by

$$B(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

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- Let $B : \ell_p(\mathbb{N}) \to \ell_p(\mathbb{N})$ be the backward shift operator defined by
$$B(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$
- λB $(\lambda \in \mathbb{C})$ is hypercyclic on $\ell_p(\mathbb{N})$ if and only if $|\lambda| > 1$.

Composition Operators

- Let Ω be a domain in $\mathbb{C}$ and let $H(\Omega)$ be the space of holomorphic functions on Ω, endowed with the topology of locally uniform convergence.

Composition Operators

- Let Ω be a domain in $\mathbb{C}$ and let $H(\Omega)$ be the space of holomorphic functions on Ω, endowed with the topology of locally uniform convergence.
- For each $\varphi \in H(\Omega)$ with $\varphi(\Omega) \subset \Omega$, let C_φ denote the composition operator defined by

$$f \overset{C_\varphi}{\mapsto} f \circ \varphi \qquad (f \in H(\Omega)).$$

Hypercyclic Composition Operators

- **(Birkhoff)** Let τ be the $\mathbb{C}$-automorphism given by $\tau(z) := z + a$ ($a \in \mathbb{C}, a \neq 0$). Then C_τ is hypercyclic on $H(\mathbb{C})$.

Hypercyclic Composition Operators

- **(Birkhoff)** Let τ be the $\mathbb{C}$-automorphism given by $\tau(z) := z + a$ $(a \in \mathbb{C}, a \neq 0)$. Then C_τ is hypercyclic on $H(\mathbb{C})$.
- **(Seidel and Walsh, 1941)** Let ϕ be the $\mathbb{D}$-automorphism given by $\phi(z) := \frac{z+a}{1+\bar{a}z}$ $(a \in \mathbb{D}, a \neq 0)$. Then C_ϕ is hypercyclic on $H(\mathbb{D})$.

Disjoint Hypercyclic Operators

(J. Bès and A. Peris (2007) also L. Bernal-González (2007))
We say that hypercyclic operators $T_1, \ldots, T_N$ $(N \geq 2)$ are **d-hypercyclic** (**d-supercyclic**) provided that the direct sum operator $T_1 \oplus \ldots \oplus T_N$ acting on X^N have a hypercyclic (supercyclic) vector in the form $(f, \ldots, f) \in X^N$.

Examples of D-Hypercyclic Operators

- **(Bernal and Bès and Peris)** If $a_1, a_2 \in \mathbb{C}$ non-zero with $a_1 \neq a_2$, and $\tau_1(z) := z + a_1$ and $\tau_2(z) := z + a_2$, then C_{τ_1}, C_{τ_2} are d-hypercyclic on $H(\mathbb{C})$.

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- **(Bernal)** If a_1 and a_2 are non-zero distinct points in $\mathbb{D}$ and $\varphi_1(z) := \frac{z+a_1}{1+\bar{a}_1 z}$ and $\varphi_2(z) := \frac{z+a_2}{1+\bar{a}_2 z}$ are their respective non-Euclidean translations, then C_{φ_1}, C_{φ_2} are d-hypercyclic on $H(\mathbb{D})$.

Problem 1

- If τ_1, τ_2 are distinct $\mathbb{C}$-automorphisms such that C_{τ_1} and C_{τ_2} are hypercyclic, then C_{τ_1}, C_{τ_2} are d-hypercyclic on $H(\mathbb{C})$.

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- If the $\mathbb{D}$-automorphisms $\varphi_1(z) := \frac{z+a_1}{1+\bar{a}_1 z}$ and $\varphi_2(z) := \frac{z+a_2}{1+\bar{a}_2 z}$ are distinct, then C_{φ_1}, C_{φ_2} are d-hypercyclic on $H(\mathbb{D})$.

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- If the $\mathbb{D}$-automorphisms $\varphi_1(z) := \frac{z+a_1}{1+\bar{a}_1 z}$ and $\varphi_2(z) := \frac{z+a_2}{1+\bar{a}_2 z}$ are distinct, then C_{φ_1}, C_{φ_2} are d-hypercyclic on $H(\mathbb{D})$.
- **Problem 1:** (Bernal-González) Let C_{φ_1} and C_{φ_2} be generated by non-elliptic automorphisms. Must they be d-hypercyclic on X, where X is a subspace of $H(\mathbb{D})$?

L. Bernal-González, Disjoint hypercyclic operators, *Studia Math.* **182** *Vol 2 (2007), 113–131.*

Problem 2

- If T is invertible, then T is hypercyclic if and only if T^{-1} is.

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- **Problem 2:** (Bès and Peris) Let T_1, T_2 be d-hypercyclic and invertible. Must T_1^{-1}, T_2^{-1} be d-hypercyclic?

J. Bès and A. Peris, Disjointness in hypercyclicity, *J. Math. Anal. Appl.* **336** *(2007) 297–315.*

Problem 3

Problem 3: When are $C_{\varphi_1}, C_{\varphi_2}$ d-hypercyclic if φ_1 and φ_2 are self maps of $\mathbb{D}$? When are they d-supercyclic?

Linear fractional Transformations

- The group $LFT(\widehat{\mathbb{C}})$ of linear fractional transformations consists of bijections of the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that are in the form

$$\varphi(z) = \frac{az+b}{cz+d}$$

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- $LFT(\mathbb{D}) = \{\varphi \in LFT(\widehat{\mathbb{C}}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$ is the subgroup consisting of self maps of the unit disc.

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- $LFT(\mathbb{D}) = \{\varphi \in LFT(\widehat{\mathbb{C}}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$ is the subgroup consisting of self maps of the unit disc.
- $Aut(\mathbb{D}) = \{\varphi \in LFT(\widehat{\mathbb{C}}) : \varphi(\mathbb{D}) = \mathbb{D}\}$ is the set of linear transformations that take $\mathbb{D}$ onto itself. These are called **automorphisms**.

Fixed Points of Linear fractional Transformations

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- Members of $LFT(\mathbb{D})$, other then the identity, have at most two fixed points.
- A fixed point of $\varphi \in LFT(\mathbb{D})$ is called **attractive** if the iterations

$$\varphi^{[n]}(z) = (\varphi \circ \ldots \circ \varphi)(z) \to \alpha$$

as $n \to \infty$ for all $z \in \mathbb{C}$.

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as $n \to \infty$ for all $z \in \mathbb{C}$.
- A **repulsive** fixed point of $\varphi \in LFT(\mathbb{D})$ is the attractive fixed point of the inverse φ^{-1}.

Classification of $LFT(\widehat{\mathbb{C}})$

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- If φ has two fixed points, then it is conjugate to a mapping in the form $\psi(z) = \lambda z$ ($|\lambda| \geq 1$ and $\lambda \neq 1$).
- Then φ is called:
  1. **Elliptic** if $|\lambda| = 1$,
  2. **Hyperbolic** if $\lambda > 1$, and
  3. **Loxodromic** if φ is neither elliptic nor parabolic.

Classification of $LFT(\mathbb{D})$ Depending on the Fixed Points

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- **Loxodromic** and **elliptic** members of $LFT(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside of the closed unit disc.

The Hardy Space

We say that a function analytic on the unit disc (i.e. it is in $H(\mathbb{D})$)

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$$

belongs to the **Hardy space** $H^2(\mathbb{D})$ if its sequence of power series coefficients is square-summable:

$$H^2(\mathbb{D}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

Linear Fractional Hypercyclicity

(Bourdon, Shapiro, Ansari, Gallardo, and Montes) If C_φ is a composition operator with $\varphi \in LFT(\mathbb{D})$, then the following are equivalent:

1. C_φ is hypercyclic on $H^2(\mathbb{D})$.
2. C_φ is supercyclic on $H^2(\mathbb{D})$.
3. φ is either a parabolic automorhism or a hyperbolic map without a fixed point in $\mathbb{D}$.

Weighted Dirichlet Spaces

- $S_\nu = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu} < \infty\}$ where $\nu \in \mathbb{R}$.

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- S_0 is the Hardy space $H^2(\mathbb{D})$.
- $S_{-\frac{1}{2}}$ is the Bergman space.
- $S_{\frac{1}{2}}$ is the Dirichlet space.

Linear Fractional Hypercyclicity on S_ν

(Gallardo and Montes, 2004) Let $\varphi \in LFT(\mathbb{D})$. Then

- C_φ is hypercyclic on S_ν iff $\nu < \frac{1}{2}$ and C_φ is hypercyclic on $H^2(\mathbb{D})$.

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- If $\nu < \frac{1}{2}$, then C_φ is supercyclic on S_ν iff it is hypercyclic on S_ν.

Linear Fractional Hypercyclicity on S_ν

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- C_φ is hypercyclic on S_ν iff $\nu < \frac{1}{2}$ and C_φ is hypercyclic on $H^2(\mathbb{D})$.
- If $\nu < \frac{1}{2}$, then C_φ is supercyclic on S_ν iff it is hypercyclic on S_ν.
- C_φ is supercyclic on $S_{\frac{1}{2}}$ iff φ is a hyperbolic non-automorphism without a fixed point in $\mathbb{D}$.

Characterization of d-Hypercyclicity

Theorem: (J. Bès, Ö. M., and A. Peris) Let $\nu < \frac{1}{2}$ and $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$. The following are equivalent:

1. $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on S_ν.
2. $C_{\varphi_1}, C_{\varphi_2}$ are d-supercyclic on S_ν.
3. φ_1 and φ_2 are either parabolic automorphisms or hyperbolic maps without fixed points in $\mathbb{D}$ and satisfy that if they have the same attractive fixed point α, the expression $\varphi_1'(\alpha) = \varphi_2'(\alpha) < 1$ does not occur.

Problems

- **Problem 1:** (Bernal) Let C_{φ_1} and C_{φ_2} be generated by distinct non-elliptic automorphisms. Must they be d-hypercyclic on $H(\mathbb{D})$?
- **Problem 2:** (Bès and Peris) Let T_1, T_2 be d-hypercyclic and invertible. Must T_1^{-1}, T_2^{-1} be d-hypercyclic?

Example

The hyperbolic maps $\varphi_j \in \mathrm{Aut}(\mathbb{D})$ $(j = 1, 2)$ given by

$$\varphi_1(z) = \frac{(3+i)z - 1 - i}{(-1+i)z + 3 - i} \quad \text{and} \quad \varphi_2(z) = \frac{(3+2i)z - 1 - 2i}{(-1+2i)z + 3 - 2i}$$

have the attractive fixed points $-i$ and $\frac{3}{5} - \frac{4}{5}i$, respectively, and have the same repellent fixed point 1. Thus, by the main theorem, $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on S_ν $(\nu < \frac{1}{2})$, while $C_{\varphi_1}^{-1} = C_{\varphi_1^{-1}}$ and $C_{\varphi_2}^{-1} = C_{\varphi_2^{-1}}$ are not d-hypercyclic since

$$(\varphi_1^{-1})'(1) = (\varphi_2^{-1})'(1) = \frac{1}{2} < 1.$$

Problem

Problem: Let $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$ be hyperbolic non-automorphisms without fixed points in $\mathbb{D}$. When are $C_{\varphi_1}, C_{\varphi_2}$ d-supercyclic on $S_{\frac{1}{2}}$?

Universality

- **(Birkhoff, 1929)** There exists an entire function f and a sequence of $\mathbb{C}$-automorphisms $\tau_n(z) := z + a_n$ ($a_n \in \mathbb{C}$ with $|a_n| \to \infty$) for which the set

$$\{C_{\tau_n}(f) : n \geq 1\} = \{f \circ \tau_n : n \geq 1\}$$

is dense in $H(\mathbb{C})$.

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  is dense in $H(\mathbb{C})$.

- **(Seidel and Walsh, 1941)** There exists a function f in $H(\mathbb{D})$ and a sequence of $\mathbb{D}$-automorphisms $\phi_n(z) := \frac{z+a_n}{1+\overline{a_n}z}$ ($a_n \in \mathbb{D}$ with $|a_n| \to 1$) so that the set

$$\{C_{\phi_n}(f) : n \geq 1\} = \{f \circ \phi_n : n \geq 1\}$$

  is dense in $H(\mathbb{D})$.

Hypercyclic and Supercyclic Sequences

- A sequence continuous linear transformations $(T_n)_{n=1}^{\infty}$ on a topological vector space X is said to be **hypercyclic** (or **universal**) provided there is some $f \in X$ so that the set

$$\{T_n(f) : n \geq 1\} = \{T_1(f), T_2(f), \ldots\}$$

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$$\{T_n(f) : n \geq 1\} = \{T_1(f), T_2(f), \dots\}$$

  is dense in X.
- $(T_n)_{n=1}^{\infty}$ is said to be **supercyclic** provided there is some $f \in X$ so that the projective orbit

$$\{\lambda T_n(f) : n \geq 0, \lambda \in \mathbb{C}\}$$

  is dense in X.

Characterization of Compositional Hypercyclicity

(Bernal and Montes, 1995)
Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $(\varphi_n) \in Aut(\Omega)^{\mathbb{N}}$. Then the following are equivalent:

1. (C_{φ_n}) is hypercyclic on $H(\Omega)$.
2. (φ_n) is a **run-away** sequence: For each compact $K \subset \Omega$, $\exists n \in \mathbb{N}$ so that $\varphi_n(K) \cap K = \emptyset$.

Compositional Hypercyclicity Equals Supercyclicity

(Bernal, Bonilla, and Calderón, 2007)

- $(\varphi_n) \in Aut(\Omega)^{\mathbb{N}}$ where $\Omega \neq \mathbb{C}$ simply connected. Then the following are equivalent:
    1. (C_{φ_n}) is hypercyclic on $H(\Omega)$.
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  1. (C_{φ_n}) is hypercyclic on $H(\Omega)$.
  2. (C_{φ_n}) is supercyclic on $H(\Omega)$.
  3. (φ_n) is run-away.
- If $\{\varphi_n(z) := a_n z + b_n : n \geq 1\} \subset Aut(\mathbb{C})$ and $0 < \inf |a_n| \leq \sup |a_n| < \infty$, then (C_{φ_n}) is hypercyclic if and only if it is supercyclic.

Disjoint Sequences of Hypercyclic Operators

We say that $N \geq 2$ sequences of continuous linear operators $(T_{1,n}), \ldots, (T_{N,n})$ on a topological vector space X are **d-hypercyclic** (**d-supercyclic**) provided that the sequence of the direct sums $(T_{1,n} \oplus \ldots \oplus T_{N,n})$ has a hypercyclic (supercyclic) vector on the diagonal of X^N.

Compositional d-Hypercyclicity Equals d-Supercyclicty

Theorem: (J. Bès and Ö. M.) Let $(\varphi_{\ell,n}) \in Aut(\Omega)^{\mathbb{N}}$ $(1 \leq \ell \leq N)$, Ω simply connected. TFAE:

1. $\left(C_{\varphi_{1,n}}\right), \ldots, \left(C_{\varphi_{N,n}}\right)$ are d-hypercyclic on $H(\Omega)$.
2. $\left(C_{\varphi_{1,n}}\right), \ldots, \left(C_{\varphi_{N,n}}\right)$ are d-supercyclic on $H(\Omega)$.
3. For each $K \in \Omega$ compact, $\exists n \in \mathbb{N}$ such that $K, \varphi_{1,n}(K), \ldots, \varphi_{N,n}(K)$ are pairwise disjoint.

Compositional Hypercyclicity Equals Supercyclicity

Corollary: If $\{\varphi_n(z) := a_n z + b_n : n \geq 1\} \subset Aut(\mathbb{C})$, then the following are equivalent:

1. (C_{φ_n}) is hypercyclic on $H(\mathbb{C})$.
2. (C_{φ_n}) is supercyclic on $H(\mathbb{C})$.
3. (φ_n) is run-away.
4. $\sup_n \min\{|b_n|, |b_n/a_n|\} = \infty$.

Non-automorphic Symbols

Theorem: (J. Bès and Ö. M.) $\phi_{\ell,n} : \Omega \to \Omega$ holomorphic $(1 \leq \ell \leq N, n \in \mathbb{N})$. TFAE:

- $(C_{\phi_{1,n}})_{n=1}^{\infty}, \ldots, (C_{\phi_{N,n}})_{n=1}^{\infty}$ are d-hypercyclic.
- $\forall K \subset \Omega$ compact, $\exists n \in \mathbb{N}$
  1. $K, \phi_{1,n}(K), \ldots, \phi_{N,n}(K)$ are pairwise disjoint.
  2. Each map $\phi_{\ell,n}|_K : K \to \Omega$ is injective $(1 \leq \ell \leq N)$.

Case $N = 1$: (Grosse-Erdmann, Mortini, 2009).

Linear Fractional Symbols

Corollary: Let $\varphi_1, \varphi_2 \in LFT(\mathbb{D})$. The following are equivalent:

1. $C_{\varphi_1}, C_{\varphi_2}$ are d-hypercyclic on $H(\mathbb{D})$.
2. $C_{\varphi_1}, C_{\varphi_2}$ are d-supercyclic on $H(\mathbb{D})$.
3. φ_1 and φ_2 have no fixed points in $\mathbb{D}$ and satisfy that If they have the same attractive fixed point α, then the expression $\varphi_1'(\alpha) = \varphi_2'(\alpha) < 1$ does not occur.

Linear Fractional Symbols

- **Corollary:** Let $\varphi_1, \ldots, \varphi_N \in LFT(\mathbb{D})$ $(N \geq 2)$. $C_{\varphi_1}, \ldots, C_{\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$ iff $\mu_1 C_{\varphi_1}, \ldots, \mu_N C_{\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$, for any non-zero scalars.

Linear Fractional Symbols

- **Corollary:** Let $\varphi_1, \ldots, \varphi_N \in LFT(\mathbb{D})$ $(N \geq 2)$. $C_{\varphi_1}, \ldots, C_{\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$ iff $\mu_1 C_{\varphi_1}, \ldots, \mu_N C_{\varphi_N}$ are d-hypercyclic on $H(\mathbb{D})$, for any non-zero scalars.
- **Problem:** Let $\mu_1, \ldots, \mu_N$ be scalars and $\varphi_1, \ldots, \varphi_N \in LFT(\mathbb{D})$ $(N \geq 2)$. When are $\mu_1 C_{\varphi_1}, \ldots, \mu_N C_{\varphi_N}$ d-hypercyclic on S_ν?

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Thanks

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