# 40 SORUDA ANALİZE GİRİŞ

## Nesin Matematik Köyü, 12-17 Eylül 2011

### PROBLEMS ON INFINITE SERIES AND INFINITE PRODUCTS

**(1)** [PROBLEM 12, p. 411.]

(a) If $(c_n)_{n\in\mathbb{N}}$ is a monotone sequence of positive terms such that $\sum_{n\in\mathbb{N}} c_n < \infty$, then $\lim_{n\to\infty} nc_n = 0$.

*Hint*: $nc_{2n} \leqslant c_{n+1} + \cdots + c_{2n}$.

(b) Assertion (a) is best possible in the sense that if $p_n \to \infty$ as $n \to \infty$, then there is a series $\sum_{n\in\mathbb{N}} c_n$ as in (a) such that $\limsup_{n\to\infty} np_nc_n = \infty$. Thus, the conclusion of (a) *cannot* be changed to read $\lim_{n\to\infty}(n\log n)c_n = 0$.

*Hint*: Write $p_0 = 0$ and choose $0 = n_0 < n_1 < n_2 < \cdots$, $p_{n_{k-1}} < p_{n_k}$ and $p_{n_k} > 4^k$ for all $k \geqslant 1$. Let $c_n = \left(nk\sqrt{p_{n_k}}\right)^{-1}$ for $n_{k-1} < n \leqslant n_k$.

**(2)** [PROBLEM 3, p. 426.]

(a) Suppose that $\{b_n\}_{n=1}^{\infty} \subseteq [0,\infty)$ is a non-increasing sequence, $b_n \to 0$, and $\sum b_n = \infty$. Then $\sum b_n\cos(n\theta)$ and $\sum b_n\sin(n\theta)$ are conditionally [not absolutely] convergent for $\theta \neq k\pi$, $k \in \mathbb{Z}$.

(a) We have, for $-\pi < \theta < \pi$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos(n\theta)}{n} = -\log\left(2\cos\frac{\theta}{2}\right)$$

and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\theta)}{n} = \frac{\theta}{2},$$

the convergence being uniform on $[-\pi+\delta, \pi-\delta]$ for any $0 < \delta < \pi$.

**(3)** [PROBLEM 4, p. 426.]

Establish the following equalities:

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}(1-\cos(nx))/n^2 = x^2/4$ for $-\pi \leqslant x \leqslant \pi$.

(b) $\sum_{n=1}^{\infty}(-1)^{n}\cos(nx)/n^2 = (3x^2-\pi^2)/12$ for $-\pi \leqslant x \leqslant \pi$.

(c) $\sum_{n=1}^{\infty}\cos(nx)/n^2 = (3x^2-6\pi x+2\pi^2)/12$ for $0 \leqslant x \leqslant 2\pi$.

*Hint*: For $0 < x < \pi$, integrate the second equality in the preceding exercise over $[0,x]$ to obtain (a) for $-\pi < x < \pi$. Then note that both sides of (a) are continuous on $[-\pi,\pi]$. For (b), integrate (a) over $[-\pi,\pi]$.

**(4)** [PROBLEM 6, p. 427.]

Let $x \in \mathbb{R}$, $x > 0$. Test each series $\sum a_n$ for convergence, where $a_n$ is given by

(a) $[\log(\log n)]^{-\log n}$ $\quad (n > 2)$,

(b) $\left(e - \left(1+\frac{1}{n}\right)^n\right)^x$,

(c) $\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n}\right)^x$.

**(5)** [PROBLEM 12, p. 443.]

Suppose that

$$a_{n+1}/a_n = 1 - \gamma/n - w_n/n^p,$$

where $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence, $p > 1$, $\gamma \in \mathbb{C}$, and $\mathrm{Re}(\gamma) \leqslant 1$. Then $\sum_{n\in\mathbb{N}} a_n$ diverges.

*Hints*: First consider $\gamma \neq 1$. Writing $z_n = nw_n/(n-\gamma)$, we see that $(z_n)_{n\in\mathbb{N}}$ is bounded and

$$\frac{a_{n+1}}{a_n} = (1-\gamma/n)(1-z_n/n^p),$$

$$\begin{aligned} a_{n+1} &= a_1 \cdot \prod_{k=1}^{n} \frac{k-\gamma}{k} \cdot \prod_{k=1}^{n} (1 - z_k/k^p) \\ &= a_1 \cdot (-1)^n \cdot \binom{\gamma-1}{n} \cdot c_{n+1}, \end{aligned}$$

where the last equality defines $c_{n+1}$. Choose $M$ and $\delta$ so that $0 < \delta \leqslant |c_n| \leqslant M$ and $|z_n| \leqslant M$ for all $n$. Then $|c_n^{-1} - c_{n+1}^{-1}| \leqslant M^2\delta^{-2}n^{-p}$, so $\sum |c_n^{-1} - c_{n+1}^{-1}| < \infty$. If $\sum a_n$ converges, then $\sum a_n c_n^{-1}$ converges (7. 36. iii), and so $\sum (-1)^n \binom{\gamma-1}{n}$ converges, contrary to (7. 46). If $\gamma = 1$, we can write

$$\frac{n+1}{n} \cdot \frac{a_{n+1}}{a_n} = 1 + u_n/n^q,$$

where $(u_n)_{n\in\mathbb{N}}$ is bounded and $q = \min\{p, 2\}$. Thus

$$a_n = a_1/n \cdot \prod_{k=1}^{n-1} \left(1 + \frac{u_k}{k^q}\right) = \frac{a_1}{n} \cdot d_n.$$

As before, $\sum |d_n^{-1} - d_{n+1}^{-1}| < \infty$, and $\sum d_n$ diverges.

**(6)** (a) If $\{\varepsilon_n\}$ is a strictly decreasing sequence with limit 0, then there is some divergent series $\sum d_n,\ d_n > 0$, such that

$$\sum \varepsilon_n d_n < \infty.$$

(b) If $\{M_n\}$ is a strictly increasing sequence with limit $\infty$, then there is some convergent series $\sum c_n,\ c_n > 0$, such that

$$\sum M_n c_n = \infty.$$

**(7)** (a) Suppose that $1 \leqslant n_1 < n_2 < \ldots$ and, for some $\beta \in \mathbb{R}$,

$$n_{k+1} - n_k \leqslant \beta(n_k - n_{k-1})$$

for all $k > 1$. If $\{a_n\}$ is a non-increasing sequence with positive terms, then

$$\sum a_n < \infty \quad \text{if and only if} \quad \sum (n_{k+1} - n_k) a_{n_k} < \infty.$$

(b) If $p > 1$ is an integer, then the sequences $n_k = p^k$ and $n_k = k^p$ satisfy the hypothesis of (a).

**(8)** [PROBLEM 12, p. 492.]

(a) Let $(a_{j,k})_{j,k=0}^{\infty}$ be an infinite matrix of complex numbers with the property that if $(s_k)_{k=0}^{\infty} \subseteq \mathbb{C}$ and $s_k \to 0$, then

(i) $\tau_j = \sum_{k=0}^{\infty} a_{j,k}s_k$ converges for all $j \geqslant 0$, and
(ii) $(\tau_j)_{j=0}^{\infty}$ is a bounded sequence.
Then
(iii) $M_j = \sum_{k=0}^{\infty} |a_{j,k}| < \infty$ for all $j \geqslant 0$, and
(iv) $(M_j)_{j=0}^{\infty}$ is a bounded sequence.

*Hints*: Assume that (iii) fails for some fixed $j$. Let $k_0$ be the smallest $k$ with $a_{j,k} \neq 0$. Write $b_k = 0$ for $0 \leqslant k < k_0$ and $b_k = \left(\sum_{n=k_0}^{k} |a_{j,n}|\right)^{-1}$ for $k \geqslant k_0$. Choose $s_k = b_k \cdot \operatorname{sgn} \overline{a_{j,k}}$. Then $s_k \to 0$ and, using (2. 41. i), $\tau_j = \sum_{k=k_0}^{\infty} b_k |a_{j,k}| = \infty$, contrary to (i).

For a fixed integer $p \geqslant 0$, let $s_p = 1$ and $s_k = 0$ for $k \neq p$. By (ii), $|a_{j,p}| = |\tau_j| \leqslant \beta_p < \infty$ for all $j \geqslant 0$. Write $B_k = \sum_{p=0}^{k} \beta_p$. Assume that (iv) is false. Let $j_0 = k_1 = 0$ and suppose that $j_{r-1}$ and $k_r$ have been chosen for some $r \in \mathbb{N}$. Choose $j_r > j_{r-1}$ such that

$$M_{j_r} > 2rB_{k_r} + r^2 + r + 1$$

and choose $k_{r+1} > k_r$ such that

$$\sum_{k=0}^{k_{r+1}} |a_{j_r,k}| > M_{j_r} - 1.$$

Then

$$\sum_{k=k_r+1}^{k_{r+1}} |a_{j_r,k}| > (M_{j_r} - 1) - B_{k_r} > rB_{k_r} + r^2 + r.$$

This defines $(j_r)_{r=0}^{\infty}$ and $(k_r)_{r=1}^{\infty}$. Now take $s_0 = 0$ and $s_k = r^{-1} \cdot \operatorname{sgn} \overline{a_{j_r,k}}$ for $k_r < k \leqslant k_{r+1}$. Then $s_k \to 0$ and

$$|\tau_{j_r}| \geqslant \sum_{k=k_r+1}^{k_{r+1}} r^{-1} \cdot |a_{j_r,k}| - \sum_{k=0}^{k_r} |a_{j_r,k}| - \sum_{k=k_{r+1}+1}^{\infty} |a_{j_r,k}|$$
$$> (B_{k_r} + r + 1) - B_{k_r} - 1 = r.$$

Thus, $\lim_{r\to\infty} |\tau_{j_r}| = \infty$, contrary to (ii).

(b) Let $(b_n)_{n=0}^{\infty} \subseteq \mathbb{C}$. A necessary and sufficient condition that $\sum_{n=0}^{\infty} b_n c_n$ be convergent for every convergent series $\sum_{n=0}^{\infty} c_n$ of complex terms is that $\sum_{n=0}^{\infty} |b_n - b_{n+1}| < \infty$.

*Hints*: For sufficiency, use (7. 36. iii). For necessity, write $s_k = c_0 + c_1 + \cdots + c_k$, note that

$$\tau_j = \sum_{n=0}^{j} b_n c_n = \sum_{k=0}^{j-1} (b_k - b_{k+1}) s_k + b_j s_j = \sum_{k=0}^{\infty} a_{j,k} s_k,$$

and use (a).

**(9)** If $\gamma \in \mathbb{C}$ and $\operatorname{Re}(\gamma) \leqslant 1$, then the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\gamma}}$$

diverges. In fact, this series is neither Abel summable nor $C_1$-summable. The fact that this series is not Abel summable follows from Littlewood's Tauberian theorem [Theorem 7.91] and that this series is not $C_1$-summable follows from Hardy's Tauberian theorem [Theorem 7.92].

**(10)** [PROBLEM 2, p. 500.]

If a series $\sum_{n=0}^{\infty} a_n$ having non-negative real terms is Abel summable to $s \in \mathbb{R}$, then it converges to $s$.

*Hint*: $\sum_{n=0}^{N} a_n r^n \leqslant s$ for $0 < r < 1$ and $N \in \mathbb{N}$.