FUNCTIONAL ANALYTIC APPROACH TO THE NON-LINEAR THEORY OF DISTRIBUTIONS (GENERALIZED FUNCTIONS)

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The so-called 'Stability Paradox' and Laurent Schwartz's 'Impossibility Theorem' was for a long time misinterpreted as "it is impossible to define a suitable rule of product for linear distributions". The impossibility result states that provided some natural requirements be satisfied, $\mathcal{D}'$: the space of Schwartz distributions can not be imbedded into a differential algebra while preserving the distributional derivative, natural properties and the Leibniz rule etc. A by-product of this result is that any imbedding of this kind should necessarily change the pointwise product of continuous functions.

However such an imbedding is possible should some concessions be made. After many such attempts some with gross concessions, finally J.F. Colombeau managed in 1984 to imbed $\mathcal{D}'$ into a differential algebra warding off the consequences of the stability paradox and the impossibility result in an optimal way. His concessions were optimal in the sense that his differential algebra $\mathcal{G}$ preserved the distributional derivative, Leibniz rule and the pointwise product of smooth functions. As for pointwise product rule of continuous functions, it was preserved within a suitable equivalence relation in the Colombeau's coupled calculus.

Colombeau's construction depends on the reflexivity of $\mathcal{C}^\infty(\mathcal{R}^n)$, thus the isomorphism of $\mathcal{E}''(\mathcal{R}^n)$ and $\mathcal{C}^\infty(\mathcal{R}^n)$, where $\mathcal{E}'$ stands for the compactly supported distributions. Because of this isomorphism, the members of $\mathcal{E}''(\mathcal{R}^n)$ can be multiplied like ordinary functions. This suggests an equivalence relation in $\mathcal{C}^\infty(\mathcal{E}'(\mathcal{R}^n))$. Accordingly in $\mathcal{C}^\infty(\mathcal{E}'(\mathcal{R}^n))/I$, where I is the kernel of this equivalence relation, the difficulty of defining a suitable product rule is tackled. Then finding out a larger ideal J we can imbed this factor space into $\mathcal{C}^\infty(\mathcal{D}(\mathcal{R}^n))/J.$ Restricting this to a strictly smaller subalgebra, namely the algebra of moderate elements, we obtain the Colombeau algebra $\mathcal{G}$ of 'non-linear distributions' which contains $\mathcal{D}'$ as a vector subspace.

In this talk after a brief review of the basic features of the Schwartz distributions the details of the above construction will be given. Colombeau's theory, although still unpopular most probably due to its complexity, has already been applied with success to different areas of mathematical physics, most notably to the quantum field theory, boundary value problems especially to those with irregular or distributional initial data. To my view it offers great potentiality in the theory of non-linear deterministic and stochastic p.d.e. and Wiener functionals.