QUESTIONS ABOUT LIMITS AND CONTINUITY

Most of the following questions have been chosen from the "limits and continuity" section of the book *An Introduction to Classical Real Analysis* by Karl R.Stromberg (1981).

Question 0.1. *A metric space X is said to be* totally bounded *if for each $r > 0$ there exists a finite set $F \subset X$ such that $dist(x, F) < r$ for each $x \in X$; i.e., $X = \cup\{B_r(y) : \quad y \in F\}$. Show that a metric space is compact if and only if it is both complete and totally bounded.*
(Hint: Given $(x_n)_{n=1}^{\infty} \subset X$, use total boundedness to choose an infinite set $A_1 \subset \mathbb{N}$ such that $\rho(x_m, x_n) < 1$ for all $m, n \in A_1$. When an infinite set A_{k-1} has been chosen , use total boundedness to choose an infinite set $A_k \subset A_{k-1}$ such that $\rho(x_m, x_n) < \frac{1}{k}$ for all $m, n \in A_k$. Next select $n_1 < n_2$... such that $n_k \in A_k$ for all k and prove that $(x_{n_k})_{k=1}^{\infty}$ is a Cauchy sequence.)

Question 0.2. *Show that the set of nowhere differentiable functions is residual in $C([0, 1])$ i.e its complement is a set of first category. (Hint: Consider the sets $A_n = \{f \in C([0, 1]) : \exists x_0 \in [0, 1] \quad s.t.|f(x)-f(x_0)| \leq n|x-x_0|, \forall x \in [0, 1]\}$)*

Question 0.3. {Banach's Fixed Point Theorem} *Let X be a complete metric space with metric ρ and let $f : X \to X$ satisfy*

$$\rho(f(x), f(y)) \leq \alpha\rho(x, y)$$

for all $x, y \in X$, where α is some constant independent of x and y with $0 < \alpha < 1$. Then there is a unique $p \in X$ such that $f(p) = p$.
{Hint: Obtain p as the limit of a Cauchy sequence. Let $x_1 \in X$ and define $x_{n+1} = f(x_n)$.}

Question 0.4. *If X is a metric space and A and B are non-empty disjoint closed subsets of X, then there is a continuous $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise.*
{Hint: Let

$$f(x) = \frac{dist(x, A)}{dist(x, A) + dist(x, B)}$$

}.

Question 0.5. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in \mathbb{R}$. *Then*
(a) $f(rx) = rf(x)$ *for* $r \in \mathbb{Q}$, $\quad x \in \mathbb{R}$\{*Hint: Fix* x *and consider successively* $r = 0, r \in \mathbb{N}, -r \in \mathbb{N}, r \in \mathbb{Q}$.\}
(b)if $f(I)$ *is bounded for some non-empty open interval* I, *then* f *is continuous at* 0\{*Hint: First show that we may suppose* $0 \in I$ *and then use (a).*\}
(c)if f *is continuous at* 0, *then* f *is continuous on* $\mathbb{R}$
(d)if f *is continuous on* $\mathbb{R}$*,then* $f(x) = ax$ *for all* $x \in \mathbb{R}$*,where* $a = f(1)$.\{*Hint: show that if two continuous functions* f *and* g *are equal on a dense subset* $D \in X$ *then they are equal on* X *and use (a).*\}
(e)Let $g : \mathbb{R} \to \mathbb{R}$ *be continuous, not identically zero, and satisfy* $g(x+y) = g(x)g(y)$ *for all* $x, y \in \mathbb{R}$. *Then* $g(x) = a^x$ *for all* $x \in \mathbb{R}$ *where* $a = g(1)$.
\{*Hint:Show that* $g(x) > 0$ *for all* x *and define* $f(x) = \log g(x)$.\}
(f) Let $h : (0, \infty) \to \mathbb{R}$ *be continuous and satisfy* $h(xy) = h(x) + h(y)$ *for all* $x > 0$ *and* $y > 0$. *Then* $h(x) = b \log x$ *for all* $x > 0$ *where* $b = h(e)$.
(g) Let $k : (0, \infty) \to \mathbb{R}$ *be continuous, not identically zero, and satisfy* $k(xy) = k(x)k(y)$ *for all* $x > 0$ *and* $y > 0$. *Then* $k(x) = x^b$ *for all* $x > 0$ *where* $b = \log k(e)$.

Question 0.6. *(a) A topological space* X *is connected if and only if every continuous function from* X *into the discrete two element space* $\{0, 1\}$ *is a constant.*
(b) Let X *be a topological space , let* $A \subset X$ *be connected, and let* $A \subset B \subset \overline{A}$. *Then* B *is connected.*
(c) Let $\mathfrak{B}$ *be a non-empty family of connected subsets of a topological space* X *such that* $\cap \mathfrak{B} \neq \emptyset$. *Then* $\cup \mathfrak{B}$ *is connected.*

Question 0.7. \{*H.Hahn*\} *If* X *is a metric space and* f *and* g *are extended real-valued functions such that* f *is lower semicontinuous on* X, g *is upper semicontinuous on* X, *and for each* $x \in X$,

$$f(x) > -\infty, \quad g(x) < \infty, \quad g(x) \leq f(x),$$

then there exists a continuous real-valued function h *on* X *such that* $g(x) \leq h(x) \leq f(x)$ *for all* $x \in X$
\{*Hint: Fill the many missing parts in the following: Choose a nondecreasing sequence* $(f_n)_{n=1}^{\infty}$ *and a nonincreasing sequence* $(g_n)_{n=1}^{\infty}$ *of continuous real-valued functions on* X *such that* $f_n(x) \to f(x)$ *and* $g_n(x) \to g(x)$ *for all* $x \in X$. *For any extended real-valued function* ϕ *on* X *, define* ϕ^+ *on* X *by*

$\phi^+(x) = \max\{\phi(x), 0\}$. *Then*

$$(g_1 - f_1)^+ \geq (g_1 - f_2)^+ \geq (g_2 - f_2)^+ \geq ... \geq (g_n - f_n)^+$$
$$\geq (g_n - f_{n+1})^+ \geq (g_{n+1} - f_{n+1})^+ \geq ...$$

and this sequence has limit 0 *at every* $x \in X$. *Therefore the alternating series* $s = (g_1 - f_1)^+ - (g_1 - f_2)^+ + (g_2 - f_2)^+ - ...$ *formed from this sequence converges at every* $x \in X$. *The partial sums* s_n *of this series are continuous and satisfy* $s_1 \geq s_3 \geq s_{5 \geq ...}$ *and* $s_2 \leq s_4 \leq s_6 \leq ...$, *so* s *is both upper and lower semicontinuous and thus continues on* X. *Write* $h = f_1 + s$. *If* $f(x) = g(x)$, *then* $f_j(x) \leq g_k(x)$ *for all* j *and* k, *so*

$$h(x) = \lim_{n\to\infty}(f_1(x) + s_{2n-1}(x)) = \lim_{n\to\infty} g_n(x) = g(x)$$

and

$$h(x) = \lim_{n\to\infty}(f_1(x) + s_{2n}(x)) = \lim_{n\to\infty} f_{n+1}(x) = f(x)$$

Suppose $g(x) < f(x)$. *If* $(g_n - f_n)^+(x)$ *is the first* 0 *term in the series for* $s(x)$, *then* $h(x) = f_n(x) \leq f(x)$ *and* $h(x) = f_n(x) \geq g_n(x) \geq g(x)$. *If the first* 0 *term is* $(g_n - f_{n+1})^+(x)$ *then* $f(x) \geq f_{n+1}(x) \geq g_n(x) = h(x) \geq g(x)$.*}*

Question 0.8. *For* $n \in \mathbb{N}$ *and* $x \geq 0$ *define* $f_1(x) = \sqrt{x}$, $\quad f_{n+1}(x) = \sqrt{x + f_n(x)}$. *Then*
(a)$0 < f_n(x) < f_{n+1}(x) < 1 + x$ *for all* $n \in \mathbb{N}$ *and* $x > 0$.
(b)$(f_n)_{n=1}^{\infty}$ *converges uniformly on* $[a, b]$ *whenever* $0 < a < b < \infty$ *but not on* $[0, 1]$.
{Hint:Use induction and Dini's Theorem which is given as:

Theorem 0.1. *Let* X *be a compact space and let* $(f_n)_{n=1}^{\infty}$ *be a sequence of real valued functions that are each continuous on* X. *Suppose that* $(f_n)_{n=1}^{\infty}$ *is nondecreasing on* X*(this means that* $f_1(x) \leq f_2(x) \leq ...$ *for each* $x \in X$*) and converges pointwise on* X *to a real valued limit* f *that is continuous on* X. *Then* $f_n \to f$ *uniformly on* X.*}*

Question 0.9. *Let* X *be a compact space and let* $L \subset C^r(X)$ *be a lattice. i.e.* $f, g \in L$ *implies* $f \wedge g$, $\quad f \vee g \in L$.
(a) If $f \in C^r(X)$ *and if for each* $x, y \in X$ *and* $\epsilon > 0$ *there is some* $h \in L$ *such that* $|f(z) - h(z)| < \epsilon$ *for* $z = x$ *and* $z = y$, *then there is an* $h \in L$ *such that* $|f(z) - h(z)| < \epsilon$ *for all* $z \in X$.
(b) If L *is a linear space (*$f, g \in L$ *and* $\alpha \in \mathbb{R}$ *imply* αf, $\quad f + g \in L$*),* L *separates the points of* X *and* $1 \in L$, *then* $\overline{L} = C^r(X)$.
{Hint:$h \in L$ *implies* $\alpha h + \beta \in L$ *for all* $\alpha, \beta \in \mathbb{R}$.*}*

Question 0.10. *For $n \in \mathbb{N}$, define f_n on $\mathbb{R}$ by $f_n(x) = x - n + 1$ if $n - 1 \leq x \leq n$, $\quad f_n(x) = n + 1 - x$ if $n \leq x \leq n + 1$ and $f_n(x) = 0$ for all other x. Then the set $F = \{f_n : n \in \mathbb{N}\}$ is a uniformly bounded equicontinuous subset of $C(\mathbb{R})$ and the sequence $(f_n)_{n=1}^{\infty} \subset F$ converges to 0 at every $x \in \mathbb{R}$, but it has no subsequence that converges uniformly on $\mathbb{R}$. This shows that we cannot delete "each compact subset of" from the following theorem and have it remain true.*

Theorem 0.2. *(Arzelà-Ascoli) Suppose that the topological space X is separable and the metric space Y is complete. Let F be a non-empty equicontinuous subset of $C(X,Y)$ having the property that for each $x \in X$, the closure of the set $\{f(x) : f \in F\}$ is a compact subset of Y. Then each sequence $(f_n)_{n=1}^{\infty} \subset F$ has a subsequence that converges pointwise on X to a function $f \in C(X,Y)$. Moreover, this convergence is uniform on each compact subset of X.*